the witness reduction technique

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presentation structure

- ❑motivation
- ❑preliminaries
- ❑what *The witness reduction technique* is about
- ❑warm up
- ❑a central question

presentation structure

- ❑other questions…
- ❑about **OptP** and **SpanP**
- ❑resources

motivation

This technique is mainly used in order to relate potential closure properties of **#P** *to complexity class hierarchy collapses!*

preliminaries: classes

language classes

- **NP**, co**NP**
- **PH**
- **UP**
- $\oplus$ **P**

function classes

- **FP**
- **#P**
- **OptP**
- **SpanP**

preliminaries: classes

language classes **(continued)**

- ❑**SPP**
- ❑**PP**
- ❑$C_{=}P$

preliminaries: operations

- addition $a + b$
- multiplication $a \cdot b$
- proper subtraction $a \ominus b = \max(\{0, a - b\})$
- proper division $a \oslash b = \lfloor a / b \rfloor$

preliminaries: operations

- **proper decrement** $a \ominus 1$
- **proper division by two** $a \oslash 2$
- **maximum** $\max(\{a, b\}) = a \text{ if } a \geq b \text{ else } b$
- **minimum** $\min(\{a, b\}) = a \text{ if } a \leq b \text{ else } b$

(pause)

$\text{Closed}(\blacksquare, \boxplus) \equiv$ the complexity class $\blacksquare$ is closed under the operation $\boxplus$

(pause)

Turing Machine ≡ Non-Deterministic and Polynomial-Time Turing Machine

warm up

$$\text{Closed}(\#\mathbf{P}, +)$$

$f, g \in \#\mathbf{P}$

$\exists M_1, M_2 : f(x) = \#acc_{M_1}(x) \wedge g(x) = \#acc_{M_2}(x)$

$$M_3(x) =_{\text{def}} M_1(x) \text{ or } M_2(x)$$

$\#acc_{M_3}(x) = f(x) + g(x)$

$\#acc_{M_3}(x) \in \#\mathbf{P} \Rightarrow f(x) + g(x) \in \#\mathbf{P}$

warm up

$$\text{Closed}(\#\mathbf{P},\cdot)$$

$f, g \in \#\mathbf{P}$

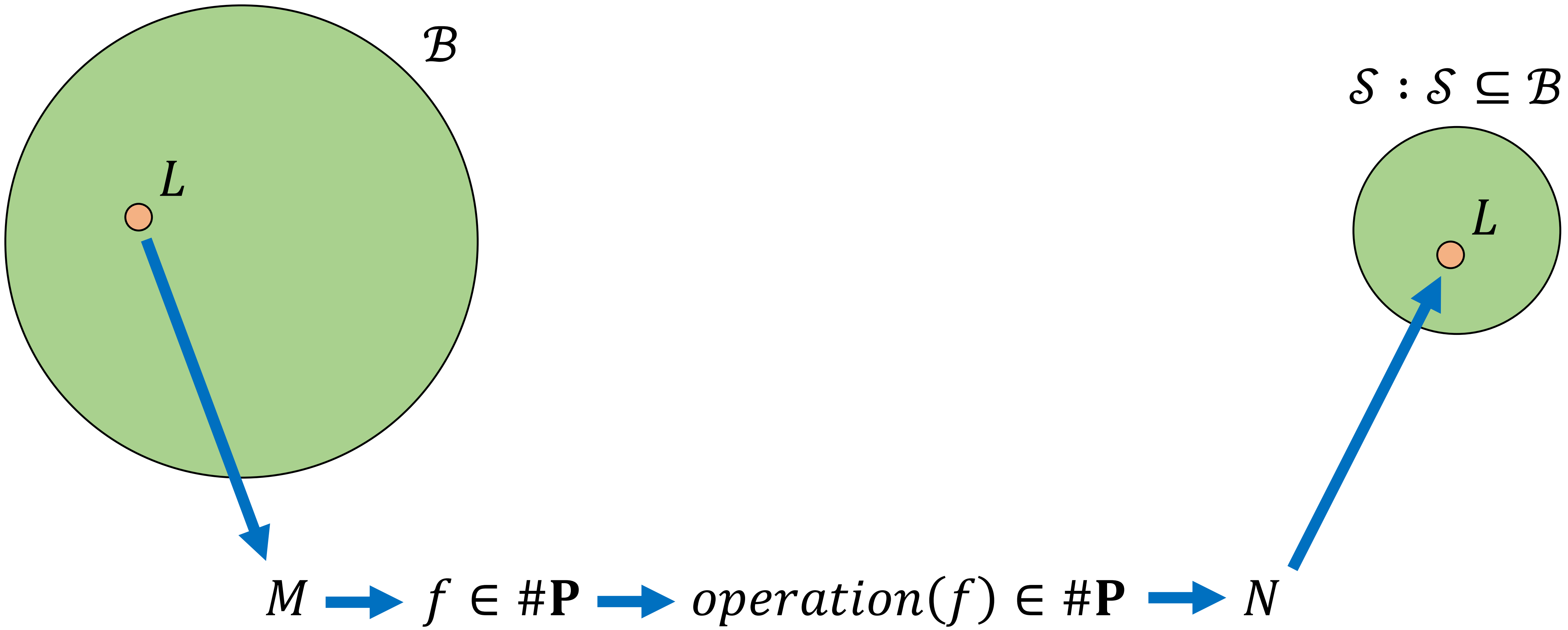
$\exists M_1, M_2 : f(x) = \#acc_{M_1}(x) \wedge g(x) = \#acc_{M_2}(x)$

