Descriptive Complexity: Inductive Definitions

Stathis Zachos, Petros Potikas, Ioannis Kokkinis and Aggeliki Chalki

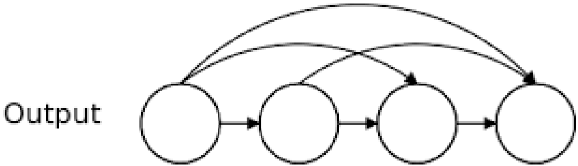
ALMA

INTER-INSTITUTIONAL GRADUATE PROGRAM "ALGORITHMS, LOGIC AND DISCRETE MATHEMATICS"

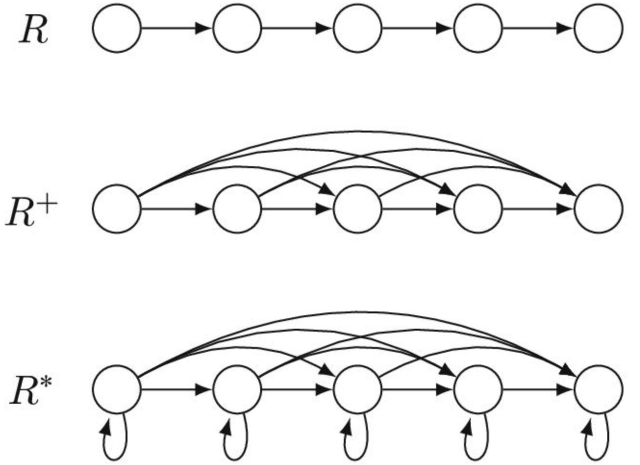
Overview

Overview

Transitive closure



Reflexive and transitive closure



- The transitive closure of a relation is not first-order definable.
- Let $\tau_g = \langle E^2, s, t \rangle$. To define the transitive closure E^+, we first define the following first-order formula:

$$\phi_{tc}(R, x, y) = E(x, y) \vee \exists z \big(E(x, z) \wedge R(z, y) \big)$$

where R ia a binary relation symbol.

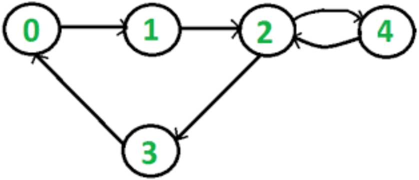
You can see this formula as an inductive definition of transitive closure:

$$E^+(x, y) = E(x, y) \vee \exists z \big(E(x, z) \wedge E^+(z, y) \big).$$

For every structure $\mathcal{A} \in STRUC[\tau_g]$, the formula ϕ_{tc} induces an operator $F_{\phi_{tc}} : \mathcal{P}(|\mathcal{A}|^2) \to \mathcal{P}(|\mathcal{A}|^2)$ defined as follows:

$$F_{\phi_{tc}}(X) = \{(a, b) \mid \mathcal{A} \models \phi_{tc}(X/R, a, b)\}$$

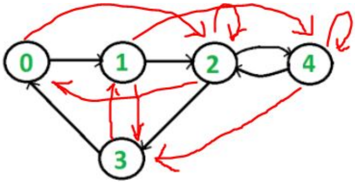
where X/R means that R is interpreted as X in ϕ_{tc}.



$$F_{\phi_{tc}}(\emptyset) = \{(a,b) | \mathcal{A} \models E(a,b) \vee \exists z \big(E(a,z) \wedge R(z,b)\big)\}$$

$$F_{\phi_{tc}}(\emptyset) = \{(0,1),(1,2),(2,3),(2,4),(3,0),(4,2)\}$$

$$F_{\phi_{tc}}(\emptyset) = \{(a,b) \in |\mathcal{A}|^2 \mid distance(a,b) = 1\}$$

