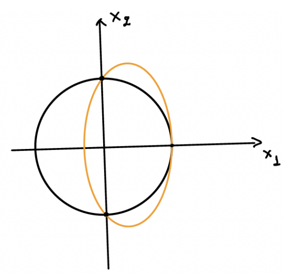
Advanced Algorithms: Solution of Problem 7

Comment. By no means your solutions are expected to be as long as the ones I am providing. Mine are long because I describe the discovery process.

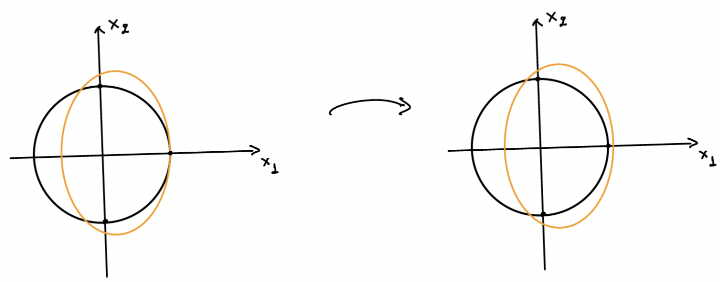
We will employ the 2^{nd} and the 4^{th} strategy (special cases and formulating questions).

Special case: $n = 2$

We will focus on ellipses[1] with centers $(a, 0)$ for $a \in (0, 1)$ and axes along the vectors $(1, 0)$ and $(0, 1)$. A natural question is "out of all these ellipses, how does the one that contains H and has minimum possible area look like?". We now claim that this ellipse - let's call it E_* - must touch H at all three points $(0, 1)$, $(0, -1)$, $(1, 0)$.



Here is why: first of all, note that either both $(0, 1)$ and $(0, -1)$ belong to the boundary of E_*, or none of them does (since E_* is constrained to be symmetric). Suppose that the second case holds. Then, for a small enough $\epsilon > 0$, if we translate E_* by ϵ towards the right, we will still contain H.



But then, we could slightly decrease the axis-lengths and still contain H, contradiction. On the other hand, if $(1, 0)$ does not belong to the boundary of E_*, we can again reach a contradiction by translating E_* towards the left.

Based on these observations, we are looking for an ellipse:

$$\frac{(x_1 - a)^2}{\lambda_1^2} + \frac{x_2^2}{\lambda_2^2} \leq 1 \tag{1}$$

[1] When we say ellipse we mean the union of the curve and its interior.

such that $a \in (0,1)$, $\lambda_1, \lambda_2 > 0$, and $(0,1)$, $(0,-1)$ and $(1,0)$ satisfy (1) with equality. This gives $\frac{(1-a)^2}{\lambda_1^2} = 1 \Rightarrow \lambda_1 = 1-a$, and

$$\frac{a^2}{\lambda_1^2} + \frac{1}{\lambda_2^2} = 1 \Rightarrow \lambda_2^2 = \frac{(1-a)^2}{1-2a}$$

which implies that $a \in (0,1/2)$ and $\lambda_2 = (1-a)/\sqrt{1-2a}$. The area of this ellipse is

$$\lambda_1 \lambda_2 \pi = \frac{(1-a)^2}{\sqrt{1-2a}} \pi$$

By taking the derivative, we get that the $a \in (0,1/2)$ that minimizes it is $a = 1/3$. For this ellipse, $\lambda_1 = 2/3, \lambda_2 = 2/\sqrt{3}$ and the area is $\frac{4}{3\sqrt{3}}\pi < e^{-1/6}$.

It is left to show that it contains H. To this end, let (x_1, x_2) with $x_1 \geq 0$ and $x_1^2 + x_2^2 \leq 1$. Then,

$$\frac{(x_1 - 1/3)^2}{(2/3)^2} + \frac{x_2^2}{(2/\sqrt{3})^2} \leq \frac{(x_1 - 1/3)^2}{(2/3)^2} + \frac{1 - x_1^2}{(2/\sqrt{3})^2}$$

The above is a convex quadratic as a function of x_1, and since $x_1 \in [0,1]$, it is maximized either at $x_1 = 0$ or at $x_1 = 1$. In both cases the above is equal to one.

General case

We consider an ellipsoid of the form

$$\frac{(x_1 - a)^2}{\lambda_1^2} + \frac{x_2^2}{\lambda_2^2} + \cdots + \frac{x_n^2}{\lambda_2^2} \ \leq \ 1$$

where $a \in (0,1), \lambda_1, \lambda_2 > 0$. Observe that due to symmetry, excluding the first axis-length, we choose all the others to be equal. Furthermore, we impose the constraint that the boundary contains $e_1, \pm e_2, \ldots, \pm e_n$, i.e., $\frac{(1-a)^2}{\lambda_1^2} = 1 \Rightarrow \lambda_1 = 1-a$ and

$$\frac{a^2}{\lambda_1^2} + \frac{1}{\lambda_2^2} = 1 \Rightarrow \lambda_2^2 = \frac{(1-a)^2}{1-2a}$$

which implies that $a \in (0,1/2)$ and $\lambda_2 = (1-a)/\sqrt{1-2a}$. Now,

$$\frac{\text{vol}(E)}{\text{vol}(B)} = \lambda_1 \lambda_2^{n-1} = (1-a) \left(\frac{1-a}{\sqrt{1-2a}} \right)^{n-1} = \frac{(1-a)^n}{(1-2a)^{\frac{n-1}{2}}}$$

By taking the derivative of the above function and setting it to zero, we get $a = \frac{1}{n+1}$. For this a,

$$\frac{\text{vol}(E)}{\text{vol}(B)} = \lambda_1 \lambda_2^{n-1} = \left(1 - \frac{1}{n+1}\right) \left(\frac{1 - \frac{1}{n+1}}{\sqrt{1 - \frac{2}{n+1}}} \right)^{n-1} =$$

and since $\frac{1-\frac{1}{n+1}}{\sqrt{1-\frac{2}{n+1}}} = \frac{n}{\sqrt{n-1}\sqrt{n+1}} = \frac{n}{\sqrt{n^2-1}}$, we have

$$\begin{aligned}\frac{\mathrm{vol}(E)}{\mathrm{vol}(B)} = \left(1-\frac{1}{n+1}\right)\left(\frac{n}{\sqrt{n^2-1}}\right)^{n-1} &= \left(1-\frac{1}{n+1}\right)\left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}} \\ &= \left(1-\frac{1}{n+1}\right)\left(1+\frac{1}{n^2-1}\right)^{\frac{n-1}{2}} \\ &\leq e^{-\frac{1}{n+1}} \cdot \left(e^{\frac{1}{n^2-1}}\right)^{\frac{n-1}{2}} \\ &= e^{-\frac{1}{2(n+1)}}\end{aligned}$$

The inequality describing our ellipsoid is

$$\left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}\sum_{i=2}^{n} x_i^2 \ \leq \ 1$$

We show now that it contains H. To this end, let $(x_1, \ldots, x_n)$ with $x_1 \geq 0$ and $x_1^2 + \ldots x_n^2 \leq 1$. Then,

$$\left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}\sum_{i=2}^{n} x_i^2 \ \leq \ \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}\left(1-x_1^2\right)$$

This is a convex quadratic as a function of x_1. Since $x_1 \in [0,1]$, the maximum is attained either at $x_1 = 0$ or at $x_1 = 1$. In both cases the last expression is equal to one. □