$$M_3(x) =_{\text{def}} M_1(x) \text{ and } M_2(x)$$

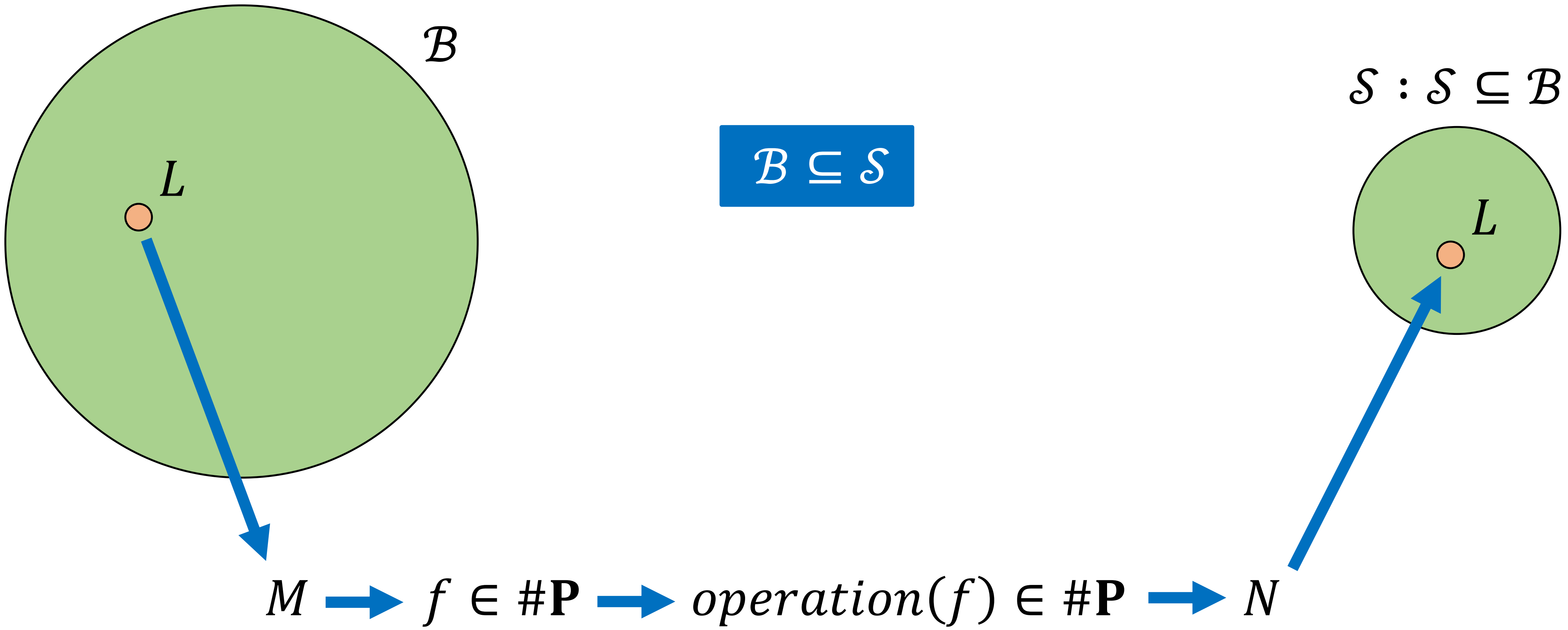
$\#acc_{M_3}(x) = f(x) \cdot g(x)$

$\#acc_{M_3}(x) \in \#\mathbf{P} \Rightarrow f(x) \cdot g(x) \in \#\mathbf{P}$

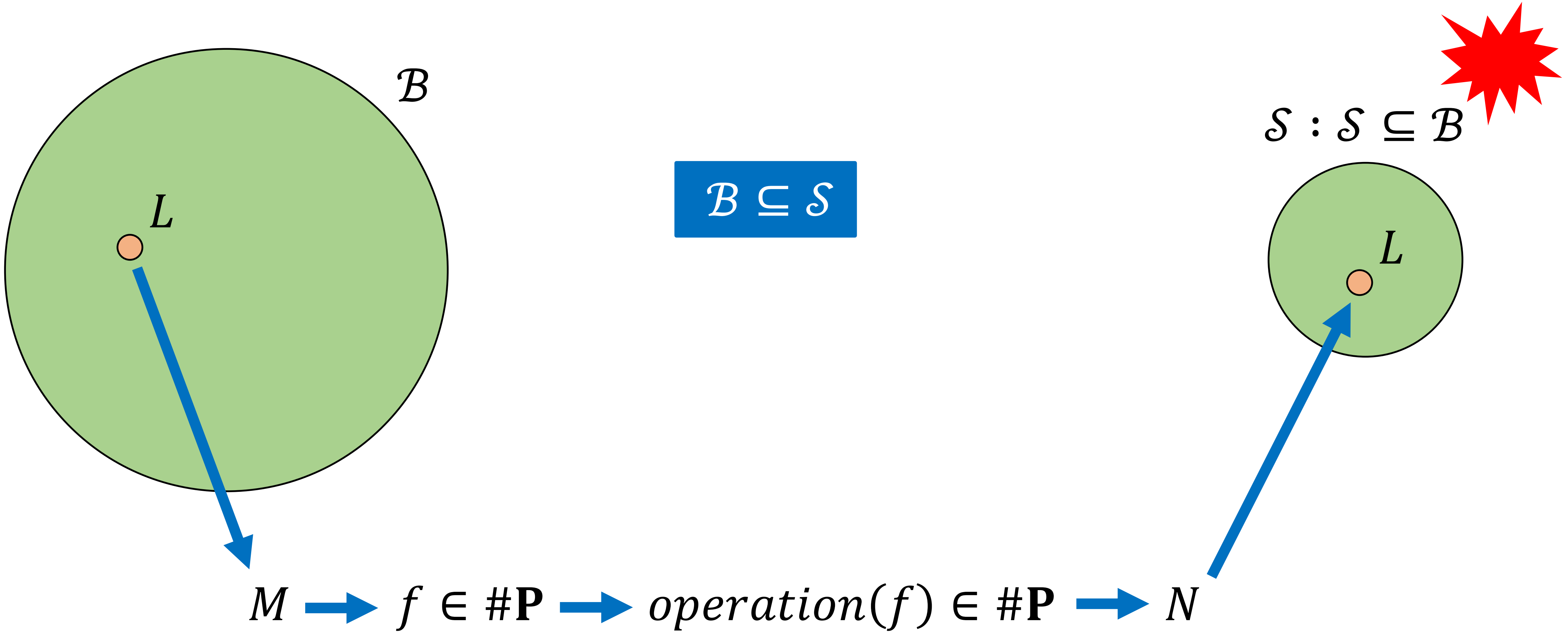
what *The witness reduction technique* is about



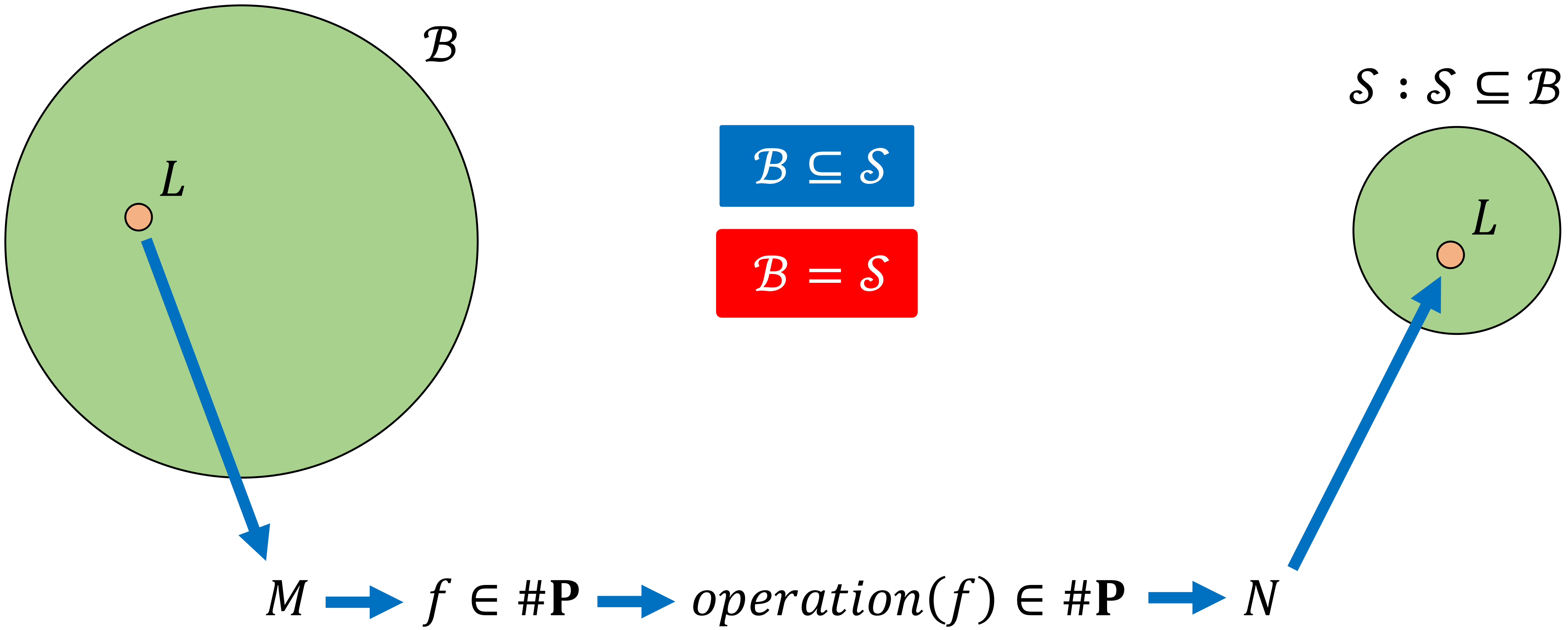
what The witness reduction technique is about



