# RSA

October 23, 2014

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EUCLID(a, b)
1.  if b=0
2.     then return a
3.     else return EUCLID(b, a mod b)
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```
EXTENDED-EUCLID(a, b)
1.  if b=0
2.     then return a
3.  (d', x', y') ← EXTENDED-EUCLID(b, a mod b)
4.  (d, x, y) ← (d', y', x' − ⌊a/b⌋y')
5.  return (d, x, y)
```

# Modular exponentiation

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MODULAR-EXPONENTIATION($x, y, N$)
1. if $y$=0: return 1
2. z=MODULAR-EXPONENTIATION($x, \lfloor y/2 \rfloor, N$)
3. if $y$ is even:
4. return $z^2 \bmod N$
5. else:
6. return $x \cdot z^2 \bmod N$

# Modular multiplicative inverse

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- Generalizing in $\mathbb{Z}$, all possible solutions for this example can be formed from $4 + (11 \cdot z), z \in \mathbb{Z}$, yielding $\{..., -18, -7, \mathbf{4}, 15, 26, ...\}$.

# Fermat Primality Test

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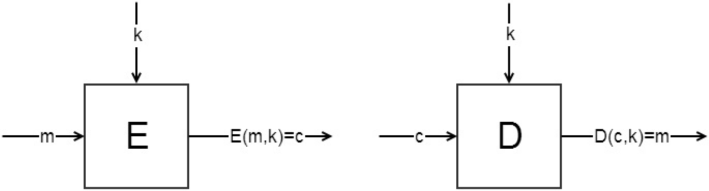
- It is possible for a composite number $N$ to pass Fermat's test for certain choices of $a$.
- Carmichael numbers: rare composite numbers that pass Fermat's test for *all* $a$ relatively prime to $N$.
- For composite $N$, *most* values of $a$ will fail the test.

# Symmetric Cryptography (1)



# Symmetric Cryptography (2)

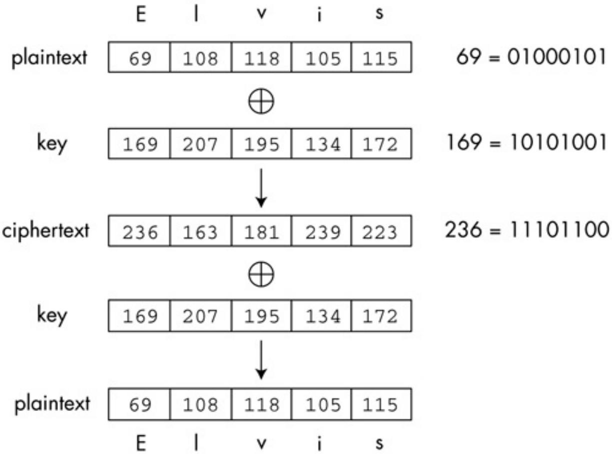
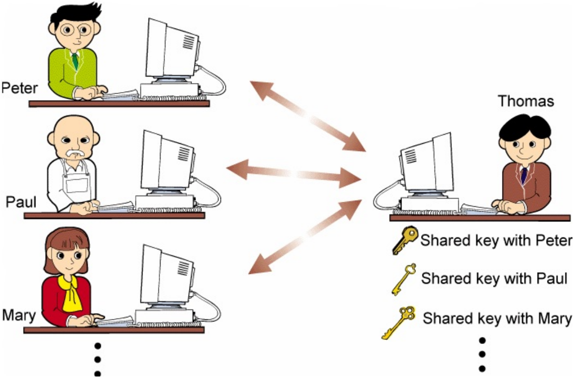
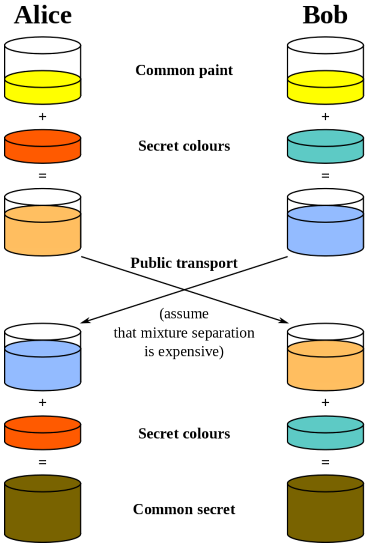



Figure : One-time pad

# Symmetric Cryptography (3)



# Diffie-Hellman-Merkle key exchange



# Asymmetric Cryptography

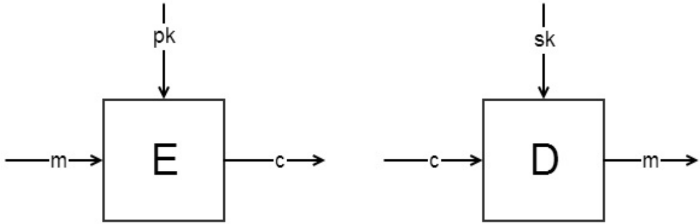



Figure : Public Key Cryptography

# Rivest, Shamir, Adleman (1977)

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Alice's **public key** is the pair of integers $(N, e)$ and her **private key** is the integer $d$.

