ALMA
ALGORITHMS

Fall 2016

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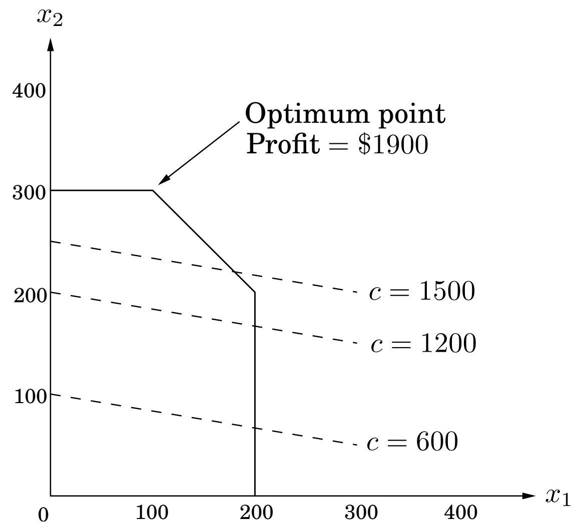
Approximation Algorithms
LP Duality

LP Duality

$$\max x_1 + 6x_2$$
$$x_1 \le 200 \quad (1)$$
$$x_2 \le 300 \quad (2)$$
$$x_1 + x_2 \le 400 \quad (3)$$
$$x_1, x_2 \ge 0$$



For **x**=(200,300)
we get a profit of 1900

A magic trick called duality

Why **x**=(200, 300) of profit of 1900, is the optimum?

Multiply (1), (2) and (3) by 0, 5, and 1, respectively, and add them

You get un upper bound of $x_1 + 6x_2 \le 1900$ on the max profit

So, **x**=(200, 300) is an optimal solution

How we get the multipliers (0, 5, 1) ?

They are the solution of another LP, called the dual of the original one

LP Duality

$$\max x_1 + 6x_2$$

$x_1 \leq 200$ (1) Multiplier y_1

$x_2 \leq 300$ (2) Multiplier y_2

$x_1 + x_2 \leq 400$ (3) Multiplier y_3

$x_1, x_2 \geq 0$

Assign a nonnegative multiplier to each constraint

Multiple each constraint by the corresponding y_i and add them

$$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$$

Thus, we get un upper bound

$$x_1 + 6x_2 \leq 200y_1 + 300y_2 + 400y_3 \quad \text{if} \quad \left\{ \begin{array}{c} y_1, y_2, y_3 \geq 0 \\ y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \end{array} \right\}$$

For y=(5,3,6) we get an upper bound of 4300 which is to loose…

We want an upper bound as tight as possible

LP Duality

$$\max x_1 + 6x_2$$

$x_1 \leq 200$ (1) Multiplier y_1

$x_2 \leq 300$ (2) Multiplier y_2

$x_1 + x_2 \leq 400$ (3) Multiplier y_3

$$x_1, x_2 \geq 0$$

We want an upper bound as tight as possible

$$x_1 + 6x_2 \leq 200y_1 + 300y_2 + 400y_3 \quad \text{if} \quad \left\{ \begin{array}{c} y_1, y_2, y_3 \geq 0 \\ y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \end{array} \right\}$$

That is,

$$\min \ 200y_1 + 300y_2 + 400y_3$$
$$y_1 + y_3 \geq 1$$
$$y_2 + y_3 \geq 6$$
$$y_1, y_2, y_3 \geq 0$$

The dual LP !

LP Duality

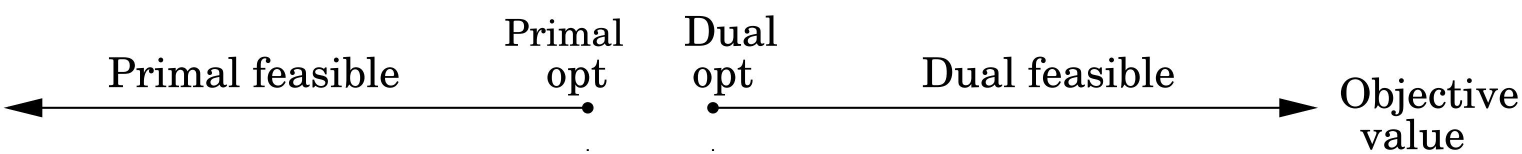
Primal

$$\max x_1 + 6x_2$$
$$x_1 \leq 200 \qquad y_1$$
$$x_2 \leq 300 \qquad y_2$$
$$x_1 + x_2 \leq 400 \qquad y_3$$
$$x_1, x_2 \geq 0$$

Dual

$$\min\ 200y_1 + 300y_2 + 400y_3$$
$$y_1 + y_3 \geq 1$$
$$y_2 + y_3 \geq 6$$
$$y_1, y_2, y_3 \geq 0$$

Any feasible value of the dual is an upper bound on the primal LP



If there is a pair of feasible primal and dual solutions of EQUAL VALUES, then they must be both be optimal; they certify each other's optimality

Primal: **x** = (100, 300)   Dual : **y**= (0, 5, 1)

both of value 1900

THIS IS ALWAYS TRUE !