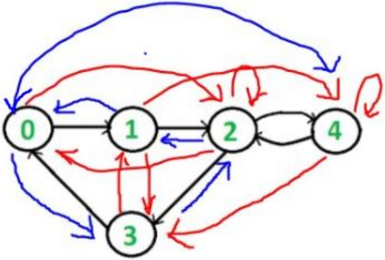


$$F_{\phi_{tc}}(F_{\phi_{tc}}(\emptyset)) =$$

$$F^2_{\phi_{tc}}(\emptyset) = \{(a,b) \mid \mathcal{A} \models E(a,b) \vee \exists z\big(E(a,z) \wedge R(z,b)\big)\}$$

$$F^2_{\phi_{tc}}\emptyset) = \{(0,1),(1,2),(2,3),(2,4),(3,0),(4,2),\\ (0,2),(1,3),(1,4),(2,0),(2,2),(3,1),(4,3),(4,4)\}$$

$$F^2_{\phi_{tc}}(\emptyset) = \{(a,b) \in |\mathcal{A}|^2 \mid d(a,b) = 1 \text{ or } d(a,b) = 2\}$$

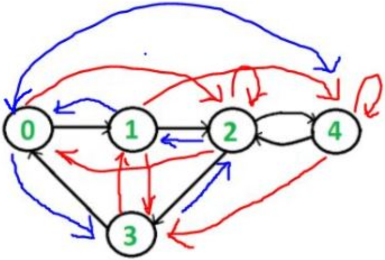


$$F_{\phi_{tc}}(F^2_{\phi_{tc}}(\emptyset)) = F^3_{\phi_{tc}}(\emptyset) =$$

$$\{(a,b) \mid \mathcal{A} \models E(a,b) \vee \exists z (E(a,z) \wedge R(z,b))\}$$

$$F^3_{\phi_{tc}}\emptyset) = \{(0,1),(1,2),(2,3),(2,4),(3,0),(4,2),\\ (0,2),(1,3),(1,4),(2,0),(2,2),(3,1),(4,3),(4,4),\\ (0,3),(0,4),(1,0),(2,1),(3,2),(4,0)\}$$

$$F^3_{\phi_{tc}}(\emptyset) = \{(a,b) \in |\mathcal{A}|^2 \mid 1 \le d(a,b) \le 3\}$$



$$F_{\phi_{tc}}(F^3_{\phi_{tc}}(\emptyset)) = F^4_{\phi_{tc}}(\emptyset) =$$
$$\{(0,1),(1,2),(2,3),(2,4),(3,0),(4,2),$$
$$(0,2),(1,3),(1,4),(2,0),(2,2),(3,1),(4,3),(4,4),$$
$$(0,3),(0,4),(1,0),(2,1),(3,2),(4,0)$$
$$(0,0),(1,1),(3,3),(3,4),(4,1)\}$$

$$F^4_{\phi_{tc}}(\emptyset) = \{(a,b) \in |\mathcal{A}|^2 \mid 1 \le d(a,b) \le 4\}$$

Here $F^4_{\phi_{tc}}(\emptyset) = |\mathcal{A}|^2$, so $F^5_{\phi_{tc}}(\emptyset) = F^4_{\phi_{tc}}(\emptyset) = |\mathcal{A}|^2$.

Thus for $n = ||\mathcal{A}||$, then

$$F^n_{\phi_{tc}}(\emptyset) = E^+ = \text{ the least fixed point of } F_{\phi_{tc}}.$$

In other words, $F^n_{\phi_{tc}}(\emptyset)$ is the minimal relation T such that

$$F_{\phi_{tc}}(T) = T.$$

- Let R be a new relation symbol of arity k.
- Let $\phi(R, x_1, ..., x_k)$ be a first-order formula that induces a monotone operator F_ϕ, i.e.

$$X \subseteq Y \Longrightarrow F_\phi(X) \subseteq F_\phi(Y)$$

Theorem

Let $\phi(R, x_1, ..., x_k)$ be a first-order formula that induces a monotone operator F_ϕ.
For any structure $\mathcal{A}$, the least fixed point of F_ϕ, symb. $\textbf{lfp}(F_\phi)$, exists.
It is equal to $F_\phi^r(\emptyset)$ where r is minimum so that $F_\phi^r(\emptyset) = F_\phi^{r+1}(\emptyset)$.
Furthermore, $r \leq n^k$, where $n = ||\mathcal{A}||$.