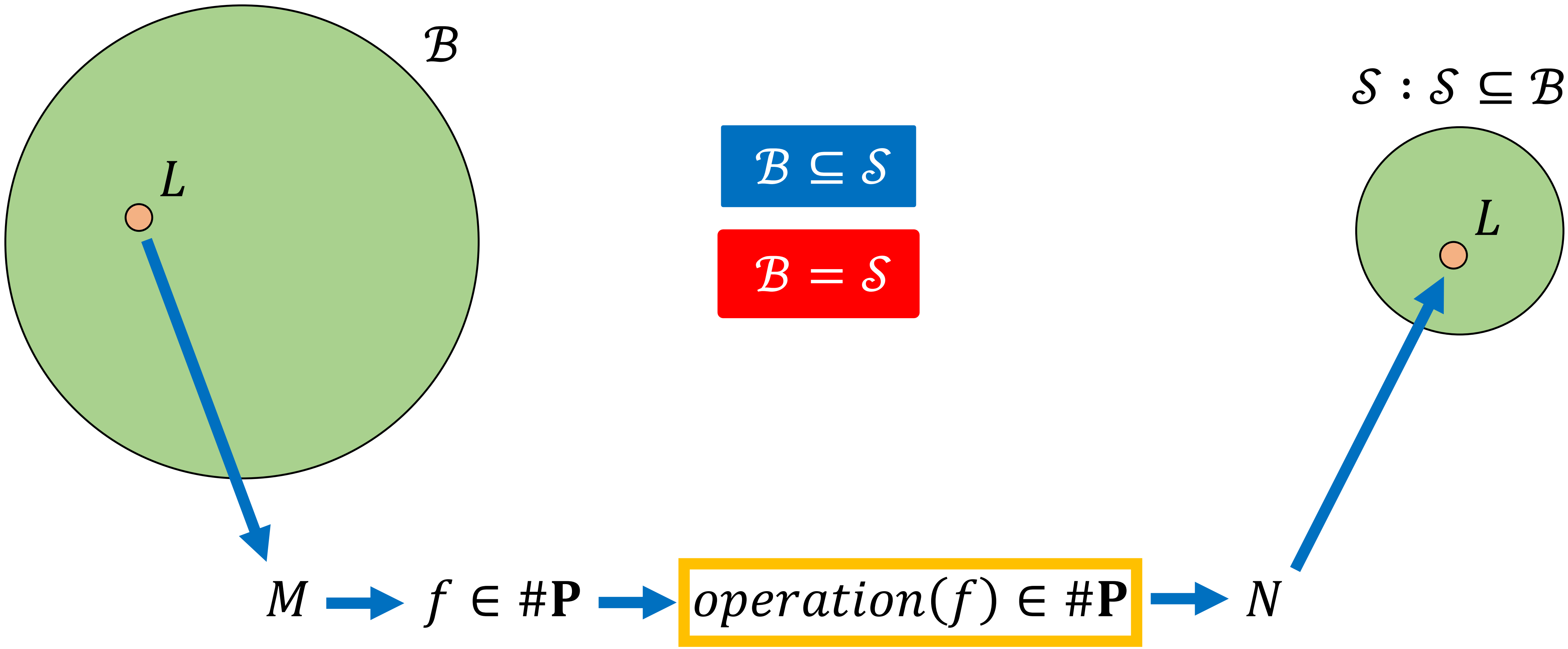
what The witness reduction technique is about



what *The witness reduction technique* is about



what *The witness reduction technique* is about



a central question

$$\text{Closed}(\#\mathbf{P}, \ominus) \stackrel{?}{=} \textbf{True}$$

a central question

1. $\text{Closed}(\#\mathbf{P}, \ominus)$
2. $\text{Closed}(\#\mathbf{P}, op)$
3. $\mathbf{UP} = \mathbf{PP}$

$op : \text{Function}(op, \mathbb{N}^2, \mathbb{N}) \wedge \text{PolyTime}(op)$

$$1 \equiv 2 \equiv 3$$

a central question: a theorem

1. $\text{Closed}(\#\mathbf{P}, \ominus)$

2. $\text{Closed}(\#\mathbf{P}, op)$

3. $\mathbf{UP} = \mathbf{PP}$

$op : \text{Function}(op, \mathbb{N}^2, \mathbb{N}) \wedge \text{PolyTime}(op)$

$$1 \equiv 2 \equiv 3$$

Proof

$$\mathbf{1 \rightarrow 3 \quad 3 \rightarrow 2 \quad 2 \rightarrow 1}$$

(pause)

$\mathbf{FP} \subseteq \#\mathbf{P}$

a theorem: proof, 1 → 3

$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{UP} = \mathbf{PP}$

a theorem: proof, 1 → 3

$$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{UP} = \mathbf{PP}$$

$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{PP} \subseteq \mathbf{UP} \wedge \mathbf{UP} \subseteq \mathbf{PP}$

$\mathbf{UP} \subseteq \mathbf{PP}$: **True**

$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{PP} \subseteq \mathbf{UP}$

$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{PP} \subseteq \text{co}\mathbf{NP} \wedge \text{co}\mathbf{NP} \subseteq \mathbf{UP}$

a theorem: proof, 1 → 3

$$\text{Closed}(\#\mathbf{P},\ominus) \rightarrow \mathbf{PP} \subseteq \text{co}\mathbf{NP} \wedge \text{co}\mathbf{NP} \subseteq \mathbf{UP}$$

$$\Rightarrow \begin{cases} \text{Closed}(\#\mathbf{P},\ominus) \rightarrow \mathbf{PP} \subseteq \text{co}\mathbf{NP} \\ \text{Closed}(\#\mathbf{P},\ominus) \rightarrow \text{co}\mathbf{NP} \subseteq \mathbf{UP} \end{cases}$$

a theorem: proof, 1 → 3

$$\begin{cases} \mathrm{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{PP} \subseteq \mathrm{co}\mathbf{NP} \\ \mathrm{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathrm{co}\mathbf{NP} \subseteq \mathbf{UP} \end{cases}$$

a theorem: proof, 1 → 3

$$\text{Closed}(\#\mathbf{P},\ominus) \rightarrow \mathbf{PP} \subseteq \text{co}\mathbf{NP}$$

$L \in \mathbf{PP}$

$$\Rightarrow L = \left\{x : |\{y : R(x,y) \wedge |y| \leq p(|x|)\}| \geq \frac{2^{p(|x|)}}{2} + 1\right\}$$

a theorem: proof, $1 \to 3$

Turing Machine $M(x)$

guess $y : |y| \leq p(|x|)$

return 1 **if** $R(x, y)$ **else** 0

a theorem: proof, $1 \to 3$

Turing Machine $M(x)$

guess $y : |y| \leq p(|x|)$

return 1 **if** $R(x, y)$ **else** 0

$\Rightarrow L(M) = L$

$\& \exists f : f(x) = \#acc_M(x) \in \#\mathbf{P}$

observe $\begin{cases} x \in L \to f(x) \geq \dfrac{2^{p(|x|)}}{2} + 1 \\ x \notin L \to f(x) \leq \dfrac{2^{p(|x|)}}{2} \end{cases}$

a theorem: proof, 1 → 3

$$g(x) = \frac{2^{p(|x|)}}{2} \in \mathbf{FP} \wedge \mathbf{FP} \subseteq \#\mathbf{P} \Rightarrow g(x) \in \#\mathbf{P}$$

$$\text{Closed}(\#\mathbf{P}, \ominus) \Rightarrow h(x) = f(x) \ominus g(x) \in \#\mathbf{P}$$

$$\Rightarrow \exists N : h(x) = \#acc_N(x)$$

a theorem: proof, 1 → 3

$$x \in L \to f(x) \geq \frac{2^{p(|x|)}}{2} + 1 \;\&\; g(x) = \frac{2^{p(|x|)}}{2}$$

$$\to h(x) = f(x) \ominus g(x) \geq 1$$

$$x \notin L \to f(x) \leq \frac{2^{p(|x|)}}{2} \;\&\; g(x) = \frac{2^{p(|x|)}}{2}$$

$$\to h(x) = f(x) \ominus g(x) = 0$$

a theorem: proof, 1 → 3

$$x \in L \rightarrow h(x) \geq 1$$

$$x \notin L \rightarrow h(x) = 0$$

a theorem: proof, 1 → 3

so $\begin{cases} x \in L \rightarrow h(x) \geq 1 \\ x \notin L \rightarrow h(x) = 0 \end{cases}$ and $\exists N : h(x) = \#acc_N(x)$

