Descriptive Complexity

Second order logic and lower bounds

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Algorithms, Logic and Discrete Mathematics

Introduction

What is second order logic?

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Let SO$\exists$ (or ESO) be the set of second order existential boolean queries.

Example

$$\Phi_{3-color} \equiv (\exists R^1)(\exists Y^1)(\exists B^1)(\forall x)[(R(x) \vee Y(x) \vee B(x)) \wedge (\forall y)(E(x,y) \rightarrow \neg(R(x) \wedge R(y)) \wedge \neg(Y(x) \wedge Y(y)) \wedge \neg(B(x) \wedge B(y)))]$$

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R(ed), Y(ellow) and B(lue) are the possible colorings for each node. $R(x)$ is 1 if the node x is colored red. Same for $Y(x)$ and $B(x)$. $E(x,y)$ is 1 if there exists an edge (x,y) on our graph.

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Remark

While first order queries can be computed on a CRAM in constant time using polynomially many processors, second order queries can be computed in constant parallel time using exponentially many processors.

Example

$$\Phi_{SAT} \equiv (\exists S)(\forall x)(\exists y)((P(x,y) \land S(y)) \lor (N(x,y) \land \neg S(y)))$$

Example

$$\Phi_{SAT} \equiv (\exists S)(\forall x)(\exists y)((P(x,y) \wedge S(y)) \vee (N(x,y) \wedge \neg S(y)))$$

Φ_{SAT} asserts that there exists a set S of variables (the set of true variables) that is a satysfying assignment for the formula. $P(x,y)$ is 1 if the variable y occurs positively in clause x, $S(y)$ is 1 if $y = 1$ in our formula and $N(x,y)$ is 1 if the variable y occurs negatively in clause x.

Another example

$Inf(f) \equiv (\forall x)(\forall y)(f(x) = f(y) \rightarrow x = y)$
$\Phi_{CLIQUE} \equiv (\exists f^1 . Inf(f))(\forall xy)((x \neq y \wedge f(x) < k \wedge f(y) < k) \rightarrow E(x,y))$

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There is a numbering of the vertices such that those vertices numbered less than k form a clique. To describe this numbering, we use the function f. $Inf(f)$ means that f is an injective function.

One last example

$\Phi_{HP} \equiv (\exists P)(\psi_1 \wedge \psi_2 \wedge \psi_3)$
$\psi_1 \equiv (\forall x)(\forall y)(P(x,y) \vee P(y,x) \vee x = y)$
$\psi_2 \equiv (\forall x)(\forall y)(\forall z)(\neg P(x,x) \wedge (P(x,y) \wedge P(y,z) \rightarrow P(x,z)))$
$\psi_3 \equiv (\forall x)(\forall y)(P(x,y) \wedge \forall z(\neg P(x,y) \vee \neg P(z,y) \rightarrow E(x,y)))$

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ψ_1: 1 if we have a path from x to y, or a path from y to x, or if x=y.
ψ_2: 1 if P is transitive but not reflexive.
ψ_3: 1 if we have a path *xy* and there is no z between x and y, then *xy* is an edge of our graph.
Thus, Φ_{HP} is true when we have a hamilton path.

SO∃ $\subseteq NP$

Given a SO∃ sentence $\Phi \equiv (\exists R_1^{r_1}) \ldots (\exists R_k^{r_k})\psi$, let τ be the vocabulary of Φ. Our task is to build an NP machine N s.t. for all $\mathcal{A} \in STRUC[\tau]$ $(\mathcal{A} \models \Phi) \Leftrightarrow (N(bin(\mathcal{A})) \downarrow)$.