Proof. Consider the sequence

$$\emptyset \subseteq F_\phi(\emptyset) \subseteq F_\phi^2(\emptyset) \subseteq F_\phi^3(\emptyset) \subseteq \ldots$$

The inclusions hold because of monotonicity of F_ϕ.
If $F_\phi^{i+1}(\emptyset) \supsetneq F_\phi^i(\emptyset)$, then $F_\phi^{i+1}(\emptyset)$ contains at least one new k-tuple from $|\mathcal{A}|^k$.
Since there are n^k such k-tuples, for some $r \leq n^k$,
$F_\phi^r(\emptyset) = F_\phi(F_\phi^r(\emptyset)) = F_\phi^{r+1}(\emptyset)$, i.e. $F_\phi^r(\emptyset)$ is a fixed point of F_ϕ.

Proof. Let S be any other fixed point of F_ϕ.
We show by induction that $F_\phi^i(\emptyset) \subseteq S$ for every i.

Base case: $F_\phi^0(\emptyset) = \emptyset \subseteq S$.
Suppose that $F_\phi^i(\emptyset) \subseteq S$. Since F_ϕ is monotone,

$$F_\phi^{i+1}(\emptyset) = F_\phi(F_\phi^i(\emptyset)) \subseteq F_\phi(S) \subseteq S.$$

Thus, $F_\phi^r(\emptyset) \subseteq S$ and $F_\phi^r(\emptyset)$ is the least fixed point of F_ϕ. □

Lemma

Testing if F_ϕ is monotone is undecidable for FO formulas ϕ.

Proof. Let Φ be an arbitrary FO sentence and consider the formula

$$\phi(S, x) = \big(S(x) \to \Phi\big).$$

- Suppose Φ is valid. Then $\phi(S, x)$ is always true and F_ϕ is monotone for every structure $\mathcal{A}$.
- Suppose that there is a non-empty structure $\mathcal{A}$ such that $\mathcal{A} \models \neg\Phi$. Then, $\phi(S, x)$ is equivalent to $\neg S(x)$ over $\mathcal{A}$, so F_ϕ is not monotone.

Therefore, F_ϕ is monotone iff Φ is valid, which is undecidale by Trakhtenbrot's theorem. $\square$

- Given a formula ϕ that contains R, we say that an ocurrence of R is *negative* if it is under the scope of an odd number of negations and is *positive* if it is under the scope of an even number of negations.
- In $\exists x \neg R(x) \vee \neg \forall y \forall z \neg \big(R(y) \wedge \neg R(z)\big)$, the first and the last occurrence of R are negative and the second occurrence of R is positive.
- A formula is *positive in R* if there are no negative occurrences of R in it.

Proposition

If $\phi(R, \overrightarrow{x})$ is positive in R, then F_ϕ is monotone.

The logic FO(LFP)

Definition

The logic FO(LFP) extends FO with the following formation rule:

- if $\phi(R, \overrightarrow{x})$ is a formula positive in R, where R is a k-ary relation symbol and $\overrightarrow{t}$ is a tuple of terms, such that $|\overrightarrow{x}| = |\overrightarrow{t}| = k$, then

$$[\mathbf{lfp}_{R,\overrightarrow{x}}\phi(R, \overrightarrow{x})](\overrightarrow{t})$$

  is a formula the free variables of which are those of $\overrightarrow{t}$.

The semantics is defined as follows:

$$\mathcal{A} \models [\mathbf{lfp}_{R,\overrightarrow{x}}\phi(R, \overrightarrow{x})](\overrightarrow{a}) \text{ iff } \overrightarrow{a} \in \mathbf{lfp}(F_\phi).$$

Examples of queries definable in FO(LFP)

<u>**Acyclicity:**</u> Let $\tau_g = \langle E^2 \rangle$ and a structure $\mathcal{A} \in STRUC[\tau_g]$.
Consider the formula $\alpha(S, x) = \forall y \big(E(y, x) \rightarrow S(y) \big)$.

