TRIPARTITE MATCHING, KNAPSACK, Pseudopolinomial Algorithms, Strong NP-completeness

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NP PROBLEMS

- TRIPARTITE MATCHING: Let B, G, H sets with $|B| = |G| = |H| = n \in \mathbb{N}$ and $T \subseteq B \times G \times H$. Is there a set of n triples in T such that no two triples have a common component?

- SET COVERING: Let $F = \{S_1, ..., S_n\}$ with $S_i \subseteq U$, where U is a finite set and positive integer B . $\exists B$ sets in F with U as their union?

- SET PACKING: Let $F = \{S_1, ..., S_n\}$ with $S_i \subseteq U$, where U is a finite set and positive integer K. $\exists K$ pairwise disjoint sets in F with U as their union?

- EXACT COVER BY 3-SETS: Let $F = \{S_1, ..., S_n\}$ with $S_i \subseteq U$, where $|U| = 3m$ for some positive integer m and $|S_i| = 3 \; \forall i \in \{1, ...n\}$. $\exists m$ disjoint sets in F with U as their union?

NP PROBLEMS

- KNAPSACK: Let $i \in \{1, ..., n\}$ items with value v_i and weight w_i, W and K positive integers. $\exists\ S \subseteq \{1, ...n\}$ such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i \geq K$?
- Bin PACKING: Let $a_1, ..., a_N, C, B \in \mathbb{N}$. Can $\{a_1, ..., a_N\}$ be partitioned in B subsets such that each subset has total sum of at most C?

SC, SP, EC3S

Before we begin proving the NP-completeness of the previous problems, we note that:

- TRIPARTITE MATCHING is a special case of EXACT COVER BY 3-SETS, where $m = n$, U is partitioned in three sets $B, G, H : |B| = |G| = |H| = n$ such that each S_i contains one element from each set.
- EXACT COVER BY 3-SETS is a special case of SET COVERING, where $|U| = 3m, |B| = m$ and $|S_i| = 3, \forall i \in \{1, ..., n\}$
- EXACT COVER BY 3-SETS is a special case of SET PACKING, where $|U| = 3m, |K| = m$ and $|S_i| = 3, \forall i \in \{1, ..., n\}$

Thus, proving NP-completeness for TRIPARTITE MATCHING gives us the NP-completeness of the other three problems with the obvious reductions.

TRIPARTITE MATCHING

$3SAT \leq_P TRIPARTITEMATCHING$

Let B be the set of boys, G of girls and H of homes. For each instance φ of 3SAT, we want a matching of each boy with a different girl and home to exist if and only if φ is satisfiable.

For the proof of the above statement we'll need two gadgets:

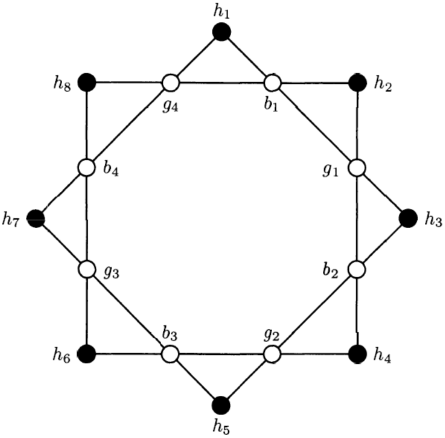
- Choice-consistency gadget:

  •$\forall$ variables x in clause φ we create k boys, k girls and $2k$ homes, where k is the maximum over the appearences of x and of $\neg x$. The boys and girls are unique for each x.

  •The boys and girls form a circle $2k$-long, with edges $\{g_k, b_1\}$, $\{b_i, g_i\}$ and $\{g_i, b_{i+1}\}$ $\forall i \in \{1, ..., k-1\}$.

  •The homes are connected with the above circle with edges $\{b_i, h_{i+1}\}$,$\{h_{i+1}, g_i\}$, $\forall i \in \{1, ..., k-1\}$ and $\{g_k, h_1\}$,$\{h_1, b_1\}$.
  •Homes h_{2i-1} correspond to occurences of x and homes h_{2k} to occurences of $\neg x$, $i \in \{1, ..., k\}$. When the number of occurences of x is different than that of $\neg x$, some homes will correspond to nothing.

CHOICE-CONSISTENCY GADGET, $k = 4$



TRIPARTITE MATCHING CONTINUED

•If a matching exists, then b_i is matched either to g_i and h_{2i} or to g_{i-1} (or g_k if $i = 1$) and h_{2i-1} $\forall i \in \{1, ..., k\}$.

•$\forall$ variables $x \in \varphi$, $T(x) = \mathit{True}$ corresponds to the matching $(b_i, g_i, h_{2i}), i \in \{1, ..., k\}$.

•$\forall$ variables $x \in \varphi$, $T(x) = \mathit{False}$ corresponds to the matching $(b_i, g_{i-1}, h_{2i-1}), i \in \{2, ..., k\}$, and (b_1, g_k, h_1), $i = 1$.

