Descriptive Complexity: First Order Reductions

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ALMA
INTER-INSTITUTIONAL GRADUATE PROGRAM "ALGORITHMS, LOGIC AND DISCRETE MATHEMATICS"

Overview

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The goal of this section is to show that $FO \subseteq L$.

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Thus, it typically can:

- store a *logn*-bit number that points to a position in the input,
- work on strings (numbers etc) of size $\mathcal{O}(\log n)$ on the work tape.

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Only a the single bit carry has to be stored.

Multiplication in Logarithmic Space (1/3)

	5	4	3	2	1	0
b			1	1	0	
a	$\times$		0	1	1	
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E.g. for column 3 we have:

$$\sum_{j+k=4} a_j b_k = 1 + 1 + 0 = 10.$$

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The carry is not necessary a single-bit!

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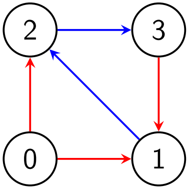
Binary Encoding of a Structure



$$G = \langle V, E, R, 0, 3 \rangle$$
$$n = 4$$
$$E = \{(1,2),(2,3)\}$$
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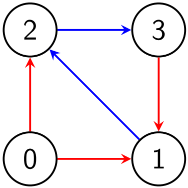

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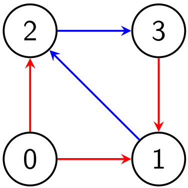

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Observe that $E(1,2)$ corresponds to bit

$$1 \cdot n + 2 + 1 = 1 \cdot 4 + 2 + 1 = 6.$$

Also $|\mathrm{bin}(G)| = n^2 + n^2 + \lceil \log n \rceil + \lceil \log n \rceil$.

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$$\mathcal{A} \models \phi \iff M(\text{bin}(\mathcal{A})) \downarrow .$$

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Also it is is easy to see that $\log f(n) = \mathcal{O}(\log n)$.

Proof

We assume that ϕ is in prenex normal form:

$$\phi = (\exists x_1)(\forall x_2)\ldots(Q_k x_k)\alpha(x_1, \ldots, x_k)$$

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In that case M can test whether $\mathcal{A} \models \phi$ by only using a pointer in the input (which is of the form $\mathrm{bin}(\mathcal{A})$).

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The above bit is '1' iff $\mathcal{A} \models R_3(c_2, \max, c_1)$.

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Induction Hypothesis: Assume that all FO-queries with $k-1$ quantifiers are logspace-computable:

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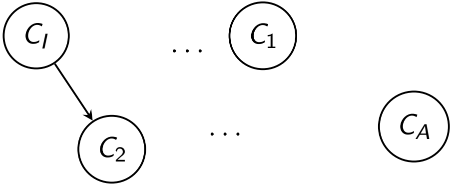

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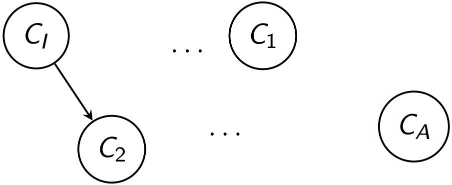

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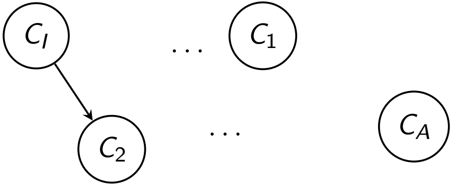

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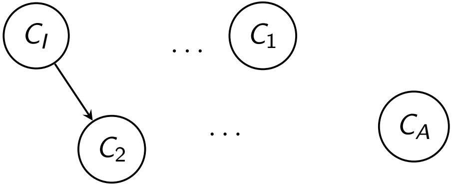

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- M accepts its input iff the accepting configuration (C_A) is reachable from the initial configuration (C_I).

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We will show that REACH is *NL*-complete via *FO*-reductions.

REACH $\in NL$

The following simple non-determinstic logspace algorithm solves REACH.

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```
b:= s;
while (not (b = t)) do {
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The above algorithm needs only store a and b, which have size $\log n$.