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To **decrypt** the ciphertext, Alice computes $m = c^d (mod\ N)$ and decodes this to obtain the original message.

# RSA Correctness (1)

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**Theorem (Fermat's Little Theorem)**

*If $p$ is a prime number and $a$ an integer such that $a$ and $p$ are relatively prime, then $a^{p-1} - 1$ is an integer multiple of $p$ or equivalently $a^{p-1} \equiv 1 (mod\ p)$.*

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**Lemma (Euclid's Lemma)**

Let $a, b$ and $d$ be integers where $d \neq 0$. Then if $d$ divides $a \cdot b$ (symbolically $d|a \cdot b$), then either $d|a$ or $d|b$.

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**Lemma (Euclid's Lemma)**

Let $a, b$ and $d$ be integers where $d \neq 0$. Then if $d$ divides $a \cdot b$ (symbolically $d | a \cdot b$), then either $d | a$ or $d | b$.

**Lemma (2)**

Let $M$ be an integer. Let $p$ and $q$ be prime numbers with $p \neq q$. Then if $a \equiv M(mod\ p)$ and $a \equiv M(mod\ q)$, then $a \equiv M(mod\ p \cdot q)$.

# RSA Correctness (2)

We need to prove that $(M^e)^d \equiv M^{ed} \equiv M(mod\ N)$.

**Proof.**

We first show that $M^{ed} \equiv M(mod\ p)$ and $M^{ed} \equiv M(mod\ q)$. The desired result follows from lemma 2.
To show $M^{ed} \equiv M(mod\ p)$, we consider two cases:
$M \equiv 0(mod\ p)$, or $M \not\equiv 0(mod\ p)$.

**Case 1.** $M \equiv 0(mod\ p)$. Then $M$ is an integer multiple of $p$, say $M = p \cdot w, w \in \mathbb{Z}$. Then $M^{ed} = (p \cdot w)^{ed} = p \cdot p^{ed-1} \cdot w^{ed}$. So both $M$ and $M^{ed}$ are integer multiples of $p$. Thus $M^{ed} \equiv M(mod\ p)$.

# RSA Correctness (2)

We need to prove that $(M^e)^d \equiv M^{ed} \equiv M(mod\ N)$.

**Proof.**

**Case 2.** $M \not\equiv 0(mod\ p)$. This means that $p$ and $M$ are relatively prime. Thus we can use Fermat's Little Theorem. We have $M^{p-1} \equiv 1(mod\ p)$.
