Advanced Algorithms: Solution of Problem 6

Observe that it suffices to prove the inequality

$$\mathbb{E}[e^{\lambda X}] \leq e^{C\lambda^2} \tag{1}$$

under the additional assumptions that $\lambda > 0$ and $|X| \leq 1$. To see why this is without loss of generality, observe that we can get the result for $\lambda < 0$ be setting $X \leftarrow -X$. For general bound B, we can set $\lambda \leftarrow \lambda \cdot B$. I will give two proofs of (1). The first will give $C = e/2$. The second will give the optimal constant $C = 1/2$.

Solution 1: $C = e/2$

The most obvious thing to do is to bound $\lambda \cdot X \leq \lambda \implies e^{\lambda \cdot X} \leq e^{\lambda} \leq e^{\lambda^2}$, for $\lambda \geq 1$. So, the inequality is proven for $\lambda \geq 1$. It remains to show it for $\lambda \in (0, 1)$. Let's think about the extreme case where λ is very small. Then, we can approximate $e^{\lambda X}$ using the first terms of the Taylor expansion of $f(y) = e^y$. From Taylor's theorem, we have that

$$e^{\lambda X} = 1 + \lambda X + \frac{e^{\xi}}{2}(\lambda X)^2$$

where ξ is between 0 and λX (notice that ξ is a random variable). Using that $\lambda, |X| \leq 1$, we get that

$$e^{\lambda X} \leq 1 + \lambda X + \frac{e}{2}\lambda^2 X^2$$

Taking expectations and using that $\mathbb{E}[X] = 0$ and $|X| \leq 1$, we get

$$\mathbb{E}[e^{\lambda X}] \leq 1 + \frac{e}{2}\lambda^2 \mathbb{E}[X^2] \leq 1 + \frac{e}{2}\lambda^2 \leq e^{\frac{e}{2}\lambda^2}$$

Solution 2: $C = 1/2$

It is intuitive that the worst-case X should be the one which deviates the most from 0 among all X's such that $\mathbb{E}[X] = 0$, $|X| \leq 1$. It is also intuitive that this random variable is 1 with probability 1/2 and -1 with probability 1/2 (we will prove this in a bit). Let's first prove (1) for this particular random variable. First of all,

$$\mathbb{E}[e^{\lambda X}] = \frac{e^{\lambda} + e^{-\lambda}}{2} = \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!}$$

On the other hand, $e^{\lambda^2/2} = \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{2^k \cdot k!}$. Since $2^k \cdot k! \leq (2k)!$, we have proved the inequality for our special case.

Now, we will show that this is indeed the worst-case X. We need to somehow upper-bound the function $f(y) = e^{\lambda y}$ using its values at the boundary of its domain: $[-1, 1]$. But this is exactly what convexity gives us! I.e.,

$$e^{\lambda X} \leq \frac{1+X}{2} e^{\lambda} + \frac{1-X}{2} e^{-\lambda}$$

Taking expectations, we get $\mathbb{E}[e^{\lambda X}] \leq \frac{e^{\lambda}+e^{-\lambda}}{2}$.