$$F_\alpha(\emptyset) = \{a \in |\mathcal{A}| \mid a \text{ is a node of in-degree } 0\} = \{1\}$$

$$F_\alpha^2(\emptyset) = F_\alpha(\emptyset) \cup \{a \in |\mathcal{A}| \mid a \text{ is a node that has incoming edges only from nodes of } F_\alpha(\emptyset)\} = \{1, 2\}$$

$$F_\alpha^3(\emptyset) = F_\alpha^2(\emptyset) \cup \{a \in |\mathcal{A}| \mid a \text{ is a node that has incoming edges only from nodes of } F_\alpha^2(\emptyset)\} = \{1, 2, 3\}$$

$$F_\alpha^4(\emptyset) = F_\alpha^3(\emptyset) \cup \{a \in |\mathcal{A}| \mid a \text{ is a node that has incoming edges only from nodes of } F_\alpha^3(\emptyset)\} = \{1, 2, 3, 4\}$$

Examples of queries definable in FO(LFP)

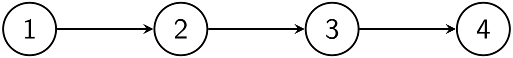
Acyclicity: Let $\tau_g = \langle E^2 \rangle$ and a structure $\mathcal{A} \in STRUC[\tau_g]$.
Consider the formula $\alpha(S, x) = \forall y \big(E(y, x) \rightarrow S(y) \big)$.

$$F_\alpha(\emptyset) = \{a \in |\mathcal{A}| \mid a \text{ is a node of in-degree } 0\} = \{1\}$$

$$F_\alpha^2(\emptyset) = \{a \in |\mathcal{A}| \mid a \text{ is a node that all paths ending in } a \text{ have length at most } 1\} = \{1, 2\}$$

$$F_\alpha^3(\emptyset) = \{a \in |\mathcal{A}| \mid a \text{ is a node that all paths ending in } a \text{ have length at most } 2\} = \{1, 2, 3\}$$

$$F_\alpha^4(\emptyset) = \{a \in |\mathcal{A}| \mid a \text{ is a node that all paths ending in } a \text{ have length at most } 3\} = \{1, 2, 3, 4\}$$



Examples of queries definable in FO(LFP)

Acyclicity:

- $F^i_\alpha(\emptyset) = \{a \in |\mathcal{A}| \mid a$ is a node that all paths ending in a have length at most $i - 1\}$
- $\mathcal{A} \models [\mathbf{lfp}_{S,x}\alpha(S,x)](a) \iff$ all the paths of $\mathcal{A}$ ending in a are of finite length
- $\mathcal{A} \models \forall u[\mathbf{lfp}_{S,x}\alpha(S,x)](u) \iff$ the graph represented by the structure $\mathcal{A}$ is acyclic.

Question: How many times do we have to apply F_α to obtain $\mathbf{lfp}(F_\alpha)$?

Examples of queries definable in FO(LFP)

Arithmetic on Successor Structures: Let $\tau_{succ} = \langle Succ^2, 0 \rangle$.
Let a structure
$\mathcal{A} = \langle \{0, 1, ..., n-1\}, \{(i, i+1) \mid 1 \leq i+1 \leq n-1\}, 0 \rangle$.

We define $\mathbf{+} = \{(i, j, k) \mid i + j = k\}$.

$$x + 0 = x$$
$$x + (y+1) = (x+y) + 1$$

Examples of queries definable in FO(LFP)

Arithmetic on Successor Structures:

$$x + 0 = x$$
$$x + (y + 1) = (x + y) + 1$$

Let R be a ternary relation symbol and $\beta_+(R, x, y, z)$ be

$$\big(y = 0 \land z = x\big) \lor \exists u \exists v \big(R(x, u, v) \land Succ(u, y) \land Succ(v, z)\big)$$

$x + 0 = x$ $\quad$ $x + u = v$ $\quad$ $y = u + 1$ $\quad$ $z = v + 1$

Examples of queries definable in FO(LFP)

Arithmetic on Successor Structures:

$$\big(y = 0 \wedge z = x\big) \vee \exists u \exists v \big(R(x,u,v) \wedge Succ(u,y) \wedge Succ(v,z)\big)$$

$$x + 0 = x \qquad x + u = v \qquad y = u + 1 \qquad z = v + 1$$

$$F_{\beta_+}(\emptyset) = \{(i,j,k) \mid (i,j,k) \text{ is of the form } (x,0,x) \text{ for some } x \in |\mathcal{A}|\}$$

$$F^2_{\beta_+}(\emptyset) = \{(i,j,k) \mid (i,j,k) \text{ is of the form } (x,0,x) \text{ or } (x,1,x+1) \text{ for some } x \in |\mathcal{A}|\}$$

Question 1: What is $F^3_{\beta_+}(\emptyset)$?
Question 2: How many times do we have to apply F_{β_+} to obtain the least fixed point of (F_{β_+})?