$\Rightarrow L(N) = L \,\&\, L \in \mathbf{NP}$

$\Rightarrow \mathbf{PP} \subseteq \mathbf{NP}$

$\Rightarrow co\mathbf{PP} \subseteq co\mathbf{NP}$

$co\mathbf{PP} = \mathbf{PP} \Rightarrow \mathbf{PP} \subseteq co\mathbf{NP}$

a theorem: proof, 1 → 3

$$\begin{cases} \text{Closed}(\#\mathbf{P},\ominus) \rightarrow \mathbf{PP} \subseteq \text{co}\mathbf{NP}\ \textbf{OK} \\ \text{Closed}(\#\mathbf{P},\ominus) \rightarrow \text{co}\mathbf{NP} \subseteq \mathbf{UP} \end{cases}$$

a theorem: proof, 1 → 3

$$\text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \text{co}\mathbf{NP} \subseteq \mathbf{UP}$$

$L \in co\mathbf{NP}$

$\Rightarrow \bar{L} \in \mathbf{NP}$

$\Rightarrow \exists M : L(M) = \bar{L}$

$\Rightarrow \exists f : f(x) = \#acc_M(x) \in \#\mathbf{P}$

a theorem: proof, $1 \rightarrow 3$

so $\exists f : f(x) = \#acc_M(x) \in \#\mathbf{P}$ and $L(M) = \bar{L}$

observe $\begin{cases} x \in \bar{L} \rightarrow f(x) \geq 1 \\ x \notin \bar{L} \rightarrow f(x) = 0 \end{cases} \Rightarrow \begin{cases} x \in L \rightarrow f(x) = 0 \\ x \notin L \rightarrow f(x) \geq 1 \end{cases}$

$1 \in \mathbf{FP} \wedge \mathbf{FP} \subseteq \#\mathbf{P} \Rightarrow 1 \in \#\mathbf{P}$

$\text{Closed}(\#\mathbf{P}, \ominus) \Rightarrow g(x) = 1 \ominus f(x) \in \#\mathbf{P}$

a theorem: proof, 1 → 3

$$g(x) = 1 \ominus f(x) \in \#\mathbf{P}$$

$$\Rightarrow \exists N : \#acc_N(x) = g(x)$$

observe $\begin{cases} x \in L \rightarrow f(x) = 0 \rightarrow g(x) = 1 \ominus 0 = 1 \\ x \notin L \rightarrow f(x) \geq 1 \rightarrow g(x) = 1 \ominus f(x) = 0 \end{cases}$

a theorem: proof, 1 → 3

$g(x) = 1 \ominus f(x) \in \#\mathbf{P}$

$\Rightarrow \exists N : \#acc_N(x) = g(x)$

observe $\begin{cases} x \in L \rightarrow g(x) = 1 \\ x \notin L \rightarrow g(x) = 0 \end{cases}$

$\Rightarrow L(N) = L \;\&\; L \in \mathbf{UP}$

$\Rightarrow co\mathbf{NP} \subseteq \mathbf{UP}$

a theorem: proof, 1 → 3

$$\begin{cases} \text{Closed}(\#\mathbf{P}, \ominus) \to \mathbf{PP} \subseteq \text{co}\mathbf{NP} \ \textbf{OK} \\ \text{Closed}(\#\mathbf{P}, \ominus) \to \text{co}\mathbf{NP} \subseteq \mathbf{UP} \ \textbf{OK} \end{cases}$$

a theorem: proof, 1 → 3

$$\begin{cases} \text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \mathbf{PP} \subseteq \text{co}\mathbf{NP} \ \textbf{OK} \\ \text{Closed}(\#\mathbf{P}, \ominus) \rightarrow \text{co}\mathbf{NP} \subseteq \mathbf{UP} \ \textbf{OK} \end{cases}$$

$\Rightarrow \mathbf{PP} \subseteq \mathbf{UP}$

$$\mathbf{UP} \subseteq \mathbf{PP} \Rightarrow \mathbf{PP} = \mathbf{UP}$$

a central question: a theorem

1. $\text{Closed}(\#\mathbf{P}, \ominus)$
2. $\text{Closed}(\#\mathbf{P}, op)$
3. $\mathbf{UP} = \mathbf{PP}$

$$1 \equiv 2 \equiv 3$$

Proof

$$\mathbf{1 \to 3} \quad \mathbf{3 \to 2} \quad \mathbf{2 \to 1}$$

a theorem: proof, 3 → 2

$$\mathbf{PP} = \mathbf{UP} \rightarrow \mathrm{Closed}(\#\mathbf{P}, op)$$

op : $\mathrm{Function}(op, \mathbb{N}^2, \mathbb{N}) \wedge \mathrm{PolyTime}(op)$

a theorem: proof, 3 → 2

$$\mathbf{PP} = \mathbf{UP} \rightarrow \text{Closed}(\#\mathbf{P}, op)$$

$f, g \in \#\mathbf{P}$

$$B_f = \{\langle x, n\rangle : f(x) \geq n\}$$

$$B_g = \{\langle x, n\rangle : g(x) \geq n\}$$

a theorem: proof, 3 → 2

$$B_f = \{\langle x, n\rangle : f(x) \geq n\}$$

$$B_g = \{\langle x, n\rangle : g(x) \geq n\}$$

$B_f \in \mathbf{PP}$ & $B_g \in \mathbf{PP}$

a theorem: proof, 3 → 2

V

$= \{\langle x, n_1, n_2\rangle : \langle x, n_1\rangle \in B_f \wedge \langle x, n_1 + 1\rangle \notin B_f \wedge \langle x, n_2\rangle$

a theorem: proof, 3 → 2

$$B_f \in \mathbf{PP} \,\&\, B_g \in \mathbf{PP} \Rightarrow \mathrm{DisjointUnion}\left(B_f, B_g\right) = B_f \uplus B_g \in \mathbf{PP}$$

$$X \uplus Y = \{0x : x \in X\} \cup \{1y : y \in Y\}$$

a theorem: proof, 3 → 2

$$B_f \in \mathbf{PP} \ \& \ B_g \in \mathbf{PP} \Rightarrow \mathrm{DisjointUnion}(B_f, B_g) = B_f \uplus B_g \in \mathbf{PP}$$

$$V \leq_{\text{bounded truth table}} B_f \uplus B_g$$

a theorem: proof, 3 → 2

$$B_f \in \mathbf{PP} \;\&\; B_g \in \mathbf{PP} \Rightarrow \text{DisjointUnion}(B_f, B_g) = B_f \uplus B_g \in \mathbf{PP}$$

$$V \leq_{\text{bounded truth table}} B_f \uplus B_g$$

$$\text{Closed}(\mathbf{PP}, \leq_{\text{bounded truth table}}) \Rightarrow V \in \mathbf{PP}$$

a theorem: proof, 3 → 2

$$V \in \mathbf{PP} \;\&\; \mathbf{PP} = \mathbf{UP} \Rightarrow V \in \mathbf{UP}$$

a theorem: proof, 3 → 2

$$V \in \mathbf{PP} \;\&\; \mathbf{PP} = \mathbf{UP} \Rightarrow V \in \mathbf{UP}$$