- Constraint gadget: For each close c in φ, we have a boy b and a girl g (different from these of the choice-consistency gadget). Thus we get three triples (b, g, h) with h ranging over the homes that correspond to the three variables of clause c.

  Claim: If any of the corresponding homes is unoccupied (from the boys and girls of the Choice-consistency gadget), it corresponds to a true literal. If no home (of the three) is unoccupied, then all three literals in c are false and b,g cannot be matched to a house.

Examples of the clause constraint

We will use two examples to check the previous claim's correctness:

Let $\varphi = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_4) \wedge (x_1 \vee x_2 \vee \neg x_3)$ and $T(x_2) = T(x_3) = T(x_4) = False$. We examine the 1st clause of φ

$\bullet Example 1$. Let $T(x_1) = True$. The corresponding homes for this clause are h_{11}, h_{21}, h_{31}, since its the first appearence for all the literals in the clause. From the choice concictency gadgets of x_1, x_2 and x_3, h_{11} is unoccupied (and the other two occupied). So, the satisfied clause corresponds to (b, g, h_{11}).

$\bullet Example 2$. Let $T(x_1) = False$. Again, the corresponding homes for this clause are h_{11}, h_{21}, h_{31}. From the choice concictency gadgets of x_1, x_2 and x_3, all the corresponding houses are occupied. Thus, there is no mathcing for the boy and girl that correspond to the unsatisfied clause.

TRIPARTITE MATCHING CONTINUED

To finish the reduction, we need to fix one more thing. Observe that if φhas m clauses, there are $3m$ occurences of the literals, so we have $|H| \geq 3m$.

We now look at the gadgets. In the Choice-consistency gadget, the number of boys (or girls) is $|H|/2$ and in the Constraint part, we have m more boys (or girls), with $m \leq |H|/3$.

Thus we have $|B| = |G| \leq |H|/2 + |H|/3 < |H|$.

- Let $l = |H| - |B|$. We add l more boys and girls with the triples (b_j, g_j, h), $j \in \{1, ..., l\}$, $\forall h \in H$. These boys and girls will occupy any house that's left unoccupied.

The polynomiality of the reduction and it's correctness are now easily checked.

KNAPSACK

We will restrict the problem for instances were
$v_i = w_i \; \forall i \in \{1, ..., n\}$ and $K = W$.
$EXACT\ COVER\ BY\ 3 - SETS \leq_P KNAPSACK$

- Let $\{S_1, ..., S_n\}$ an instance of EXACT COVER BY 3-SETS. Then, we have $|S_i| = 3 \; \forall i \in \{1, ...n\}$ and we are asked if there exist m disjoint S_i that cover $U = \{1, ..., 3m\}$.
- We think the given sets as vectors in $\{0, 1\}^{3m}$. We have $3m$ bits and the numbers in the set corresbond to the positions of the three 1's.
- We would like to see them as binary integers and their union as the binary integer addition, so our target would have been the all-one vector. Then, for $K = 2^n - 1$, the reduction would have been complete.
- But, binary integer addition has *carry*.

Examples

Let $m = 3$.

•For $\{3, 4, 8\}$ and $\{1, 2, 5\}$, we'd like the addition of the corresponding vectors to give us their union $\{1, 2, 3, 4, 5, 8\}$.

Indeed, $001100010(2^6 + 2^5 + 2) + 110010000(2^8 + 2^7 + 2^4) = 111110010(2^8 + 2^7 + 2^6 + 2^5 + 2^4 + 2)$

•On the other hand, we have $\{3, 4, 8\} \cup \{3, 4, 5\} = \{3, 4, 5, 8\}$ but $001100010 + 001110000 = 011010010$ which corresponds to the set $\{2, 3, 5, 8\}$

KNAPSACK CONTINUED

- We think the integers in base $n+1$.
- Thus, $\forall i \in \{1, ..., n\}$, the set S_i corresponds to integer $w_i = \sum_{j \in S_i} (n+1)^{3m-j}$.
- Setting $K = \sum_{j=0}^{3m-1} (n+1)^j$ completes the reduction.

Proof.

•We first observe that the problems with carrying are corrected, since we need $n+1$ 1's in the same position to encounter this problem in base $n+1$ and we have only n vectors.

•Suppose we have a cover $\{S_1, ..., S_m\}$. Then for $S = \{1, ..., m\}$ we have: $\bigcup_{i=1}^{m} S_i = \{1, ..., 3m\}$, which gives us $\sum_{i=1}^{m} w_i = \sum_{j=0}^{3m-j} (n+1)^j$, the all-one vector.

•On the other hand, supposing that $\exists S : \sum_{i \in S} w_i = \sum_{j=0}^{3m-j} (n+1)^j$ and keeping in mind that the base $n+1$ prevents carrying, we get $|S| = m$ and $\{S_i | i \in S\}$ is an exact cover.