LP Duality

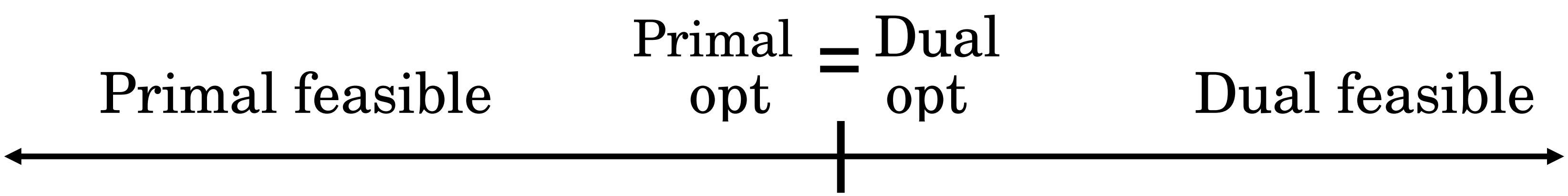
Primal

$$\max x_1 + 6x_2$$
$$x_1 \le 200 \quad y_1$$
$$x_2 \le 300 \quad y_2$$
$$x_1 + x_2 \le 400 \quad y_3$$
$$x_1, x_2 \ge 0$$

Dual

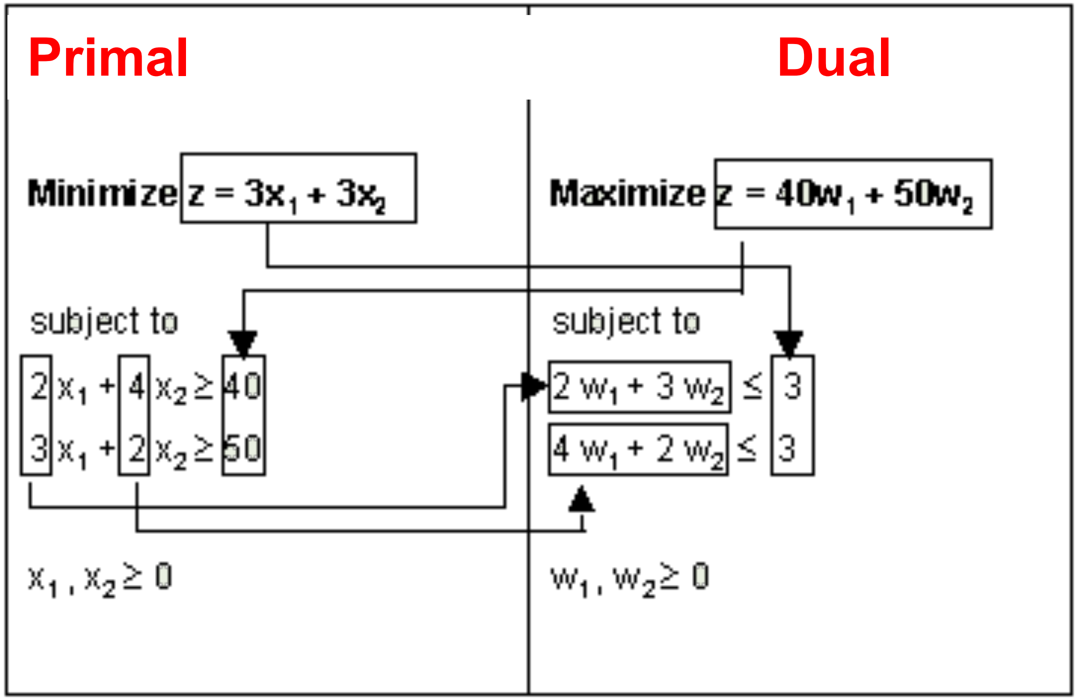
$$\min \ 200y_1 + 300y_2 + 400y_3$$
$$y_1 + y_3 \ge 1$$
$$y_2 + y_3 \ge 6$$
$$y_1, y_2, y_3 \ge 0$$

The optimal values of the a primal LP and its and dual coincide



LP Duality

We can easily write the dual of any LP



- Introduce a multiplier for each primal constraint
- Write a constraint in the dual for every variable of the primal: the sum is required to be above/below the objective coefficient of the corresponding primal variable
- Optimize the sum of the multipliers weighted by the right-hand sides of the constraints of the primal