$\Rightarrow \exists N : L(N) = V$

$\&\; \forall x, n_1, n_2 : \#acc_N(\langle x, n_1, n_2 \rangle) = 0 \text{ or } 1$

a theorem: proof, 3 → 2

V

$= \{\langle x, n_1, n_2\rangle : \langle x, n_1\rangle \in B_f \wedge \langle x, n_1 + 1\rangle \notin B_f \wedge \langle x, n_2\rangle$

a theorem: proof, 3 → 2

Turing Machine $M(x)$

guess i, j

if $N(\langle x, i, j \rangle) = yes$ **then**

guess $k : k \in \{1, \ldots, op(i, j)\}$

accept

return k

else reject

a theorem: proof, 3 → 2

Turing Machine $M(x)$

guess i, j

if $N(\langle x, i, j \rangle) = yes$ **then**

guess $k : k \in \{1, \ldots, op(i, j)\}$

accept

return k

else reject

$\forall x\ \exists i, j : i = f(x)\ \&\ j = g(x)$

$\Rightarrow \forall x\ \exists i, j : \langle x, i, j \rangle \in V$

$\Rightarrow \forall x\ \exists i, j : N(\langle x, i, j \rangle) = yes$

$\Rightarrow \forall x\ \exists i, j : \#acc_N(\langle x, i, j \rangle) \geq 1$

$\Rightarrow \forall x\ \exists i, j : \#acc_N(\langle x, i, j \rangle) = 1$

a theorem: proof, 3 → 2

$$x, i, j : \#acc_N(\langle x, i, j\rangle) = 1$$

$\Rightarrow N(\langle x, i, j\rangle) = yes$

$\Rightarrow \langle x, i, j\rangle \in V$

$\Rightarrow i = f(x) \wedge j = g(x)$

$\Rightarrow op(i, j) = op\big(f(x), g(x)\big)$

$\Rightarrow k \in \big\{1, \dots, op\big(f(x), g(x)\big)\big\}$

a theorem: proof, $3 \rightarrow 2$

Turing Machine $M(x)$

guess i, j

if $N(\langle x, i, j \rangle) = yes$ **then**

guess $k : k \in \{1, \dots, op(f(x), g(x))\}$

accept

return k

else reject

a theorem: proof, 3 → 2

$\#acc_M(x) = ?$

a theorem: proof, 3 → 2

$$\#acc_M(x) = \#acc_N(\langle x, i, j \rangle) \cdot op\big(f(x), g(x)\big)$$

a theorem: proof, 3 → 2

$$\#acc_M(x) = \#acc_N(\langle x, i, j\rangle) \cdot op\big(f(x), g(x)\big)$$

$$\Rightarrow \#acc_M(x) = 1 \cdot op\big(f(x), g(x)\big)$$

a theorem: proof, 3 → 2

Turing Machine $M(x)$

guess i, j

if $N(\langle x, i, j\rangle) = yes$ **then**

guess $k : k \in \{1, \ldots, op(i, j)\}$

accept

return k

else reject

$\forall x\ \exists i, j : i = f(x)\ \&\ j = g(x)$

$\Rightarrow \forall x\ \exists i, j : \langle x, i, j\rangle \in V$

$\Rightarrow \forall x\ \exists i, j : N(\langle x, i, j\rangle) = yes$

$\Rightarrow \forall x\ \exists i, j : \#acc_N(\langle x, i, j\rangle) \geq 1$

$\Rightarrow \forall x\ \exists i, j : \#acc_N(\langle x, i, j\rangle) = 1$

a theorem: proof, 3 → 2

$$\#acc_M(x) = \#acc_N(\langle x, i, j \rangle) \cdot op\big(f(x), g(x)\big)$$

$$\Rightarrow \#acc_M(x) = 1 \cdot op\big(f(x), g(x)\big) = op\big(f(x), g(x)\big)$$

$$\Rightarrow op\big(f(x), g(x)\big) \in \#\mathbf{P}$$

$$\Rightarrow \text{Closed}(\#\mathbf{P}, op)$$

a central question: a theorem

1. $\text{Closed}(\#\mathbf{P}, \ominus)$
2. $\text{Closed}(\#\mathbf{P}, op)$
3. **$\mathbf{UP} = \mathbf{PP}$**

$$1 \equiv 2 \equiv 3$$

Proof

$$\mathbf{1 \to 3 \; 3 \to 2 \; 2 \to 1}$$

a theorem: proof, 2 → 1

$$\text{Closed}(\#\mathbf{P}, op) \rightarrow \text{Closed}(\#\mathbf{P}, \ominus)$$

$$op : \text{Function}(op, \mathbb{N}^2, \mathbb{N}) \wedge \text{PolyTime}(op)$$

a theorem: proof, 2 → 1

$\text{Closed}(\#\mathbf{P}, op) \rightarrow \text{Closed}(\#\mathbf{P}, \ominus)$

True

$op : \text{Function}(op, \mathbb{N}^2, \mathbb{N}) \wedge \text{PolyTime}(op)$

a central question: a theorem

1. $\text{Closed}(\#\mathbf{P}, \ominus)$

2. $\text{Closed}(\#\mathbf{P}, op)$

3. $\mathbf{UP} = \mathbf{PP}$

$$1 \equiv 2 \equiv 3$$

Proof

$$\mathbf{1 \rightarrow 3 \quad 3 \rightarrow 2 \quad 2 \rightarrow 1}$$

a central question: a theorem

1. $\text{Closed}(\#\mathbf{P}, \ominus)$
2. $\text{Closed}(\#\mathbf{P}, op)$
3. $\mathbf{UP} = \mathbf{PP}$

$$1 \equiv 2 \equiv 3$$

Proved!

a central question: consequences

$$\mathbf{UP} = \mathbf{PP} \rightarrow$$

$$\mathbf{UP} = \mathbf{NP} = co\mathbf{NP} = \mathbf{PH} = \oplus\, \mathbf{P} = \mathbf{PP} \cup \mathbf{PP}^{\mathbf{PP}} \cup \mathbf{PP}^{\mathbf{PP}^{\mathbf{PP}}} \cup \cdots$$

an alternative way

1. $\mathrm{Closed}(\#\mathbf{P}, \oslash)$
2. $\mathrm{Closed}(\#\mathbf{P}, op)$
3. $\mathbf{UP} = \mathbf{PP}$

$op : \mathrm{Function}(op, \mathbb{N}^2, \mathbb{N}) \wedge \mathrm{PolyTime}(op)$

$$1 \equiv 2 \equiv 3$$

we can use $\oslash$ instead of $\ominus$

other questions…

What are the potential closure properties of **#P** *that do not imply that* **#P** *is closed under any polynomial-time operation?*

other questions...: some implications

$$\text{Closed}(\#\mathbf{P}, \ominus 1) \rightarrow co\mathbf{NP} \subseteq \mathbf{SPP}$$

$$\text{Closed}(\#\mathbf{P}, \oslash 2) \rightarrow \oplus \mathbf{P} = \mathbf{SPP}$$

$$\text{Closed}(\#\mathbf{P}, \min) \rightarrow \mathbf{NP} = \mathbf{UP}$$

$$\text{Closed}(\#\mathbf{P}, \max) \vee \text{Closed}(\#\mathbf{P}, \min) \rightarrow \mathrm{C}_{=}\mathbf{P} = \mathbf{SPP}$$