Theorem

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- Let N be the logspace nondeterministic Turing Machine that accepts a subset of STRUCT$[\sigma]$
- We will construct an FO-reduction I : STRUCT$[\sigma] \to$ STRUCT$[\tau_g]$ such that for all $\mathcal{A} \in$ STRUCT$[\sigma]$

$$N(\mathrm{bin}(\mathcal{A})) \downarrow \Longleftrightarrow I(\mathcal{A}) \in \mathrm{REACH}.$$

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- Let $a = max\{a_i \mid 1 \leq i \leq r\}$ and $k = 1 + a + c$
- The reduction I will be a k-ary FO-query. I.e. the universe of $I(\mathcal{A})$ will consist of k-tuples from $\mathcal{A}$'s elements.

Proof Sketch

- A configuration of N can be encoded in a k-tuple of variables, i.e. $C = (p, r_1, \ldots, r_a, w_1, \ldots, w_c)$, where p and all the r_i's and w_i's are elements of the universe, i.e. $\log n$-bit numbers.

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- The w_i's contain the contents of N's worktape
- p encodes the current state of N, which R_i or which c_i the input head is looking at and a pointer for the work tape
- since the number of states, relations and constants is independent of the input size and a pointer for the work tape needs $\mathcal{O}(\log \log n)$ bits, p has enough space to store all the necessary information (for large enough n).

Proof Sketch

- Now we build the desired k-ary FO-reduction

$$I = \lambda_{C,C'} \langle true, \phi_N, \alpha, \omega \rangle.$$

- True in the above relation means that the set of nodes in the graph $I(\mathcal{A})$ is equal to the set of all possible configurations
- Formulas ϕ_N, α and ω represent the edge relation, the source node s and the target node t in the created graph
- $\mathcal{A} \models \alpha(C)$ iff C is the unique initial configuration of N
- $\mathcal{A} \models \omega(C)$ iff C is the unique accepting configuration

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  The above information can be extracted from the k variables $p, r_1, \ldots, r_a, w_1, \ldots, w_c$ that describe C and C'. $\square$

Overview

1. $FO \subseteq L$
2. NL-Completeness
3. L-Completeness
4. P-Completeness
5. On FO-Reductions

A slight modification of REACH gives us a natural complete problem for L.

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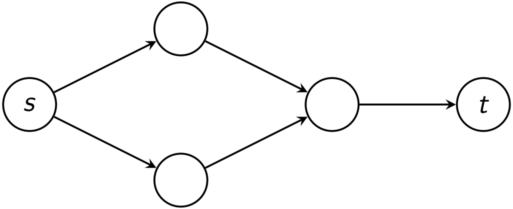
The deterministic version of REACH is the following problem:

$$\text{REACH}_d = \{(G, s, t) \mid G \text{ is directed and there is a deterministic path from } s \text{ to } t\}.$$

The path P is deterministic if for every $(x, y) \in P$, (x, y) is the unique edge leaving x.

Difference Between REACH and REACH_d



It holds that $(G, s, t) \in \text{REACH}$ but $(G, s, t) \notin \text{REACH}_d$

$\text{REACH}_d \in L$

The following logspace algortihm answers REACH_d:

$$
\begin{array}{l}
b := s;\ \ i := 0;\ \ n = |G|; \\
\textbf{while}\ \Big[(b \neq t) \wedge (i < n) \wedge (\exists! a)E(b,a)\Big]\ \textbf{do}\ \{ \\
\qquad b := \text{ the unique } a \text{ for which } E(b,a) \\
\qquad i := i + 1 \\
\} \\
\textbf{if}\ b = t\ \textbf{then}\ \text{accept}\ \textbf{else}\ \text{reject.}
\end{array}
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The definition of REACH_d was made such that the following theorem holds.

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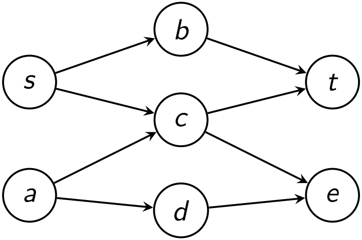
- If x is existential and for some edge (x, z) we have $P^G(z, y)$, then $P^G(x, y)$
- If x is universal, x has at least one outgoing edge and for all edges (x, z) we have $P^G(z, y)$, then $P^G(x, y)$



Universal Nodes: s and a

Existential Nodes: b, c, d, t, e

P^G	s	a	b	c	d	e	t
s	1	0	0	0	0	0	1
a	0	1	0	0	0	1	0
b	0	0	1	0	0	0	1
c	0	0	0	1	0	1	1
d	0	0	0	0	1	1	0
e	0	0	0	0	0	1	0
t	0	0	0	0	0	0	1

Alternating Reachability

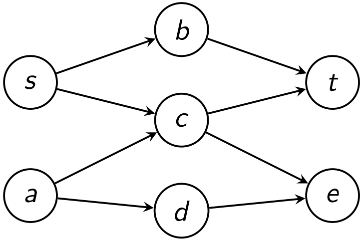
We define $\mathrm{REACH}_a = \left\{ (G, s, t) \;\middle|\; P^G(s, t) \right\}$.
The previous graph G with nodes s and t is a yes instance for REACH_a.