From the way the decryption key $d$ is defined above, we have $ed - 1 = (p-1)\cdot(q-1)\cdot k,\ k \in \mathbb{Z}$. We then have:

$$\begin{aligned} M^{ed} &= M^{ed-1}\cdot M \\ &= M^{(p-1)\cdot(q-1)\cdot k}\cdot M \\ &= (M^{p-1})^{(q-1)\cdot k}\cdot M \\ &\equiv (1)^{(q-1)\cdot k}\cdot M \pmod p \qquad \text{(apply Fermat's Little Theorem)} \\ &\equiv M \pmod p \end{aligned}$$

# RSA Correctness (2)

We need to prove that $(M^e)^d \equiv M^{ed} \equiv M(mod\ N)$.

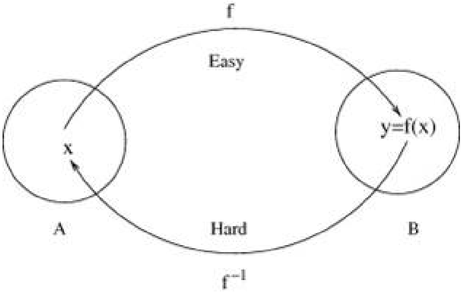
**Proof.**

In a similar way we can show that $M^{ed} = M(mod\ q)$.

By Lemma 2, it follows that $M^{ed} \equiv M(mod\ N = p \cdot q)$.

$\square$

# One-way & Trapdoor fuctions (1)

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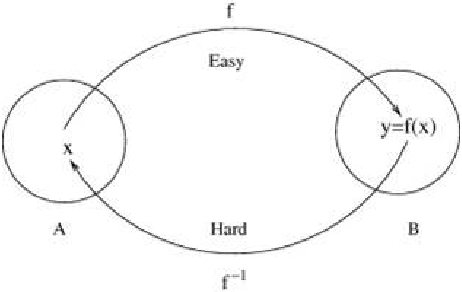


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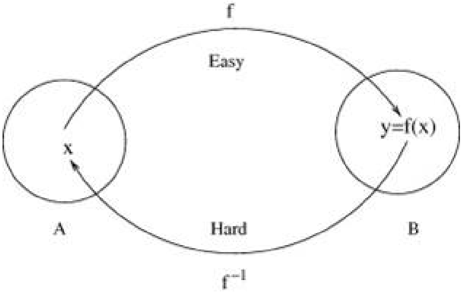


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Do one-way functions exist?
Yes, if $\mathbf{P} \neq \mathbf{NP}$.

# One-way & Trapdoor fuctions (2)

**Candidates for one-way functions:**

- Multiplication and factoring
- The Rabin function (modular squaring)
- Discrete exponential and logarithm
- Cryptographically secure hash functions
- Elliptic curves

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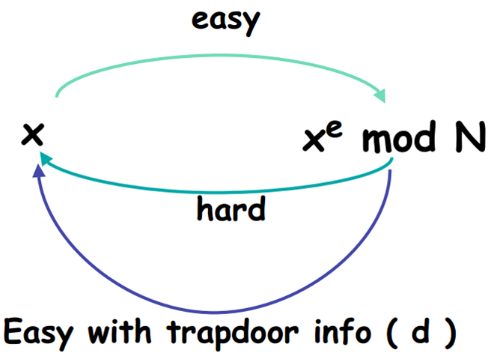
As of 2004, the best known trapdoor function candidates are the RSA and Rabin functions. Both are written as exponentiation modulo a composite number, and both are related to the problem of prime factorization.

# One-way & Trapdoor fuctions (4)

Indeed, exponentiation modulo $N$ is a one-way permutation on $(\mathbb{Z}/N\mathbb{Z})^\times$ when $e$ is co-prime to $\phi(N)$. The private key allows the permutation to be efficiently inverted and it is the trapdoor.



# Key Length and Encryption Strength

| **p,q** | **N** | **time to crack** |
|---|---|---|
| 256 bits | 512 bits | few weeks |
| 512 bits | 1024 bits | 50-100 years |
| 1024 bits | 2048 bits | $>$100 years |
| 2048 bits | 4096 bits | $\approx$ age of the universe |