Examples of queries definable in FO(LFP)

A Game on Graphs: Let G be a graph and s be a distinguished start node. There are also two players: player **I** and player **II**.
At round i:

- player **I** selects a node b_i,
- player **II** selects a node c_i

such that (a, b_1) and $(b_i, c_i), (c_i, b_{i+1})$ are edges of the graph for every i.

The player that cannot make a legal move loses the game.

Examples of queries definable in FO(LFP)

A Game on Graphs: Let $\tau_g = \langle E^2 \rangle$ and $\mathcal{A} \in STRUC[\tau_g]$.
Let S be a unary relation and $\psi(S, x)$ be

$$\forall y \Big(E(x, y) \rightarrow \exists z \big(E(y, z) \wedge S(z) \big) \Big)$$

$F_\psi(\emptyset) = \{u \mid u \text{ is a node of out-degree } 0\}$

$$F_\psi^2(\emptyset) = F_\psi(\emptyset) \cup \{u \mid \text{for every move from } u \text{ there is a move to a node with out-degree } 0\}$$

Examples of queries definable in FO(LFP)

A Game on Graphs:

Reformulating this,

$F_\psi(\emptyset) = \{u \mid \text{if the game starts from } u \text{ then player } \mathbf{I} \text{ loses}\}$

$$F^2_\psi(\emptyset) = \{u \mid \text{if the game starts from } u \text{ then player } \mathbf{I} \text{ loses}\} \cup \{u \mid \text{for every move of player } \mathbf{I} \text{ from } u \text{ there is a move of player } \mathbf{II} \text{ such that player } \mathbf{I} \text{ loses}\}$$

Examples of queries definable in FO(LFP)

A Game on Graphs:

- In general, $F_\psi(X)$ is the set of nodes u such that no matter where player **I** moves from u, then player **II** can move to some node in X.
- $F_\psi^i(\emptyset)$ consists of the nodes from which player **II** has a winning strategy in at most $i-1$ rounds.
- $\mathcal{A} \models [\mathbf{lfp}_{S,x}\psi(S,x)](s)$ iff player **II** has a winning strategy from node s in $\mathcal{A}$.

Examples of queries definable in FO(LFP)

Alternating Reachability:

Let $\tau_{ag} = \langle E^2, \forall^1, s, t \rangle$ and $\mathcal{G} \in STRUC[\tau_{ag}]$.

$P^{\mathcal{G}}$ is the smallest binary relation that satisfies the following:

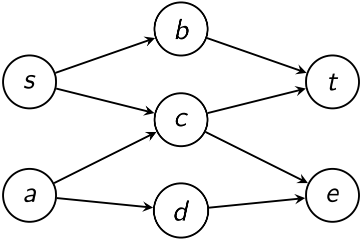
- $P^{\mathcal{G}}(x, x)$
- If x is existential and for some edge (x, z) we have $P^{\mathcal{G}}(z, y)$, then $P^{\mathcal{G}}(x, y)$
- If x is universal, x has at least one outgoing edge and for all edges (x, z) we have $P^{\mathcal{G}}(z, y)$, then $P^{\mathcal{G}}(x, y)$

Examples of queries definable in FO(LFP)

Alternating Reachability:

Let P be a binary relation symbol and ϕ_{ap} be

$$x = y \vee \Big(\exists z\big(E(x,z) \wedge P(z,y)\big) \wedge \big(\forall(x) \to \forall z\big(E(x,z) \to P(z,y)\big)\big)\Big)$$