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Proof

Let $\mathcal{A}$ be an input and $n = ||\mathcal{A}||$. N nondeterministically chooses whether to write 0 or 1 and writes down a string of length n^{r_1} representing R_1, and similarly for $R_2, \dots, R_k$. After this polynomial number of steps, we have an expanded structure $\mathcal{A}' = (\mathcal{A}, R_1, \dots, R_k)$. N should accept iff $\mathcal{A}' \models \psi$. This can be tested in logspace, so certainly in NP. Also, N accepts A iff there is some choice of relations R_1 through R_k such that $(\mathcal{A}, R_1, \dots, R_k) \models \psi$. □

Second order games

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SO∃ (monadic) games

Let $\mathcal{A}, \mathcal{B}$ be structures of the same vocabulary. For $c, m \in N$, define the so (monadic) c-color, m-move game on $\mathcal{A}, \mathcal{B}$ as follows.

1. Samson (spoiler) chooses c monadic relation $C_1^{\mathcal{A}}, C_2^{\mathcal{A}}, \ldots, C_c^{\mathcal{A}}$ on $|\mathcal{A}|$.
2. Delilah (duplicator) chooses c monadic relation $C_1^{\mathcal{B}}, C_2^{\mathcal{B}}, \ldots, C_c^{\mathcal{B}}$ on $|\mathcal{B}|$.
3. The two players play the m-move Ehrenfeucht–Fraïssé game.

Remark: The coloring phase is not symmetic.

Theorem

The following are equivalent:

1. For any formula $\Phi \equiv (\exists C_1^1 \ldots C_c^1)\phi$, with ϕ FO of quantifier rank m, $\mathcal{A} \models \Phi \Rightarrow \mathcal{B} \models \Phi$.
2. Delilah has a winning strategy for the SO(monadic) c, m game on $\mathcal{A}, \mathcal{B}$.

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Proof

Assume 1 and let $C_1^{\mathcal{A}}, C_2^{\mathcal{A}}, \dots, C_c^{\mathcal{A}}$ be Samson's move in the coloring phase. Let ϕ be the conjuction of all quantifier rank m sentences that are true of $(\mathcal{A}, C_1^{\mathcal{A}}, C_2^{\mathcal{A}}, \dots, C_c^{\mathcal{A}})$. By 1, $B \equiv (\exists C_1^1 \dots C_c^1)\phi$. Thus, Delilah can play $C_1^{\mathcal{B}}, C_2^{\mathcal{B}}, \dots, C_c^{\mathcal{B}}$. It follows that $(\mathcal{A}, C_1^{\mathcal{A}}, C_2^{\mathcal{A}}, \dots, C_c^{\mathcal{A}}) \equiv_m (\mathcal{B}, C_1^{\mathcal{B}}, C_2^{\mathcal{B}}, \dots, C_c^{\mathcal{B}})$.
Conversely, suppose 1 is false and that $A \models \Phi$, but $B \models \neg\Phi$. $(\mathcal{A}, C_1^{\mathcal{A}}, C_2^{\mathcal{A}}, \dots, C_c^{\mathcal{A}})$ and $(\mathcal{B}, C_1^{\mathcal{B}}, C_2^{\mathcal{B}}, \dots, C_c^{\mathcal{B}})$ disagree on the quantifier rank m, so Samson is the winner. □

$SO\exists$(monadic) Ehrenfeucht–Fraïssé games give a complete methodology for determining whether a boolean query is expressible in $SO\exists$(monadic). Since $SO\exists$(monadic) Ehrenfeucht–Fraïssé game is still fairly difficult for Delilah to play, Ajtai and Fagin invented an equivalent game.

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Ajtai-Fagin game

Let $I \subseteq STRUC[\tau]$ be a boolean query. Define the game as follows:

1. Samson chooses c, m.
2. Delilah chooses a structure $\mathcal{A} \in STRUC[\tau]$, s.t. $\mathcal{A} \in I$.
3. Samson chooses c monadic relations $C_1^{\mathcal{A}}, C_2^{\mathcal{A}}, \ldots, C_c^{\mathcal{A}}$ on $|\mathcal{A}|$.
4. Delilah chooses a structure $\mathcal{B} \in STRUC[\tau]$, s.t. $\mathcal{B} \notin I$. She also chooses c monadic relations $C_1^{\mathcal{B}}, C_2^{\mathcal{B}}, \ldots, C_c^{\mathcal{B}}$ on $|\mathcal{B}|$.
5. The two players play the Ehrenfeucht–Fraïssé game.

