Stable Matching

Selected Topics in Algorithms

ΑΛΜΑ, ΣΗΜΜΥ

Matchings

Match (optimally) a set of applicants to a set of open positions.

- Applicants to summer internships
- Applicants to graduate school
- Medical school graduate applicants to residency programs
- Eligible males wanting to marry eligible females

Input: males and females with their preference lists

- Every male has a preference list for women
- Every female has a preference list for men

Output: a matching with specific properties

Stablity and Instability

Consider a matching S between men and women

Unstable Pair

Male x and female y are unstable in S if:

- x prefers y to its matched female
- y prefers x to its matched male

Stable Matching

S is stable if there are no unstable pairs in S.

Formulating the Problem

Consider a set $M = \{m_1, \ldots, m_n\}$ of n men and a set $W = \{w_1, \ldots, w_n\}$ of n women.

- A matching S is a set of ordered pairs, each from $M \times W$, s.t. each member of M and each member of W appears in at most one pair in S.
- A perfect matching S' is a matching s.t. each member of M and each member of W appears in exactly one pair in S'.
- Each man $m \in M$ ranks all of the women; m prefers w to w' if m ranks w higher than w'. We refer to the ordered ranking of m as his preference list.
- Each woman ranks all of the men in the same way.
- An instability results when a perfect matching S contains two pairs (m, w) and (m', w') s.t. m prefers w' to w and w' prefers m to m'.

GOAL: A perfect matching with no instabilities.

An Example

Is the assignment X-C, Y-B, Z-A stable?

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

No. Bertha and Xavier would hook up.

Questions About Stable Marriage

1. Does there exist a stable matching for every set of preference lists?
2. Given a set of preference lists, can we efficiently construct a stable matching if there is one?

The Gale-Shapley Algorithm

Initially set all $m \in M$ and $w \in W$ to free.
While $\exists m$ who is free and hasn't proposed to every $w \in W$ **do**
- Choose such a man m;
- w is highest ranked in m's preference list to whom m has not yet proposed
- **If** w is free
  then (m, w) become engaged
  else let m' be her current match
- **If** w prefers m' to m
  then m remains free
  else (m, w) become engaged and m' becomes free

endWhile
return the set S of engaged pairs

But Does it Work?

Some Axioms

- w remains engaged from the point at which she receives her first proposal
- the sequence of partners with which w is engaged gets increasingly better (in terms of her preference list)
- the sequence of women to whom m proposes get increasingly worse (in terms of his preference list)

Men propose to women in decreasing order of preference (men "optimistic").

Once a woman is matched, she never becomes unmatched (only "trades up").

Termination

Theorem

The G-S algorithm terminates after at most n^2 iterations of the while loop.

What is a good measure of progress?

- the number of free men?
- the number of engaged couples?
- the number of proposals made?

Proof by counting proposals

- Each iteration consists of one man proposing to a woman he has never proposed to before.
- After each iteration of the while loop, the number of proposals increases by one
- Every man proposes at most once to a woman: $|proposals| \leq n^2$

A Perfect Matching Returned

Theorem

The set S returned at termination is a perfect matching.

Proof

- It is a matching since it only trades pairs with the same woman
- Women only trade up, thus once matched, remain matched.
- There is no free man at the end: He has proposed to all women so all of them should be matched.

Theorem

If the algorithm return a matching S, then S is a stable matching.

Proof (by contradiction)

- Let pairs (m, w) and (m', w') in S be s.t.
  - m prefers w' to w, i.e., $w' >_m w$, and
  - w' prefers m to m', i.e., $m >_{w'} m'$.
- m proposed to w' in the past and at some point got rejected for m''.
- In the preference list of w': $m'' >_{w'} m$ and $m' \geq_{w'} m''$.
- m is below m' in the preference list of w', contradiction.

Summary

The Gale-Shapley algorithm guarantees to find a stable matching.

- Are there multiple stable matchings?
- If multiple stable matchings, which to choose??
- Which one does the algorithm find? (Any properties?)

Understanding the Solution

For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

An instance with two stable matchings:
A-X, B-Y, C-Z
A-Y, B-X, C-Z

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Proposer Optimal Solution Returned

- Man m and woman w are valid partners if there exists some stable matching in which they are matched
- A man-optimal matching is one in which every man receives the best valid partner

- Claim 1: All executions of GS yield man-optimal assignment, which is a stable matching.
- Claim 2: All executions of GS yield woman-pessimal assignment, which is a stable matching (i.e., each woman receives the worst possible valid partner).

Claim 1: man-optimality

By contradiction: Let S' be a stable matching where m is better off.

- Let (m, w) be a pair in S'
- In the algorithm m proposed to w and got rejected for some m', thus

$$m' >_w m$$

- Assume this is the first rejection by a valid partner
- Let (m', w') be a pair in $S' \Rightarrow w'$ valid for m'
- m gets 1st rejection (by valid partner) $\Rightarrow m'$ proposed to w with no prior rejection by w' (who is valid for m'), thus

$$w >_{m'} w'$$

- S' not stable: $[(m, w) \in S']$ & $[(m', w') \in S']$ & $[m' >_w m]$ & $[w >_{m'} w']$

Claim 2: woman-pessimality

By contradiction: Let S be the algorithm's matching

- Let $(m, w) \in S$ and m not worst valid for w.
- Exists S' with $(m', w) \in S'$ and

$$m >_w m'$$

- Let $(m, w') \in S'$ be partner of m in S'. By man optimality

$$w >_m w'$$

- S' not stable: $[(m, w) \in S'] \ \& \ [(m', w') \in S'] \ \& \ [m' >_w m] \ \& \ [w >_{m'} w']$

Incentives - Strategy Proofness

Slight extension where players can mark others as unacceptable

- Truthtelling is still proposer-optimal
- Proposal-receivers may benefit by misreporting

Truthful reporting

Albert	Diane	Emily
Bradley	Emily	Diane

Diane	Bradley	Albert
Emily	Albert	Bradley

Albert	Diane	Emily
Bradley	Emily	Diane

Diane	Bradley	Albert
Emily	Albert	Bradley

Strategic reporting

Albert	Diane	Emily
Bradley	Emily	Diane

Diane	Bradley	⊘
Emily	Albert	Bradley

Albert	Diane	Emily
Bradley	Emily	Diane

Diane	Bradley	⊘
Emily	Albert	Bradley

Impossibility results

There is no matching mechanism that

1. is strategy proof for both sides and
2. always results in a stable outcome (given revealed preferences)

Consider a many-to-one extension where "men" can have up to q "women" (classes and students)

These problems look very similar yet

- No algorithm exists s.t. truthtelling is dominant strategy for "men"

Leaving Bipartite Graphs

Consider the stable roommate problem. $2n$ people each rank the others from 1 to $2n-1$. The goal is to assign roommate pairs so that none are unstable.

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

A-B, C-D $\Rightarrow$ B-C unstable
A-C, B-D $\Rightarrow$ A-B unstable
A-D, B-C $\Rightarrow$ A-C unstable

Observation: a stable matching doesn't always exist.

Irving 1985

There exists an algorithm returning a matching or deciding non existence.
(Builds on Gale-Shapley ideas and work by McVitie and Wilson '71)