Advanced Algorithms: Solution of Problem 1

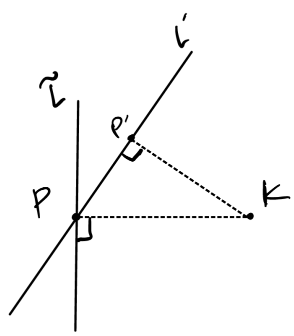
Comment. By no means your solutions are expected to be as long as the ones I am providing. Mine are long because I describe the discovery process.

Exercise 1

Caution: what follows is a process of discovering the alternative definition of L_*, not a formal proof of the equivalence.

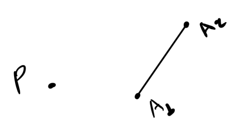
The simplest examples

We can often get an insight by considering the *simplest* examples. What is the simplest convex compact set $K \subseteq \mathbb{R}^2$? A single point! In this case, let $\widetilde{L}$ be the line that passes through P and is perpendicular to the segment[1] $[P, K]$. Then, $L_* = \widetilde{L}$. To see why, take another line L' passing through P but not through K, and let P' be the projection of K on L':



Since the triangle $\triangle PP'K$ is right, we have $d(L', K) < d(\widetilde{L}, K)$.

Let's consider a K that is slightly more interesting: a line segment $[A_1, A_2]$.



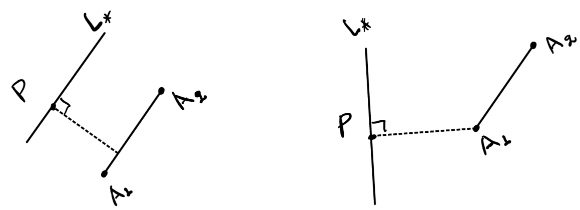
In case you didn't solve Problem 1, I strongly encourage you to stop reading and try to do Exercise 1, assuming K is a line segment.

[1] Remember that for two points A, B, we use $[A, B]$ to denote the line segment connecting A and B.

At this point, you have probably come up with the following:

Proposition 1. *Let P_* be the point in $[A_1, A_2]$ that is closest to P. If P_* is strictly between A_1, A_2, then L_* is the line passing through P that is parallel to the segment. If $P_* = A_i$, then L_* is the line passing through P that is perpendicular to $[P, A_i]$.*

The proof is left as an exercise. Here is an illustration of the proposition:

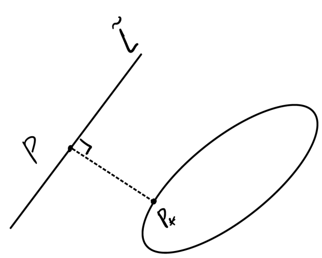


The general case

Even though the sets K that we studied were super-simple, they motivate the following: for a general convex, compact set $K \subseteq \mathbb{R}^2$, and point P outside of K, consider a point $P_* \in K$ that is closest to P, i.e.,

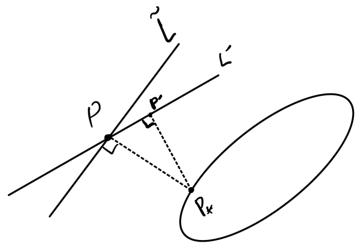
$$\|P - P_*\| = \inf\{\|P - P'\| \mid P' \in K\} \tag{1}$$

Such P_* exists because K is compact and the function $f : K \to \mathbb{R}$ defined as $f(P') = \|P - P'\|$ is continuous (see the theorem from Analysis stated at the hint). Now, which line is L_*? The line $\widetilde{L}$ passing through P that is perpendicular to $[P, P_*]$ seems to be a reasonable candidate for L_*:



It can be proven that $L_* = \widetilde{L}$, which gives us the alternative definition. Although I will not prove this equality here, I will provide intuition about why it holds.

Intuition. First of all, from the last figure, it looks like $d(\widetilde{L}, K) = \|P - P_*\|$ (which again, can be proven to be true). Now, this equality directly implies that $L_* = \widetilde{L}$. Here is why: consider another line L' passing through P, without intersecting K.



Let P' be the projection of P_* on L'. Then, $d(L', P_*) = \|P' - P_*\| < \|P - P_*\|$. Now, since $d(L', K) \leq d(L', P_*)$, we have $d(L', K) < d(\widetilde{L}, K)$.

Exercise 2

The alternative definition we found for Exercise 1 tells us how to construct a line L satisfying the requirements of the theorem. Let P_* be a point in K that is closest to P, i.e., P_* satisfies 1. Let L be the line passing through P that is perpendicular to $[P, P_*]$. Suppose $L \cap K \neq \emptyset$, and let $P' \in L \cap K$. Since K is convex, we have $[P_*, P'] \subseteq K$. Also, the angle $\angle P'PP_*$ is right. Consider the altitude PP'' in the triangle $\triangle P'PP_*$. Since $P'' \in K$ and $\|P'' - P\| < \|P_* - P\|$, we have a contradiction.