Ajtai-Fagin methodology theorem

Let $I \subseteq STRUC[\tau]$ be a boolean query. Then, the following are equivalent:

1. Delilah has a winning strategy for the Ajtai-Fagin game on I.
2. $I \notin \mathrm{SO}\exists(\text{monadic})$.

Ajtai-Fagin methodology theorem

Let $I \subseteq STRUC[\tau]$ be a boolean query. Then, the following are equivalent:

1. Delilah has a winning strategy for the Ajtai-Fagin game on I.
2. $I \notin$ SO$\exists$(monadic).

Proof

Suppose $I = MOD[\Phi]$, where $MOD[\Phi] = \{\mathcal{A} | \mathcal{A} \models \phi\}$, $\Phi \equiv (\exists C_1^1 \dots C_c^1)\phi$.

1. Samson chooses c, m.
2. Let $\mathcal{A} \in I$ be chosen by Delilah.
3. Samson choose colorings that satisfy ϕ.
4. Delilah then chooses a structure $\mathcal{B} \notin I$, so $\mathcal{B} \models \neg\Phi$.

It follows that Samson is the winner.

Proof (continue)

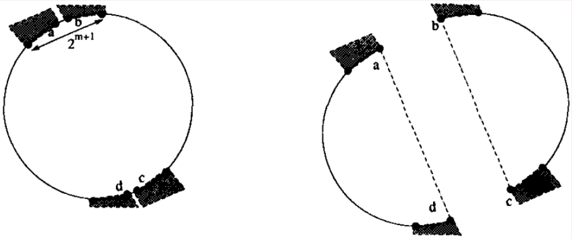
Conversely, suppose $I \notin \mathrm{SO}\exists$(monadic). Let Samson choose c, m. Let Ψ be the disjunction of all the sentences in T, where T are the sentences that are not satisfied by any structure in I. By assumption, $\mathcal{A} \in I$, so $\mathcal{A} \models \neg\Psi$. Delilah should play this $\mathcal{A}$, and as it is in I, Delilah is the winner. $\square$

Theorem

CONNECTED $\notin$ SO$\exists$(monadic)(wo$\leq$).

Proof

Suppose that Samson chooses c, m and that Delilah responds with a sufficiently large cycle, as in the following figure. Note that the node a (with its neighborhood) has the same coloring with c (with its neighborhood). The same holds also for b and d. So, to construct $\mathcal{B}$ (right), Delilah has to delete the edges *ab* and *cd* and connect a with d and b with c, so they will have the "same" neighborhood as before. □



We saw before that we cannot express connected with SO∃(monadic)(wo$\leq$). But we can express $\overline{CONNECTED}$ with SO∃(monadic)(wo$\leq$) as follows:

$$\overline{CONNECTED} \equiv$$
$$(\exists S^1)[(\exists xy)(S(x) \wedge \neg S(y)) \wedge (\forall xy)((S(x) \wedge E(x,y)) \rightarrow S(y))].$$

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Remark

SO∃(monadic)(wo≤) is not closed under complementation.

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Kanellakis observation

The undirected reachability query is expressible in SO∃(monadic)(wo$\leq$).

Proof

To express the existence of an undirected path s, t, we assert the existence of a set of vertices S s.t.:

1. s and t $\in S$.
2. s, t have unique neighbors in S.
3. All the other members of S have exactly 2 neighbors.

These three condition are FO expressible. □

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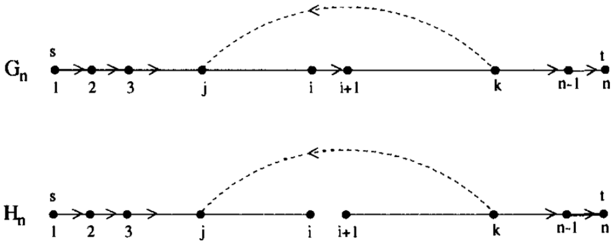
Remark

But this does not hold for directed graphs!