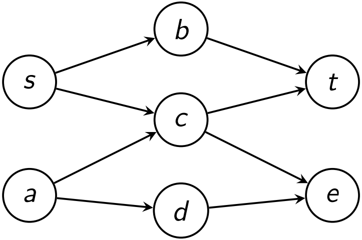


Universal Nodes: s and a

Existential Nodes: b, c, d, t, e

Examples of queries definable in FO(LFP)

$$x = y \vee \Big(\exists z\big(E(x,z) \wedge P(z,y)\big) \wedge \big(\forall(x) \to \forall z\big(E(x,z) \to P(z,y)\big)\big)\Big)$$



Universal Nodes: s and a

Existential Nodes: b, c, d, t, e

$$F_{\phi_{ap}}(\emptyset) = \{(x,x) \mid x \in V\}$$

$$F^2_{\phi_{ap}}(\emptyset) = \{(x,x) \mid x \in V\} \cup \{(b,t),(c,t),(c,e),(d,e)\}$$

$$F^3_{\phi_{ap}}(\emptyset) = \{(x,x) \mid x \in V\} \cup \{(b,t),(c,t),(c,e),(d,e)\} \cup \{(s,t),(a,e)\}$$

Examples of queries definable in FO(LFP)

Alternating Reachability:

Let P be a binary relation symbol and ϕ_{ap} be

$$x = y \vee \Big(\exists z\big(E(x,z) \wedge P(z,y)\big) \wedge \big(\forall(x) \to \forall z\big(E(x,z) \to P(z,y)\big)\big)\Big)$$

$\mathcal{G} \models [\mathbf{lfp}_{P,x,y}\phi_{ap}(P,x,y)](s,t)$ iff

there is an alternating path from s to t in $\mathcal{G}$

Overview

Proposition

FO(LFP) is closed under first order reductions.

For any two queries A and B such that $B \in \text{FO(LFP)}$ and $A \leq_{fo} B$ we have that $A \in \text{FO(LFP)}$.

FO(LFP)=P

Theorem

Over finite, ordered structures,

$$\mathrm{FO(LFP)} = \mathrm{P}$$

Proof. (FO(LFP) $\subseteq$ P): Let $\mathcal{A}$ be an input structure, let $n = ||\mathcal{A}||$ and let $[\mathbf{lfp}_{R,\overrightarrow{x}}\phi(R, \overrightarrow{x})](\overrightarrow{t})$ be a least fixed point formula.
To decide if $\mathcal{A} \models [\mathbf{lfp}_{R,\overrightarrow{x}}\phi(R, \overrightarrow{x})](\overrightarrow{a})$, we have to find $\mathbf{lfp}(F_\phi)$.
So, we have to evaluate the first order query defined by ϕ at most n^k times.
First order queries can be evaluated in L, so in P as well.

FO(LFP)=P

Proof. (P $\subseteq$ FO(LFP)):

1. FO(LFP) includes the query REACH_a, which is complete for the class P under first-order reductions.
2. FO(LFP) is closed under first-order reductions.
3. From 1 and 2, for any polynomial-time query A, we have that $A \in$ FO(LFP). □

Normal Form Theorem

Corollary

Let ϕ be any formula in FO(LFP). Then there exists a first-order formula ψ and a tuple of constants $\overline{c}$ such that over finite, ordered structures,

$$\phi \equiv [\textbf{lfp}\psi](\overline{c})$$

- The use of ordering is required in the proof that REACH_a is P-complete under first order reductions.
- If we consider FO(LFP) on unordered structures then it does not define all polynomial-time properties.
- For NP and coNP we have logics that capture them over all structures.
- Is there a logic that captures P without the additional restriction to ordered structures?

Gurevich's Conjecture

There is no logic that captures P over the class of all finite structures.

This conjecture is stronger than the P $\neq$ NP conjecture!