Closed(#P,$\ominus$ 1) → *co*NP ⊆ SPP

$L \in \mathbf{NP}$

$\Rightarrow \exists M : L(M) = L$

$\Rightarrow \exists f : f(x) = \#acc_M(x) \in \#\mathbf{P}$

$$1 \in \mathbf{FP} \wedge \mathbf{FP} \subseteq \#\mathbf{P} \Rightarrow 1 \in \#\mathbf{P}$$

$$\text{Closed}(\#\mathbf{P},\ominus) \Rightarrow g(x) = f(x) \ominus 1 \in \#\mathbf{P}$$

$\Rightarrow \exists N : g(x) = \#acc_N(x)$

Closed(#P,⊖ 1) → *co*NP ⊆ SPP

$\exists N : g(x) = f(x) \ominus 1 = \#acc_N(x)$

$$N'(x) = \neg N(x)$$

$\Rightarrow \#acc_{N'}(x) = \#rej_N(x) = 2^{p(|x|)} - g(x) \in \#\mathbf{P}$

$$\text{Closed}(\#\mathbf{P}, +) \Rightarrow h(x) = f(x) + \left(2^{p(|x|)} - g(x)\right) \in \#\mathbf{P}$$

$\Rightarrow \exists K : \#acc_K(x) = h(x) = f(x) + 2^{p(|x|)} - (f(x) \ominus 1)$

$\text{Closed}(\#\text{P}, \ominus 1) \rightarrow co\text{NP} \subseteq \text{SPP}$

$$\exists K : \#acc_K(x) = h(x) = f(x) + 2^{p(|x|)} - (f(x) \ominus 1)$$

$$x \notin L \Rightarrow f(x) = 0 \Rightarrow f(x) \ominus 1 = 0 \Rightarrow h(x) = 2^{p(|x|)}$$

$$x \in L \Rightarrow f(x) > 0 \Rightarrow f(x) \ominus 1 = f(x) - 1 \Rightarrow h(x) = 2^{p(|x|)} + 1$$

Closed(#P,⊖ 1) → *co*NP ⊆ SPP

$\exists K : \#acc_K(x) = h(x) = f(x) + 2^{p(|x|)} - (f(x) \ominus 1)$

$$x \notin L \Rightarrow h(x) = 2^{p(|x|)}$$

$$x \in L \Rightarrow h(x) = 2^{p(|x|)} + 1$$

$\Rightarrow L(K) = L \;\&\; L \in \mathbf{SPP}$

$\Rightarrow \mathbf{NP} \subseteq \mathbf{SPP}$

$\Rightarrow co\mathbf{NP} \subseteq co\mathbf{SPP}$

$\text{Closed}(\#\text{P}, \ominus 1) \rightarrow co\text{NP} \subseteq \text{SPP}$

so $co\text{NP} \subseteq co\text{SPP}$

$co\text{SPP} = \text{SPP} \Rightarrow co\text{NP} \subseteq \text{SPP}$

other questions...: some implications

$$\text{Closed}(\#\mathbf{P}, \ominus 1) \rightarrow co\mathbf{NP} \subseteq \mathbf{SPP}$$ **OK**

$$\text{Closed}(\#\mathbf{P}, \oslash 2) \rightarrow \oplus\mathbf{P} = \mathbf{SPP}$$

$$\text{Closed}(\#\mathbf{P}, \min) \rightarrow \mathbf{NP} = \mathbf{UP}$$

$$\text{Closed}(\#\mathbf{P}, \max) \vee \text{Closed}(\#\mathbf{P}, \min) \rightarrow \mathrm{C}_{=}\mathbf{P} = \mathbf{SPP}$$

Closed(#P,⊘ 2) →⊕ P = SPP

$$\oplus\,\mathbf{P} \subseteq \mathbf{SPP} \land \mathbf{SPP} \subseteq \oplus\,\mathbf{P}$$

$L \in \oplus\,\mathbf{P}$

$\Rightarrow \exists M : L(M) = L$

$\Rightarrow \exists f : f(x) = \#acc_M(x) \in \#\mathbf{P}$

$$2 \in \mathbf{FP} \land \mathbf{FP} \subseteq \#\mathbf{P} \Rightarrow 2 \in \#\mathbf{P}$$

$$\text{Closed}(\#\mathbf{P}, \oslash) \Rightarrow f(x) \oslash 2 \in \#\mathbf{P}$$

Closed(#P,⊘ 2) →⊕ P = SPP

$$\text{Closed}(\#\mathbf{P},\cdot) \Rightarrow 2 \cdot (f(x) \oslash 2) \in \#\mathbf{P}$$

$$g(x) = \#rej_M(x) = 2^{p(|x|)} - \#acc_M(x) = 2^{p(|x|)} - f(x)$$

$$\Rightarrow g(x) = \#acc_{\neg M}(x) \in \#\mathbf{P}$$

$$\text{Closed}(\#\mathbf{P},+) \Rightarrow h(x) = g(x) + 2 \cdot (f(x) \oslash 2) \in \#\mathbf{P}$$

$$\Rightarrow \exists N : \#acc_N(x) = h(x) = 2^{p(|x|)} - f(x) + 2 \cdot (f(x) \oslash 2)$$

$\text{Closed}(\#\text{P}, \oslash 2) \rightarrow \oplus\text{P} = \text{SPP}$

$$h(x) = 2^{p(|x|)} - f(x) + 2 \cdot (f(x) \oslash 2)$$

$$x \notin L \Rightarrow \text{Even}(f(x)) \Rightarrow 2 \cdot (f(x) \oslash 2) = f(x) \Rightarrow h(x) = 2^{p(|x|)}$$

$$x \in L \Rightarrow \text{Odd}(f(x)) \Rightarrow 2 \cdot (f(x) \oslash 2) = f(x) - 1 \Rightarrow h(x) = 2^{p(|x|)} - 1$$

Closed(#P, $\oslash$ 2) $\rightarrow \oplus$ P = SPP

$$h(x) = 2^{p(|x|)} - f(x) + 2 \cdot (f(x) \oslash 2)$$

$$x \notin L \rightarrow h(x) = 2^{p(|x|)}$$

$$x \in L \rightarrow h(x) = 2^{p(|x|)} - 1$$

$\Rightarrow L(N) = L \,\&\, L \in co\mathbf{SPP}$

$\Rightarrow \oplus \mathbf{P} \subseteq co\mathbf{SPP}$

$co\mathbf{SPP} = \mathbf{SPP} \Rightarrow \oplus \mathbf{P} \subseteq \mathbf{SPP}$

$\text{Closed}(\#P, \oslash 2) \rightarrow \oplus P = SPP$

$\oplus P \subseteq SPP \wedge SPP \subseteq \oplus P$

True

$\text{Closed}(\#\text{P}, \oslash 2) \rightarrow \oplus\text{P} = \text{SPP}$

$$\left.\begin{array}{r}\oplus\text{P} \subseteq \text{SPP} \\ \text{SPP} \subseteq \oplus\text{P}\end{array}\right\} \Rightarrow \oplus\text{P} = \text{SPP}$$

other questions...: some implications

$$\mathrm{Closed}(\#\mathbf{P}, \ominus 1) \rightarrow co\mathbf{NP} \subseteq \mathbf{SPP}$$ **OK**

$$\mathrm{Closed}(\#\mathbf{P}, \oslash 2) \rightarrow \oplus \mathbf{P} = \mathbf{SPP}$$ **OK**

$$\mathrm{Closed}(\#\mathbf{P}, \min) \rightarrow \mathbf{NP} = \mathbf{UP}$$

$$\mathrm{Closed}(\#\mathbf{P}, \max) \vee \mathrm{Closed}(\#\mathbf{P}, \min) \rightarrow \mathrm{C}_{=}\mathbf{P} = \mathbf{SPP}$$