REACH $\notin$ SO$\exists$(monadic)(wo $\leq$)



Observe that H_n is identical to G_n except for the edge $(i, i+1)$. Let Samson begin by playing c, m. Delilah now plays one of the random graphs G_n (random because a backedge like (k, j) exist with some probability $p(n)$). After that, Samson colors G_n with c colors, and Delilah plays H_n.

$G_n^c \sim_m H_n^c$

At first move, Delilah answers any v played by Samson with a vertex v' with the same color. From then on, Delilah answers almost any move of Samson's according to her winning strategy in the game on $(G_n, v), (G_n, v')$.

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Remark

There is one exception in this process. If Samson play something near g_i (recall that H_n does not have the edge $(i, i+1)$), Delilah should play either a vertex w which is far away from g_i or a vertex w with the same color as g_i having a backedge to g_{i+1}.

Lower bounds including ordering

Schwentick proved the lower bound via the following game: Fix constants c, m in the first move of Samson. Delilah answers by playing A_n, which we will describe now. Let S_n be the set of permutation of n elements. Let $s = \pi_1, \ldots, \pi_r$ be a sequence of elements of S_n. Define the graph $P_s^n = (V_s^n, E_s^n)$ as follows:

$V_s^n = \{1, \ldots, r+1\} \times \{1, \ldots, n\}$,

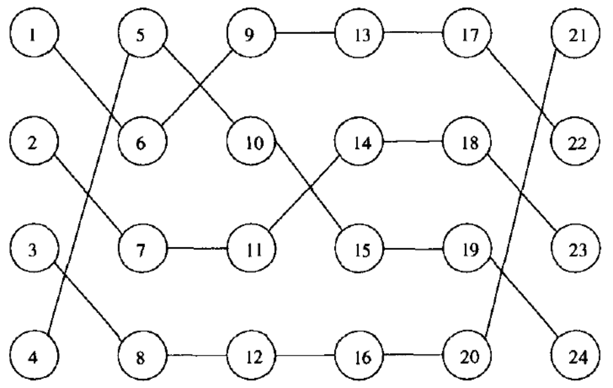
$E_s^n = \{(< i, j >, < i, \pi_i(j) >) | i \in [r], j \in [n]\}$.

Permutations

Reminder. If the permutation has rank 4, then e, the identity permutation, is the following: $(1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 3, 4 \rightarrow 4)$. Also, (12) means $(1 \rightarrow 2, 2 \rightarrow 1, 3 \rightarrow 3, 4 \rightarrow 4)$, where the node before the arrow corresponds to the left columns item and the node after the arrow corresponds to the right columns item in our problem.

Example

Let the graph P_s^4 where $s = (1234), (12), (23), e, (1234)$. For the first $s = (1234)$, we observe that the 1st node (1) of the first column is connected with the 2nd node (6) of the second column, etc.



Lemma

Let $\mathcal{B}_n^c$ result from A_n^c by replacing any number of parts P_{Q_i} by the part P_{Q_j}, for pairs $\sigma_i, \sigma_j \in \mathcal{A}$. Then $\mathcal{A}_n^c \sim_m \mathcal{B}_n^c$.

Lemma

Let $\mathcal{B}_n^c$ result from A_n^c by replacing any number of parts P_{Q_i} by the part P_{Q_j}, for pairs $\sigma_i, \sigma_j \in \mathcal{A}$. Then $\mathcal{A}_n^c \sim_m \mathcal{B}_n^c$.

This lemma tells us that if we transplant P_{Q_i} with P_{Q_j}, then the colors of the graphs remain the same. But, even if the colors remain the same, the structure of the graphs is not the same, as $\mathcal{A}_n$ is connected but $\mathcal{B}_n$ is not. We conclude that connectivity is not expressible in SO$\exists$(monadic)(wo$\leq$).

Bibliography

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Thank you!