Overview

Definition

Let R be a k-ary relation symbol, let $\phi(R, \overrightarrow{x})$ be a formula which is positive in R and let $\mathcal{A}$ be a structure of size n.
Define the *depth of* ϕ *in* $\mathcal{A}$, symb. $|\phi^{\mathcal{A}}|$, to be the minimum r such that

$$\mathcal{A} \models \left(F_{\phi}^{r}(\emptyset) \leftrightarrow F_{\phi}^{r+1}(\emptyset)\right)$$

Definition

Define the *depth of* ϕ, symb. $|\phi|$, as a function of n equal to the maximum depth of ϕ in $\mathcal{A}$ for any structure $\mathcal{A}$ of size n:

$$|\phi|(n) = \max_{||\mathcal{A}||=n} \{|\phi^{\mathcal{A}}|\}$$

Example

- Let $\tau_g = \langle E^2 \rangle$. Then the formula

$$\phi_{rtc}(R, x, y) \;\equiv\; x = y \vee \exists z \big(E(x, z) \wedge R(z, y) \big)$$

  defines inductively the reflexive transitive closure, E^*, of E.
- $|\phi_{rtc}|(n) = n$

Example

- Consider the following alternate inductive definition of E^*:

$$\phi^*(R, x, y) \ \equiv \ x = y \vee E(x, y) \vee \exists z \big(R(x, z) \wedge R(z, y)\big).$$

- $|\phi^*|(n) = \lceil logn \rceil + 1$

Definition

Let IND[t(n)] be the subset of FO(LFP) in which only fixed points of first-order formulas ϕ for which $|\phi|$ is $\mathcal{O}(t(n))$ are included.

$$\mathrm{FO(LFP)} = \bigcup_{k=1}^{\infty} \mathrm{IND}[\mathrm{n}^{\mathrm{k}}].$$

Proposition

NL $\subseteq$ IND[logn]

Proof.

- REACH is expressible as $[\mathbf{lfp}_{R,x,y}\phi^*(R,x,y)](s,t)$ and is thus in IND[logn].
- REACH is NL-complete under first-order reductions.
- IND[logn] is closed under first order reductions.

Hence NL $\subseteq$ IND[logn]. □

Remark. In chaprer 5, the depth of nesting of recursive calls is connected to the parallel time needed to evaluate such a recursive definition. In particular IND[logn] $=$ AC^1.

Overview

- Consider an arbitrary operator $F : \mathcal{P}(|\mathcal{A}^k|) \to \mathcal{P}(|\mathcal{A}^k|)$.
- Consider the sequence $F(\emptyset)$, $F^2(\emptyset)$, $F^3(\emptyset)$,...
- There are two possibilities
  1. The sequence reaches a fixed point, i.e. for some $n \in \mathbb{N}$ we have $F^n(\emptyset) = F^{n+1}(\emptyset)$ and thus for all $m > n$, $F^m(\emptyset) = F^n(\emptyset)$. In this case $n \leq 2^{||\mathcal{A}^k||}$.
  2. The sequence does not reach a fixed point.
- We define the partial fixed point of F as

$$\mathbf{pfp}(F) = \begin{cases} F^n(\emptyset), & \text{if } F^n(\emptyset) = F^{n+1}(\emptyset) \\ \emptyset, & \text{if } F^n(\emptyset) \neq F^{n+1}(\emptyset) \text{ for all } n \leq 2^{||\mathcal{A}^k||} \end{cases}$$

The logic FO(PFP)

Definition

The logic FO(PFP) extends FO with the following formation rule:

- if $\phi(R, \overrightarrow{x})$ is a formula, where R is a k-ary relation symbol and $\overrightarrow{t}$ is a tuple of terms, such that $|\overrightarrow{x}| = |\overrightarrow{t}| = k$, then

$$[\mathbf{pfp}_{R,\overrightarrow{x}}\phi(R, \overrightarrow{x})](\overrightarrow{t})$$

is a formula the free variables of which are those of $\overrightarrow{t}$.

The semantics is defined as follows:

$$\mathcal{A} \models [\mathbf{pfp}_{R,\overrightarrow{x}}\phi(R, \overrightarrow{x})](\overrightarrow{a}) \text{ iff } \overrightarrow{a} \in \mathbf{pfp}(F_\phi).$$

Theorem

Over finite, ordered structures,

$$\mathrm{FO}(\mathrm{PFP}) = \mathrm{PSPACE}$$