LP Duality

We can easily write the dual of any LP

Primal

$$
\begin{array}{ll}
\text{minimize} & 7x_1 + x_2 + 5x_3 \\
\text{subject to} & x_1 - x_2 + 3x_3 \geq 10 \\
& 5x_1 + 2x_2 - x_3 \geq 6 \\
& x_1, x_2, x_3 \geq 0
\end{array}
$$

Dual

$$
\begin{array}{ll}
\text{maximize} & 10y_1 + 6y_2 \\
\text{subject to} & y_1 + 5y_2 \leq 7 \\
& -y_1 + 2y_2 \leq 1 \\
& 3y_1 - y_2 \leq 5 \\
& y_1, y_2 \geq 0
\end{array}
$$

- Introduce a multiplier for each primal constraint
- Write a constraint in the dual for every variable of the primal: the sum is required to be above/below the objective coefficient of the corresponding primal variable
- Optimize the sum of the multipliers weighted by the right-hand sides of the constraints of the primal

LP Duality

Maximize $z = \sum_{j=1}^{n} c_j x_j,$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m),$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

Minimize $v = \sum_{i=1}^{m} b_i y_i,$

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \ldots, n),$$

$$y_i \geq 0 \qquad (i = 1, 2, \ldots, m).$$

Dual variables \ *Primal variables*	j: $x_1 \geqq 0$	$x_2 \geqq 0$	$x_3 \geqq 0$	$\cdots$	$x_n \geqq 0$	*Primal relation*	Min v
$y_1 \geqq 0$	a_{11}	a_{12}	a_{13}	$\cdots$	a_{1n}	$\leqq$	b_1
$y_2 \geqq 0$	a_{21}	a_{22}	a_{23}	$\cdots$	a_{2n}	$\leqq$	b_2
$\vdots$ i	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$y_m \geqq 0$	a_{m1}	a_{m2}	a_{m3}	$\cdots$	a_{mn}	$\leqq$	b_m
Dual Relation	$\geqq$	$\geqq$	$\geqq$		$\geqq$		
Max z	c_1	c_2	c_3	$\cdots$	c_n		

LP Duality

We can easily write the dual of any LP

Primal

$$\max \; \mathbf{c}^T\mathbf{x}$$

$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{x} \geq 0$$

Dual

$$\min \; \mathbf{y}^T\mathbf{b}$$

$$\mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T$$

$$\mathbf{y} \geq 0$$

- Introduce a multiplier for each primal constraint
- Write a constraint in the dual for every variable of the primal: the sum is required to be above the objective coefficient of the corresponding primal variable
- Optimize the sum of the multipliers weighted by the right-hand sides of the constraints of the primal

LP Duality theorems

Consider the next LP and its dual to state the theorem

min $\sum_{j=1}^{n} c_j x_j$

$\sum_{j=1}^{n} a_{ij} x_j \geq b_i,$ 1≤i≤m

$x_j \geq 0,$ 1≤j≤n

max $\sum_{i=1}^{m} b_i y_i$

$\sum_{i=1}^{m} a_{ij} y_i \leq c_j,$ 1≤j≤n

$y_i \geq 0,$ 1≤i≤m

LP Duality theorems

Duality Theorem

The primal program has finite optimum iff its dual has finite optimum. Moreover, if **x**∗ **y**∗ are optimal solutions for the primal and dual programs, respectively, then **their values coincide,**

i.e., $$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Week Duality Theorem

If **x** and **y** are feasible solutions for the primal and dual program, then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$$

LP Duality theorems

Week Duality Theorem

If **x** and **y** are feasible solutions for the primal and dual program,

then $$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$$

<u>Proof.</u>

Since **y** is dual feasible and $x_j \geqq 0$

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j$$

Similarly, since **x** is primal feasible and $y_i \geqq 0$

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{m} b_i y_i$$

The theorem follows by observing that

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i$$

min $\sum_{j=1}^{n} c_j x_j$

$\sum_{j=1}^{n} a_{ij} x_j \geq b_i,$ 1≤i≤m

$x_j \geq 0,$ 1≤j≤n

max $\sum_{i=1}^{m} b_i y_i$

$\sum_{i=1}^{m} a_{ij} y_i \leq c_j,$ 1≤j≤n

$y_i \geq 0,$ 1≤i≤m

LP Duality theorems

- By the LP duality theorem, **x** and **y** are both optimal iff
  $$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$$ holds with equality
- Clearly it happens iff both
  $$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j \quad \text{and} \quad \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{m} b_i y_i$$
  hold with equality

Complementary slackness Theorem

Let **x** and **y** be primal and dual feasible solutions, respectively.
Then, x and y are both optimal iff the following two conditions are satisfied:

Primal complementary slackness conditions

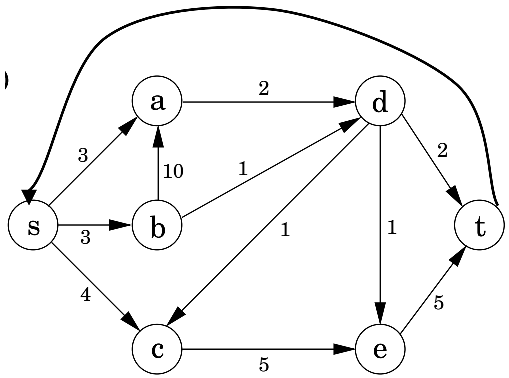
For each $1 \leq j \leq n$: *either* $x_j = 0$ *or* $\sum_{i=1}^{m} a_{ij} y_i = c_j$; *and*

Dual complementary slackness conditions

For each $1 \leq i \leq m$: *either* $y_i = 0$ *or* $\sum_{j=1}^{n} a_{ij} x_j = b_i$.

Min-Max relations and LP Duality

Maximum flow



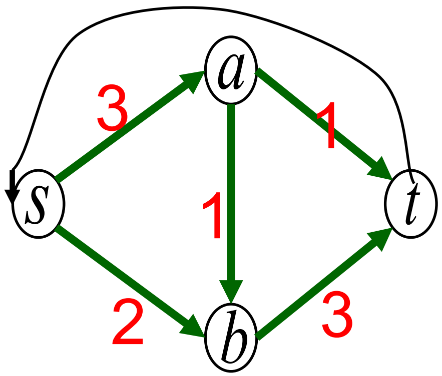
G=(V,E)
Source node s
Sink node t
Trick: Add an arc (t,s) to G of infinite capacity

(LP)

$$
\begin{aligned}
\text{maximize} \quad & f_{ts} \\
\text{subject to} \quad & f_{ij} \le c_{ij}, & (i,j) \in E \\
& \sum_{j:\ (j,i)\in E} f_{ji} - \sum_{j:\ (i,j)\in E} f_{ij} \le 0, & i \in V \\
& f_{ij} \ge 0, & (i,j) \in E
\end{aligned}
$$

Min-Max relations and LP Duality

Maximum flow



(LP)

$$
\begin{array}{lll}
\text{maximize} & f_{ts} & \\
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& \displaystyle\sum_{j:\ (j,i)\in E} f_{ji} - \sum_{j:\ (i,j)\in E} f_{ij} \le 0, & i \in V \\
& f_{ij} \ge 0, & (i,j) \in E
\end{array}
$$

$$
\max f_{ts} = \max \left[\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \overset{\mathbf{o}}{} \overset{f}{\begin{pmatrix} f_{sa} \\ f_{sb} \\ f_{ab} \\ f_{at} \\ f_{bt} \\ f_{ts} \end{pmatrix}} \right] \quad \text{s.t.} \quad \overset{A}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix}} \overset{\mathbf{f}}{\begin{pmatrix} f_{sa} \\ f_{sb} \\ f_{ab} \\ f_{at} \\ f_{bt} \\ f_{ts} \end{pmatrix}} \le \overset{\mathbf{c}}{\begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}
$$

Min-Max relations and LP Duality

Dual LP

$$\min \sum_{(i,j)\in E} c_{ij} d_{ij}$$

$$\min \left[\overset{\mathbf{c}}{\begin{pmatrix} 3 & 2 & 1 & 1 & 3 & 0 & 0 & 0 & 0 \end{pmatrix}} \begin{pmatrix} y_{sa} \\ y_{sb} \\ y_{ab} \\ y_{at} \\ y_{bt} \\ z_a \\ z_b \\ z_s \\ z_t \end{pmatrix} \right] \quad \text{s.t.} \quad \overset{A^T}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}} \begin{pmatrix} y_{sa} \\ y_{sb} \\ y_{ab} \\ y_{at} \\ y_{bt} \\ z_a \\ z_b \\ z_s \\ z_t \end{pmatrix} \geq \overset{\mathbf{o}}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

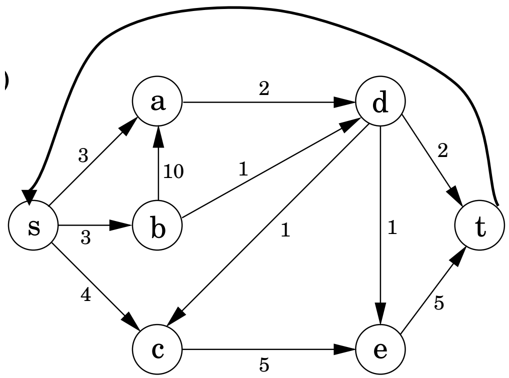
$$\text{minimize} \quad \sum_{(i,j)\in E} c_{ij} d_{ij}$$

$$\begin{aligned} \text{subject to} \quad & d_{ij} - p_i + p_j \geq 0, && (i,j) \in E \\ & p_s - p_t \geq 1 \\ & d_{ij} \geq 0, && (i,j) \in E \\ & p_i \geq 0, && i \in V \end{aligned}$$

d=y
z=p

Min-Max relations and LP Duality

Maximum flow



G=(V,E)
Source node s
Sink node t
Trick: Add an arc (t,s) to G of infinite capacity

