Tangent Hyperplane

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Let $A \subseteq \mathbb{R}^n$ open, and $f : A \to \mathbb{R}$ a C^1 function. Let x_0 in A. In this note, we explain *why* the equation of the tangent hyperplane of G_f at $(x_0, f(x_0))$ is

$$x_{n+1} = f(x_0) + \nabla f(x_0) \cdot (x - x_0)$$

where $x = (x_1, \dots, x_n)$.

1 Curves in $\mathbb{R}^n$

Definition 1. A parameterized curve is a C^1 function $\gamma : I \to \mathbb{R}^n$, where $I \subseteq \mathbb{R}$ is an open interval.

C^1 means here that all the components γ_i are C^1. The motivation behind this definition is that we think of γ as describing the trajectory of a certain particle in $\mathbb{R}^n$, i.e., $t \in I$ represents time and $\gamma(t)$ is the position of the particle at time t. For example, $\gamma(t) = (\cos(t), \sin(t))$, $t \in (0, 2\pi)$ parameterizes the unit circle.

Definition 2. If a parameterized curve $\gamma : I \to \mathbb{R}^n$ satisfies $\gamma'(t) \neq 0$ for all $t \in I$, i.e., it always has non-zero velocity, then it is called regular.

2 Tangent vectors

Definition 3. Let $\gamma : I \to \mathbb{R}^n$ be a parameterized curve. Let $t_0 \in I$. Consider the following limits:

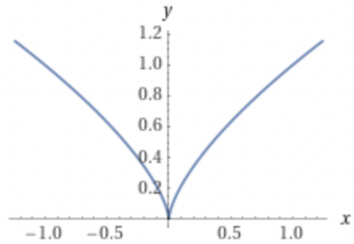
$$L_+ := \lim_{t \to t_0^+} \frac{\gamma(t) - \gamma(t_0)}{\|\gamma(t) - \gamma(t_0)\|}\ , \qquad L_- := \lim_{t \to t_0^-} \frac{\gamma(t_0) - \gamma(t)}{\|\gamma(t_0) - \gamma(t)\|}$$

If they both exist and are equal, then we call this limiting vector the *unit tangent vector* of γ at t_0.

Proposition 4. *If $\gamma : I \to \mathbb{R}^n$ is a regular parameterized curve, then for every $t \in I$, the unit tangent vector exists and equals $\frac{\gamma'(t)}{\|\gamma'(t)\|}$.*

Proof. Left as exercise (divide and multiply with $t - t_0$). □

So, for regular γ the velocities are tangent to the curve. Here is an example that shows what can go wrong when γ is nonregular. Let $\gamma(t) = (t^3, t^2)$. It traces the curve $y = |x|^{2/3}$ (see figure below). Here $\gamma'(0) = 0$, but at this point, the tangent vector is not defined. As you can see from the graph, $L_+ = (0, 1)$ while $L_- = (0, -1)$.

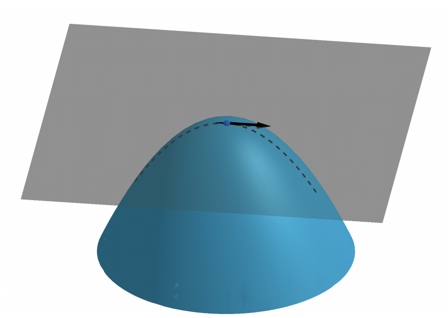


We will only consider regular parameterized curves. The reason is that almost always (except in some uninteresting to us, pathological cases like the one above) if we have a nonregular parameterized curve $\gamma : I \to \mathbb{R}^n$, then *the curve it traces*, i.e., the set $\gamma(I)$, can be parameterized by a regular one. This means that there exists $\tilde{\gamma} : \tilde{I} \to \mathbb{R}^n$ regular, such that $\tilde{\gamma}(\tilde{I}) = \gamma(I)$.

3 Tangent Hyperplane

Let $A \subseteq \mathbb{R}^n$ open, and $f : A \to \mathbb{R}$ a C^1 function. Let x_0 in A. We want to develop a reasonable definition of the tangent hyperplane H of G_f on $(x_0, f(x_0))$, that generalizes the $n = 2$ and $n = 3$ cases. It is natural to impose the following requirements on H: Let $P = (x_0, f(x_0))$. We want

1. $P \in H$
2. Take any regular parameterized curve $\gamma : I \to \mathbb{R}^n$ that lies on G_f (i.e., the range of γ is subset of G_f), and also is such that $0 \in I$ and $\gamma(0) = P$. Then, we must have $P + \gamma'(0) \in H$.



Take one such γ, i.e., regular, lies on G_f, and passes from P at $t = 0$. We can write $\gamma(t) = (u(t), f(u(t))$, where $u(t) = (\gamma_1(t), \ldots, \gamma_n(t))$. Observe that $u(0) = x_0$. From chain rule, we have $\gamma'(0) = (u'(0), \nabla f(x_0) \cdot u'(0))$. Now, the set of all these possible $\gamma'(0)$ is a linear subspace of $\mathbb{R}^{n+1}$ and this linear subspace is exactly the hyperplane

$$x_{n+1} = \nabla f(x_0) \cdot x$$

where $x = (x_1, \ldots, x_n)$. This is proved by considering the curves $\gamma(t) = (x_0 + tv, f(x_0 + tv))$, where v is some arbitrary vector in $\mathbb{R}^n$ (check this!). Now, what is left is to translate this hyperplane to contain $(x_0, f(x_0))$. To this end, we add a constant term d:

$$x_{n+1} = \nabla f(x_0) \cdot x + d$$

and we need $f(x_0) = \nabla f(x_0) \cdot x_0 + d$, and so $d = f(x_0) - \nabla f(x_0) \cdot x_0$. Substituting, we get

$$x_{n+1} = f(x_0) + \nabla f(x_0) \cdot (x - x_0)$$