$\mathrm{Closed}(\#\mathbf{P}, \min) \rightarrow \mathbf{NP} = \mathbf{UP}$

$$\mathrm{Closed}(\#\mathbf{P}, \min) \rightarrow \mathbf{NP} \subseteq \mathbf{UP} \wedge \mathbf{UP} \subseteq \mathbf{NP}$$

$L \in \mathbf{NP}$

$\Rightarrow \exists M : L(M) = L$

$\Rightarrow \exists f : f(x) = \#acc_M(x) \in \#\mathbf{P}$

$$1 \in \mathbf{FP} \wedge \mathbf{FP} \subseteq \#\mathbf{P} \Rightarrow 1 \in \#\mathbf{P}$$

$$\mathrm{Closed}(\#\mathbf{P}, \min) \Rightarrow g(x) = \min(\{f(x), 1\}) \in \#\mathbf{P}$$

Closed(#P, min) → NP = UP

$$g(x) = \min(\{f(x), 1\}) \in \#\mathbf{P}$$

$$\Rightarrow \exists N : \#acc_N(x) = g(x)$$

$$x \in L \to f(x) \geq 1 \to g(x) = 1$$

$$x \notin L \to f(x) = 0 \to g(x) = 0$$

$\textbf{Closed}(\#\mathbf{P}, \min) \rightarrow \mathbf{NP} = \mathbf{UP}$

$g(x) = \min(\{f(x), 1\}) \in \#\mathbf{P}$

$\Rightarrow \exists N : \#acc_N(x) = g(x)$

$x \in L \rightarrow g(x) = 1$

$x \notin L \rightarrow g(x) = 0$

$\Rightarrow L(N) = L \,\&\, L \in \mathbf{UP}$

$\Rightarrow \mathbf{NP} \subseteq \mathbf{UP}$

$\text{Closed}(\#\mathbf{P}, \min) \rightarrow \mathbf{NP} = \mathbf{UP}$

$$\text{Closed}(\#\mathbf{P}, \min) \rightarrow \mathbf{NP} \subseteq \mathbf{UP} \land \mathbf{UP} \subseteq \mathbf{NP}$$

True

$\mathbf{Closed(\#P, min)} \rightarrow \mathbf{NP = UP}$

$$\left.\begin{array}{l}\mathbf{NP} \subseteq \mathbf{UP} \\ \mathbf{UP} \subseteq \mathbf{NP}\end{array}\right\} \Rightarrow \mathbf{NP} = \mathbf{UP}$$

other questions...: some implications

$$\text{Closed}(\#\mathbf{P}, \ominus 1) \rightarrow co\mathbf{NP} \subseteq \mathbf{SPP}\ \textbf{OK}$$

$$\text{Closed}(\#\mathbf{P}, \oslash 2) \rightarrow \oplus\mathbf{P} = \mathbf{SPP}\ \textbf{OK}$$

$$\text{Closed}(\#\mathbf{P}, \min) \rightarrow \mathbf{NP} = \mathbf{UP}\ \textbf{OK}$$

$$\text{Closed}(\#\mathbf{P}, \max) \vee \text{Closed}(\#\mathbf{P}, \min) \rightarrow \mathrm{C}_{=}\mathbf{P} = \mathbf{SPP}$$

other questions...: some implications

$\text{Closed}(\#\mathbf{P}, \ominus 1) \rightarrow co\mathbf{NP} \subseteq \mathbf{SPP}$ **OK**

$\text{Closed}(\#\mathbf{P}, \oslash 2) \rightarrow \oplus\, \mathbf{P} = \mathbf{SPP}$ **OK**

$\text{Closed}(\#\mathbf{P}, \min) \rightarrow \mathbf{NP} = \mathbf{UP}$ **OK**

$\text{Closed}(\#\mathbf{P}, \max) \vee \text{Closed}(\#\mathbf{P}, \min) \rightarrow \text{C}_{=}\mathbf{P} = \mathbf{SPP}$

about OptP and SpanP

about OptP

1. $\text{Closed}(\mathbf{OptP}, \ominus)$
2. $\text{Closed}(\mathbf{OptP}, op)$
3. **$\mathbf{NP} = co\mathbf{NP}$**

$op : \text{Function}(op, \mathbb{N}^2, \mathbb{N}) \wedge \text{PolyTime}(op)$

$$1 \equiv 2 \equiv 3$$

about OptP: theorem 2

1. $\text{Closed}(\textbf{OptP}, \ominus)$
2. $\text{Closed}(\textbf{OptP}, op)$
3. $\textbf{NP} = co\textbf{NP}$

$op : \text{Function}(op, \mathbb{N}^2, \mathbb{N}) \wedge \text{PolyTime}(op)$

$$1 \equiv 2 \equiv 3$$

Proof

$$\mathbf{3 \rightarrow 2} \quad \mathbf{2 \rightarrow 1} \quad \mathbf{1 \rightarrow 3}$$

(pause)

$\mathbf{FP} \subseteq \mathbf{OptP}$

theorem 2: proof, 3 → 2

$$\mathbf{NP} = co\mathbf{NP} \rightarrow \mathrm{Closed}(\mathbf{OptP}, op)$$

$$op : \mathrm{Function}(op, \mathbb{N}^2, \mathbb{N}) \wedge \mathrm{PolyTime}(op)$$

theorem 2: proof, 3 → 2

$$\mathbf{NP} = co\mathbf{NP} \to \text{Closed}(\mathbf{OptP}, op)$$

$f, g \in \mathbf{OptP}$

$\Rightarrow \exists N_f, N_g \; \forall x : f(x) = \text{MaxOutput}(N_f, x) \wedge$

$g(x) = \text{MaxOutput}(N_g, x)$

theorem 2: proof, 3 → 2

$$\mathbf{NP} = co\mathbf{NP} \rightarrow \text{Closed}(\mathbf{OptP}, op)$$

$f, g \in \mathbf{OptP}$

$\Rightarrow \exists N_f, N_g\ \forall x : f(x) = \text{MaxOutput}(N_f, x) \land$

$g(x) = \text{MaxOutput}(N_g, x)$

$$L_f = \{\langle x, i\rangle : f(x) > i\}$$

$$L_g = \{\langle x, i\rangle : g(x) > i\}$$

theorem 2: proof, 3 → 2

$\Rightarrow L_f, L_g \in \mathbf{NP}$

$\mathbf{NP} = co\mathbf{NP} \Rightarrow L_f, L_g \in co\mathbf{NP} \Rightarrow \overline{L_f}, \overline{L_g} \in \mathbf{NP}$

$$\overline{L_f} = \{\langle x, i \rangle : f(x) \leq i\}$$

$$\overline{L_g} = \{\langle x, i \rangle : g(x) \leq i\}$$

$\Rightarrow \exists N_1, N_2 : L(N_1) = \overline{L_f} \wedge L(N_2) = \overline{L_g}$

theorem 2: proof, 3 → 2

Turing Machine $M(x)$

$w_f = \text{AGuessedOutput}(N_f, x)$

$w_g = \text{AGuessedOutput}(N_g, x)$

if $N_1(\langle x, w_f \rangle) \wedge N_2(\langle x, w_g \rangle)$ **then return** $op(w_f, w_g)$

else return 0

theorem 2: proof, 3 → 2

Turing Machine $M(x)$

$$w_f = \mathrm{AGuessedOutput}(N_f, x) \leq f(x) = \mathrm{MaxOutput}(N_f, x)$$

$$w_g = \mathrm{AGuessedOutput}(N_g, x) \leq g(x) = \mathrm{MaxOutput}(N_g, x)$$

if $N_1(\langle x, w_f \rangle) \wedge N_2(\langle x, w_g \rangle)$ **then return** $op(w_f, w_g)$