(LP)

$$\begin{array}{lll} \text{maximize} & f_{ts} & \\ \text{subject to} & f_{ij} \le c_{ij}, & (i,j) \in E \\ & \sum_{j:\ (j,i)\in E} f_{ji} - \sum_{j:\ (i,j)\in E} f_{ij} \le 0, & i \in V \\ & f_{ij} \ge 0, & (i,j) \in E \end{array}$$

(Dual LP)

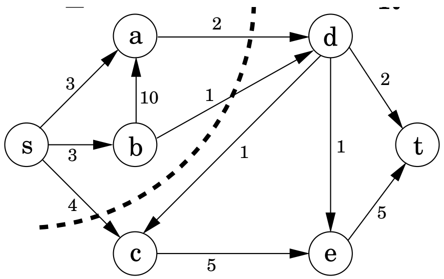
$$\begin{array}{lll} \text{minimize} & \sum_{(i,j)\in E} c_{ij} d_{ij} & \\ \text{subject to} & d_{ij} - p_i + p_j \ge 0, & (i,j) \in E \\ & p_s - p_t \ge 1 & \\ & d_{ij} \ge 0, & (i,j) \in E \\ & p_i \ge 0, & i \in V \end{array}$$

(IP)

$$\begin{array}{lll} \text{minimize} & \sum_{(i,j)\in E} c_{ij} d_{ij} & \\ \text{subject to} & d_{ij} - p_i + p_j \ge 0, & (i,j) \in E \\ & p_s - p_t \ge 1 & \\ & d_{ij} \in \{0,1\}, & (i,j) \in E \\ & p_i \in \{0,1\}, & i \in V \end{array}$$

Min-Max relations and LP Duality

Minimum Cut



s-t cut: X,V-X: s∈X, t∈V-X

(IP)

$$\text{minimize} \quad \sum_{(i,j)\in E} c_{ij} d_{ij}$$

$$\begin{aligned}
\text{subject to} \quad & d_{ij} - p_i + p_j \geq 0, & (i,j) \in E \\
& p_s - p_t \geq 1 & \\
& d_{ij} \in \{0,1\}, & (i,j) \in E \\
& p_i \in \{0,1\}, & i \in V
\end{aligned}$$

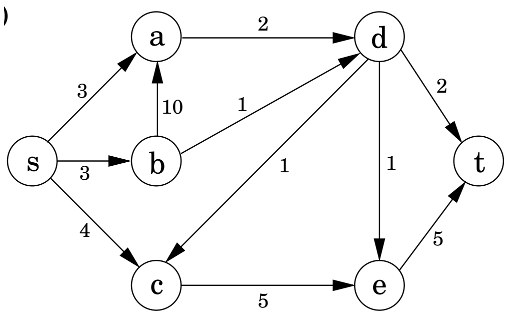
Claim 1. This is an IP formulation of the minimum cut problem

Proof.

- d*, p*: an optimal solution to (IP)
- $p_s^* - p_t^* \geq 1$ is satisfied only for $p_s^* = 1$ and $p_t^* = 0$.
- let the cut (X,V-X) where X is the set of vertices with $p_i^* = 1$
- consider an arc (i,j) with i∈X and j∈V-X
  - Since $p_i^* = 1$ and $p_j^* = 0$, it is $d_{ij}^* \geq 1$, that is $d_{ij}^* = 1$
- for all remaining edges d_{ij}^* can be set to either 0 or 1
  - but in order to minimize the objective value it must be set to 0
- thus, the objective value is equal to the capacity of the cut (X,V-X) and (X,V-X) is a minimum s–t cut

Min-Max relations and LP Duality

Maximum flow



(LP)

$$\begin{aligned} \text{maximize} \quad & f_{ts} \\ \text{subject to} \quad & f_{ij} \le c_{ij}, && (i,j) \in E \\ & \sum_{j:\ (j,i)\in E} f_{ji} - \sum_{j:\ (i,j)\in E} f_{ij} \le 0, && i \in V \\ & f_{ij} \ge 0, && (i,j) \in E \end{aligned}$$

(Dual LP)

$$\begin{aligned} \text{minimize} \quad & \sum_{(i,j)\in E} c_{ij} d_{ij} \\ \text{subject to} \quad & d_{ij} - p_i + p_j \ge 0, && (i,j) \in E \\ & p_s - p_t \ge 1 \\ & d_{ij} \ge 0, && (i,j) \in E \\ & p_i \ge 0, && i \in V \end{aligned}$$

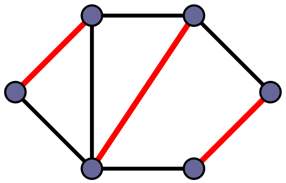
Claim 2. This dual LP has always an integral optimal solution (Proof?)