$\text{REACH}_a \in P$

make QUEUE empty; mark(t); **insert** t into QUEUE;
while QUEUE not empty **do** {
    remove first element, x, from QUEUE;
    for each unmarked vertex y such that E(y,x) **do** {
        delete edge (y,x);
        if y is existential or y has no outgoing edges **then**
            { mark(y); **insert** y into QUEUE}
        }
    };
if s is marked **then** accept **else** reject;

Remember: t is the target node, s is the source node and we wish to test whether there is an alternating path from s to t.

A Run of the Algorithm on G



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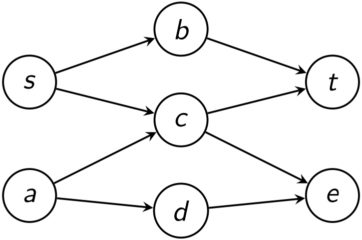

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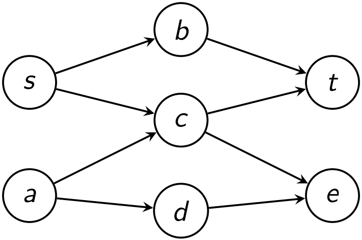

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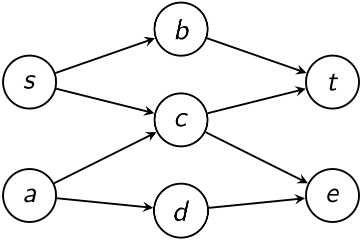

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- b is removed from QUEUE and (s, b) is deleted

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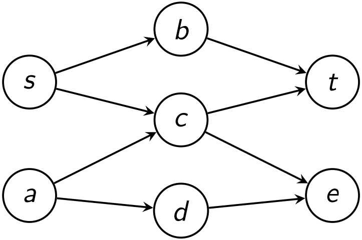

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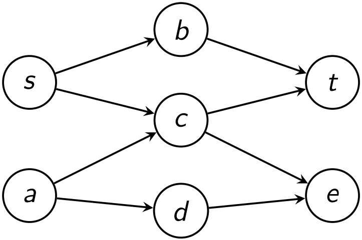

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- c is removed from QUEUE and (a, c) and (s, c) are deleted
- s is marked and added in QUEUE (success!)

A Natural Complete Problem for $ASPACE[\log n]$

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Proof Sketch.

The same construction as before works. The L-Turing machine is now an $ASPACE\,[\log n]$-Turing Machine. We have to make sure that the universal states are mapped to universal nodes. □

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Closure under FO-reductions

Let C be a complexity class and $\mathcal{L}$ be a language that we use to express problems (e.g. FO,$SO\exists$).

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$\mathcal{L}$ is closed under FO-reductions if whenever B is expressible in $\mathcal{L}$ and $A \leq_{fo} B$ then A is expressible in $\mathcal{L}$ too.

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Despite the limited power of *FO*-logic almost all the complexity classes and languages that we will consider are closed under *FO*-reductions.

At least for complexity classes one should suspect that, since almost all of them are closed under logspace reductions and $FO \subseteq L$.

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Let $\mathcal{L}$ be a language and let C be a complexity class.

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4. Express T in $\mathcal{L}$.

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Let $\mathcal{L}$ be a language and let C be a complexity class. In order to show that $\mathcal{L} = C$, i.e. that a query belongs in C if and only if it is expressible in $\mathcal{L}$, we do the following:

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2. Find a boolean query T that is complete for C via *FO*-reductions.
3. Show that $\mathcal{L}$ is closed under *FO*-reductions.
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From 2-4 we can show that $C \subseteq \mathcal{L}$.

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From 2-4 we can show that $C \subseteq \mathcal{L}$. Indeed, let $A \in C$. Then $A \leq_{fo} T$, hence A is expressible in $\mathcal{L}$.

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- $FO \subseteq L$
- Reachability problems are typically the natural complete problems for space complexity classes.
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- A strategy for finding the descriptive analogues of complexity classes.