else return 0

theorem 2: proof, 3 → 2

$$N_1(\langle x, w_f \rangle) \wedge N_2(\langle x, w_g \rangle)$$

$$\Rightarrow f(x) \leq w_f \wedge g(x) \leq w_g \Rightarrow w_f \geq f(x) \wedge w_g \geq g(x)$$

$$\Rightarrow w_f = f(x) \wedge w_g = g(x)$$

$$\Rightarrow op(w_f, w_g) = op(f(x), g(x))$$

theorem 2: proof, 3 → 2

Turing Machine $M(x)$

$w_f = \mathrm{AnOutput}(N_f, x)$

$w_g = \mathrm{AnOutput}(N_g, x)$

if $N_1(\langle x, w_f \rangle) \wedge N_2(\langle x, w_g \rangle)$ **then return** $op(f(x), g(x))$

else return 0

theorem 2: proof, 3 → 2

$$\forall x \, \exists w_f, w_g : w_f = f(x) \,\&\, w_g = g(x)$$

$$\Rightarrow \forall x \, \exists w_f, w_g : N_1(\langle x, w_f \rangle) \wedge N_2(\langle x, w_g \rangle)$$

$\Rightarrow \forall x \, \exists w_f, w_g$: the **if** condition is **True**

$$\Rightarrow \forall x \, \exists w_f, w_g : \text{Output}(M, x) = op\big(f(x), g(x)\big)$$

theorem 2: proof, 3 → 2

Turing Machine $M(x)$

$w_f = \text{AGuessedOutput}(N_f, x)$

$w_g = \text{AGuessedOutput}(N_g, x)$

if $N_1(\langle x, w_f \rangle) \wedge N_2(\langle x, w_g \rangle)$ **then return** $op(f(x), g(x))$

else return 0

theorem 2: proof, 3 → 2

$\Rightarrow \forall x : \text{MaxOutput}(M, x) = op\big(f(x), g(x)\big)$

$\Rightarrow op\big(f(x), g(x)\big) \in \mathbf{OptP}$

$\Rightarrow \text{Closed}(\mathbf{OptP}, op)$

about OptP: theorem 2

1. $\text{Closed}(\mathbf{OptP}, \ominus)$

2. $\text{Closed}(\mathbf{OptP}, op)$

3. $\mathbf{NP} = co\mathbf{NP}$

$$1 \equiv 2 \equiv 3$$

Proof

$$\mathbf{3 \rightarrow 2}\ \mathbf{2 \rightarrow 1}\ \mathbf{1 \rightarrow 3}$$

theorem 2: proof, 2 → 1

$$\text{Closed}(\mathbf{OptP}, op) \rightarrow \text{Closed}(\mathbf{OptP}, \ominus)$$

$$op : \text{Function}(op, \mathbb{N}^2, \mathbb{N}) \land \text{PolyTime}(op)$$

theorem 2: proof, 2 → 1

$$\text{Closed}(\mathbf{OptP}, op) \rightarrow \text{Closed}(\mathbf{OptP}, \ominus)$$

True

$$op : \text{Function}(op, \mathbb{N}^2, \mathbb{N}) \wedge \text{PolyTime}(op)$$

about OptP: theorem 2

1. $\text{Closed}(\mathbf{OptP}, \ominus)$

2. $\text{Closed}(\mathbf{OptP}, op)$

3. $\mathbf{NP} = co\mathbf{NP}$

$$1 \equiv 2 \equiv 3$$

Proof

$$\mathbf{3 \rightarrow 2} \quad \mathbf{2 \rightarrow 1} \quad \mathbf{1 \rightarrow 3}$$

theorem 2: proof, 1 → 3

$$\text{Closed}(\mathbf{OptP}, \ominus) \rightarrow \mathbf{NP} = co\mathbf{NP}$$

theorem 2: proof, 1 → 3

$$\text{Closed}(\mathbf{OptP}, \ominus) \rightarrow \mathbf{NP} = co\mathbf{NP}$$

$L \in \mathbf{NP}$

$\Rightarrow \exists M : L(M) = L$

$\Rightarrow \text{MaxOutput}(M, x) = \begin{cases} 1, \text{if } x \in L \\ 0, \text{if } x \notin L \end{cases}$

$\Rightarrow \exists f : f(x) = \text{MaxOutput}(M, x) \in \mathbf{OptP}$

theorem 2: proof, 1 → 3

$$1 \in \mathbf{FP} \wedge \mathbf{FP} \subseteq \mathbf{OptP} \Rightarrow 1 \in \mathbf{OptP}$$

$$\mathrm{Closed}(\mathbf{OptP}, \ominus) \Rightarrow h(x) = 1 \ominus f(x) \in \mathbf{OptP}$$

$$\Rightarrow \exists N\ \forall x : \mathrm{MaxOutput}(N, x) = h(x)$$

observe $h(x) = \begin{cases} 1 \ominus 1 = 0, \text{if } x \in L \\ 1 \ominus 0 = 1, \text{if } x \notin L \end{cases}$

theorem 2: proof, 1 → 3

so $h(x) = \begin{cases} 0, \text{if } x \in L \\ 1, \text{if } x \notin L \end{cases}$

theorem 2: proof, 1 → 3

so $h(x) = \begin{cases} 0, \text{if } x \in L \\ 1, \text{if } x \notin L \end{cases} \Rightarrow h(x) = \begin{cases} 1, \text{if } x \in \bar{L} \\ 0, \text{if } x \notin \bar{L} \end{cases}$

$\Rightarrow L(N) = \bar{L}\ \&\ \bar{L} \in \mathbf{NP}$

$\Rightarrow L \in co\mathbf{NP}$

$\Rightarrow \mathbf{NP} \subseteq co\mathbf{NP}$

$\Rightarrow co\mathbf{NP} \subseteq co(co\mathbf{NP})$

theorem 2: proof, 1 → 3

so $co\mathbf{NP} \subseteq co(co\mathbf{NP})$

$\Rightarrow co\mathbf{NP} \subseteq \mathbf{NP}$

$$\left.\begin{matrix} \mathbf{NP} \subseteq co\mathbf{NP} \\ co\mathbf{NP} \subseteq \mathbf{NP} \end{matrix}\right\} \Rightarrow \mathbf{NP} = co\mathbf{NP}$$

about OptP: theorem 2

1. $\text{Closed}(\mathbf{OptP}, \ominus)$
2. $\text{Closed}(\mathbf{OptP}, op)$
3. $\mathbf{NP} = co\mathbf{NP}$

$$1 \equiv 2 \equiv 3$$

Proof

$$\mathbf{3 \rightarrow 2 \; 2 \rightarrow 1 \; 1 \rightarrow 3}$$

about OptP: theorem 2

1. $\text{Closed}(\mathbf{OptP}, \ominus)$

2. $\text{Closed}(\mathbf{OptP}, op)$

3. $\mathbf{NP} = co\mathbf{NP}$

$$1 \equiv 2 \equiv 3$$

Proved!

about SpanP

1. $\text{Closed}(\mathbf{SpanP}, \ominus)$
2. $\text{Closed}(\mathbf{SpanP}, op)$
3. $\mathbf{NP} = \mathbf{PH} = \mathbf{PP}^{\mathbf{NP}}$

$op : \text{Function}(op, \mathbb{N}^2, \mathbb{N}) \land \text{PolyTime}(op)$

$$1 \equiv 2 \equiv 3$$

open issue

Find complexity classes $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ such that,

$$\mathrm{Closed}(\#\mathbf{P}, \ominus\, 1) \leftrightarrow \boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{B}}$$

resources

- Hemaspaandra, L. A. and Ogihara, M., *The Complexity Theory Companion*, Springer, 2002, pp. 91-108

Thank(*You*)