Hence,

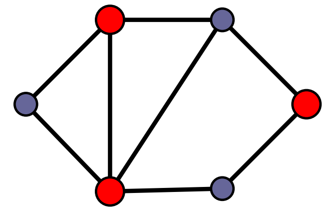
- the maximum flow is equal to the minimum fractional cut (by the duality theorem)
- the latter equals to the capacity of an (integral) minimum cut (by Claims 1 and 2)
- the maximum flow is equal to minimum cut (Max-flow Min-cut theorem)

Min-Max relations and LP Duality

Max Matching ($M \subseteq E$)



Min Vertex Cover ($C \subseteq V$)



(IP)

$$\max \sum_{e \in E} x_e$$

$$\text{s.t.} \quad \sum_{e: v \in e} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \in \{0,1\} \qquad \forall e \in E$$

(IP)

$$\min \sum_{v \in V} y_v$$

$$\text{s.t.} \quad y_v + y_u \geq 1 \qquad \forall \{v,u\} \in E$$

$$y_v \in \{0,1\} \qquad \forall v \in V$$

(LP relax.)

$$\max \sum_{e \in E} x_e$$

$$\text{s.t.} \quad \sum_{e: v \in e} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

(Dual LP)

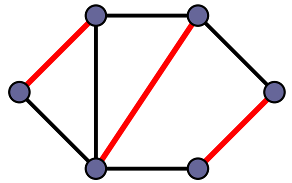
$$\min \sum_{v \in V} y_v$$

$$\text{s.t.} \quad y_v + y_u \geq 1 \qquad \forall \{v,u\} \in E$$

$$y_v \geq 0 \qquad \forall v \in V$$

Min-Max relations and LP Duality

Max Matching (M $\subseteq$ E)

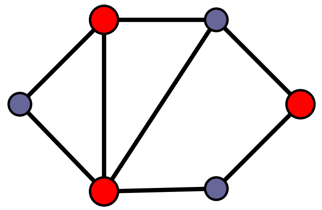


$$\max \sum_{e \in E} x_e$$

$$\text{s.t.} \quad \sum_{e: v \in e} x_e \leq 1 \qquad \forall v \in V$$

$$x_e \geq 0 \qquad \forall e \in E$$

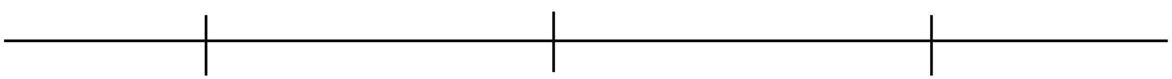
Min Vertex Cover (C $\subseteq$ V)



$$\min \sum_{v \in V} y_v$$

$$\text{s.t.} \quad y_v + y_u \geq 1 \qquad \forall \{v, u\} \in E$$

$$y_v \geq 0 \qquad \forall v \in V$$



|M*| (integer)

Opt primal = Opt dual (fractional)

|C*| (integer)

|M*| $\leq$ |C*|

Set Cover via the Primal-Dual schema

Primal

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{m} w_j x_j \\ \text{subject to} \quad & \sum_{j:e_i \in S_j} x_j \geq 1, \qquad i = 1, \ldots, n, \\ & x_j \in \{0,1\} \qquad j = 1, \ldots, m. \end{aligned}$$

n: # elements
m: # subsets

x_j: for each subset
y_i: for each element

Dual

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} y_i \\ \text{subject to} \quad & \sum_{i:e_i \in S_j} y_i \leq w_j, \qquad j = 1, \ldots, m, \\ & y_i \geq 0, \qquad i = 1, \ldots, n. \end{aligned}$$

Set Cover via the Primal-Dual schema

- We start with the dual solution **y**=0; this is a feasible since $w_j \geq 0$ for all j
- We also have an infeasible primal solution I = $\varnothing$
- As long as there is some element e_i not covered by I
  - we look at all the sets S_j that contain e_i, and consider the amount by which we can increase the dual variable y_i and **y** is still feasible
  - this amount is $\epsilon = \min_{j:e_i \in S_j} \left(w_j - \sum_{k:e_k \in S_j} y_k \right)$
  - we increase y_i by ε; this makes some dual constraint associated with some set S_l tight; that is, after increasing y_i we have for this set S_l $\sum_{k:e_k \in S_\ell} y_k = w_\ell.$
  - we add the set S_l to our cover (by adding to I)

```
y ← 0
I ← ∅
while there exists e_i ∉ ⋃_{j∈I} S_j do
    Increase the dual variable y_i until there is some ℓ such that ∑_{j:e_j∈S_ℓ} y_j = w_ℓ
    I ← I ∪ {ℓ}
return I
```

Set Cover via the Primal-Dual schema

Algorithm Primal-Dual is an f-approximation one for the set cover problem

Proof.

We add a set S_j to our cover only when its dual inequality is tight,

that is $w_j = \sum_{i:e_i \in S_j} y_i$, *for any* $j \in I$

Thus, $\sum_{j \in I} w_j = \sum_{j \in I} \sum_{i:e_i \in S_j} y_i = \sum_{i=1}^{n} y_i \cdot |\{j \in I : e_i \in S_j\}|$

Since, $|\{j \in I : e_i \in S_j\}| \leq f$ we get $\sum_{j \in I} w_j \leq f \cdot \sum_{i=1}^{n} y_i$

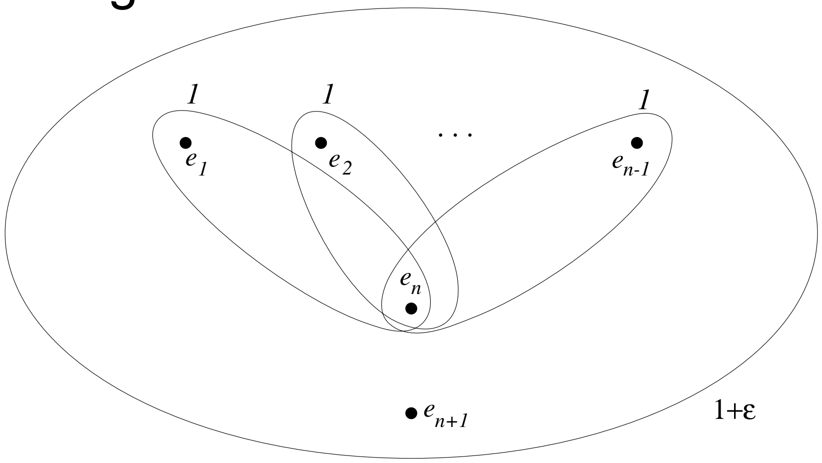
Let Z^*_{LP} be the optimal value of the LP relaxation

Then, by the week duality theorem, $\sum_{i=1}^{n} y_i \leq Z^*_{LP}$

and as $Z^*_{LP} \leq \mathrm{OPT}$ we get $\sum_{j \in I} w_j \leq f \cdot \sum_{i=1}^{n} y_i \leq f \cdot \mathrm{OPT}$

Set Cover via the Primal-Dual schema

Tightness



S consists of:
$n-1$ sets of cost 1, $\{e_1,e_n\},\ldots,\{e_{n-1},e_n\}$, and
one set of cost $1+\varepsilon$, $\{e_1,\ldots,e_{n+1}\}$, for a small $\varepsilon>0$

$f=n$, since e_n appears in all n sets, $f = n$.

- Suppose the algorithm raises y_{e_n} in the first iteration
- When y_{e_n} is raised to 1, all sets $\{e_i,e_n\}$, $i = 1,\ldots,n-1$ go tight
- They are all picked in the cover, thus covering the elements $e_1, \ldots, e_n$
- In the second iteration, $y_{e_{n+1}}$ is raised to ε and the set $\{e_1, \ldots, e_{n+1}\}$ goes tight
- We get a set cover of cost of $n+\varepsilon$, whereas the optimum has cost $1+\varepsilon$
- That is an approximation ratio of $n=f$

Primal-Dual schema

Primal

$$\text{minimize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = 1, \ldots, m$$

$$x_j \geq 0, \quad j = 1, \ldots, n$$

Dual

$$\text{maximize} \quad \sum_{i=1}^{m} b_i y_i$$

$$\text{subject to} \quad \sum_{i=1}^{m} a_{ij} y_i \leq c_j, \quad j = 1, \ldots, n$$

$$y_i \geq 0, \quad i = 1, \ldots, m$$

Complementary slackness Theorem

Let **x** and **y** be primal and dual feasible solutions, respectively.
Then, x and y are both optimal iff the following two conditions are satisfied:

Primal complementary slackness conditions

For each $1 \leq j \leq n$: *either* $x_j = 0$ *or* $\sum_{i=1}^{m} a_{ij} y_i = c_j$; *and*

Dual complementary slackness conditions

For each $1 \leq i \leq m$: *either* $y_i = 0$ *or* $\sum_{j=1}^{n} a_{ij} x_j = b_i$.

Primal-Dual schema

Primal

$$\text{minimize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \qquad i = 1, \ldots, m$$

$$x_j \geq 0, \qquad j = 1, \ldots, n$$

Dual

$$\text{maximize} \quad \sum_{i=1}^{m} b_i y_i$$

$$\text{subject to} \quad \sum_{i=1}^{m} a_{ij} y_i \leq c_j, \qquad j = 1, \ldots, n$$

$$y_i \geq 0, \qquad i = 1, \ldots, m$$

Primal–dual schema:

Ensure one set of conditions and suitably relax the other

Capture both situations by relaxing both conditions

Primal complementary slackness conditions

Let $\alpha \geq 1$.

For each $1 \leq j \leq n$: either $x_j = 0$ or $c_j/\alpha \leq \sum_{i=1}^{m} a_{ij} y_i \leq c_j$.

Dual complementary slackness conditions

Let $\beta \geq 1$.

For each $1 \leq i \leq m$: either $y_i = 0$ or $b_i \leq \sum_{j=1}^{n} a_{ij} x_j \leq \beta \cdot b_i$

If primal conditions are ensured, we set α = 1

If dual conditions are ensured, we set β = 1

Primal-Dual schema

If **x** and **y** are primal and dual feasible solutions satisfying the conditions stated above then $\sum_{j=1}^{n} c_j x_j \leq \alpha \cdot \beta \cdot \sum_{i=1}^{m} b_i y_i.$

Proof.

$$\sum_{j=1}^{n} c_j x_j \leq \alpha \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \alpha \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i$$

$$\leq \alpha\beta \sum_{i=1}^{m} b_i y_i \ .$$

The first and second inequalities follow from the primal and dual conditions
The equality follows by simply changing the order of summation.

Primal-Dual schema

If **x** and **y** are primal and dual feasible solutions satisfying the conditions stated above then

$$\sum_{j=1}^{n} c_j x_j \le \alpha \cdot \beta \cdot \sum_{i=1}^{m} b_i y_i.$$

- Start with a primal infeasible solution and a dual feasible solution; these are usually the trivial solutions x = 0 and y = 0.
- Iteratively improve the feasibility of the primal solution, and the optimality of the dual solution,
- At the end a primal feasible solution is obtained and all conditions stated above, with a suitable choice of α and β, are satisfied.
- The primal solution is always integral
- The improvements to the primal and the dual go hand-in-hand:
  - the current primal solution is used to determine the improvement to the dual, and vice versa.
- The cost of the dual solution is used as a lower bound on OPT, and by the fact above, the approximation guarantee of the algorithm is $\alpha\beta$.

Set Cover revisited

Primal

$$\text{minimize} \quad \sum_{S \in \mathcal{S}} c(S) x_S$$

$$\text{subject to} \quad \sum_{S:\ e \in S} x_S \geq 1, \qquad e \in U$$

$$x_S \geq 0, \qquad S \in \mathcal{S}$$

Dual

$$\text{maximize} \quad \sum_{e \in U} y_e$$

$$\text{subject to} \quad \sum_{e:\ e \in S} y_e \leq c(S), \qquad S \in \mathcal{S}$$

$$y_e \geq 0, \qquad e \in U$$

Set Cover revisited

We choose α=1 and β=f

Primal complementary slackness: $\forall S \in \mathcal{S} : x_S \neq 0 \Rightarrow \sum_{e:\ e \in S} y_e = c(S).$

Set S is tight if $\sum_{e:\ e \in S} y_e = c(S)$

We increment the primal variables integrally;

so, we can state the conditions as: Pick only tight sets in the cover

To maintain dual feasibility, we are not allowed to overpack any set

Dual complementary slackness: $\forall e : y_e \neq 0 \Rightarrow \sum_{S:\ e \in S} x_S \leq f$

We will find an integral (0/1) solution for **x;**

so, each element with a nonzero dual value can be covered at most f times

Since each element is in at most f sets, this condition is trivially satisfied for all elements.

Set Cover revisited

Algorithm (the same as before)

1. **Initialization:** $\boldsymbol{x} \leftarrow \mathbf{0}$; $\boldsymbol{y} \leftarrow \mathbf{0}$
2. Until all elements are covered, do:

   Pick an uncovered element, say e, and raise y_e until some set goes tight.

   Pick all tight sets in the cover and update $\boldsymbol{x}$.

   Declare all the elements occurring in these sets as "covered".
3. Output the set cover $\boldsymbol{x}$.

The Algorithm achieve an approximation ratio of f

Proof.

- There will be no uncovered elements and no overpacked sets at the end
- The primal and dual solutions will both be feasible
- They satisfy the relaxed complementary slackness conditions with α=1 and β=f
- The approximation ratio is f