Another example

•Let $m = 3$, $U = \{1,2,3,4,5,6,7,8,9\}$, $S_1 = \{1,3,4\}$, $S_2 = \{2,3,4\}$, $S_3 = \{2,5,6\}$, $S_4 = \{6,7,8\}$, $S_5 = \{7,8,9\}$

•Since we have five S_i's, $n = 5$ and our base is $n + 1 = 6$

•From the reduction that we described we get:

$K = \sum_{j=0}^{3\cdot 3-1} 6^j = 6^8 + 6^7 + 6^6 + 6^5 + 6^4 + 6^3 + 6^2 + 6^1 + 6^0$

$w_1 = \sum_{j\in S_1} 6^{9-j} = 101100000 = 6^8 + 6^6 + 6^5$

$w_2 = \sum_{j\in S_2} 6^{9-j} = 011100000 = 6^7 + 6^6 + 6^5$

$w_3 = \sum_{j\in S_3} 6^{9-j} = 010011000 = 6^7 + 6^4 + 6^3$

$w_4 = \sum_{j\in S_4} 6^{9-j} = 000001110 = 6^3 + 6^2 + 6^1$

$w_5 = \sum_{j\in S_5} 6^{9-j} = 000000111 = 6^2 + 6^1 + 6^0$

•We have $w_1 + w_3 + w_5 = K$ and $S_1 \cup S_3 \cup S_5 = U$, an exact cover by 3-sets.

BIN PACKING

TRIPARTITE MATCHING $\leq_P$ *BIN PACKING*

- Let $B = \{b_1, ..., b_n\}$, $G = \{g_1, ..., g_n\}$, $H = \{h_1, ..., h_n\}$, $T = \{t_1, ..., t_m\} \subseteq B \times G \times H$. We want to know if there exist n triples in T: each boy, girl and home is contained in one and only one such triple.
- We want to construct an instance of BIN PACKING, that is items $a_1, ..., a_N$, a capacity C and B bins.
- We construct one item for each triple and one for each occurence of a boy, a girl and a home in each such triple. Thus, $N = 4m$. The items corresponding to the boy b_i are $b_i[1], ..., b_i[N(b_i)], \forall i \in \{1, ..., n\}$, where $N(b_i)$ is the number of occurences of b_i in the triples. The same goes for the items corresponding to girls and homes. Finally, the items corresponding to triples are $t_1, ..., t_m$.
- The sizes of these items are shown in the following table. $M = 100n$, $C = 40M^4 + 15$ and $B = m$.

ITEMS IN BIN PACKING

1.pdf

Item	Size
first occurrence of a boy $b_i[1]$	$10M^4 + iM + 1$
other occurrences of a boy $b_i[q], q > 1$	$11M^4 + iM + 1$
first occurrence of a girl $g_j[1]$	$10M^4 + jM^2 + 2$
other occurrences of a girl $g_j[q], q > 1$	$11M^4 + jM^2 + 2$
first occurrence of a home $h_k[1]$	$10M^4 + kM^3 + 4$
other occurrences of a home $h_k[q], q > 1$	$8M^4 + kM^3 + 4$
triple $(b_i, g_j, h_k) \in T$	$10M^4 + 8-$ $-iM - jM^2 - kM^3$

Proof that $3DM \leq_P BP$

- Suppose that there is a way to fit these items into m bins.

  •Observe that the capacity C is always just enough to fit a triple and one occurence of each of it's three members, provided they are either all three or none, a first occurence. Also the sum of all items' sizes is mC.

  •All items' sizes are between $1/5$ and $1/3$ of C. Thus, each bin must contain four items.

  •$C \equiv 15 \text{mod} M$, and there is only one way $(\text{mod} M)$ to create 15 out of $1, 2, 4, 8$ with four items, even if repetitions are allowed. And that is to take all numbers, one time each. It follows that each bin will get a triple $\{b_i, g_j, h_k\}$, a boy b'_i, a girl g'_j and a home h'_k.

  •$C \equiv 15 \text{mod} M^2$ as well, so $i = i'$. Equivalently, taking $C \text{mod} M^3$ and M^4, we get $j = j'$ and $h = h'$.

  •It follows that there are n bins with only first occurences. The n triples in these bins form a TRIPARTITE MATCHING.
- The opposite direction is obvious.

- Proposition: Every instance of KNAPSACK can be solved in $O(nW)$.

Proof.

•Let $V(w,i)$ be the maximum value over the first i items such that their total weight is exactly w. We compute it in a table as follows:

• $V(w,0) = 0 \ \forall w$

• $V(w, i+1) = max\{V(w,i), v_{i+1} + V(w - w_{i+1}, i)\}$

•An instance of KNAPSACK is a 'yes' instance iff the table contains an entry greater than or equal to K. □

Pseudopolynomial Algorithms and Strong NP-Completeness, continued

- The $O(nW)$ complexity does not contradict the fact that the knapsack is NP-complete, since W, unlike n, is not polynomial in the length of the input to the problem. The length of the W input to the problem is proportional to the number of bits in W, $log W$, not to W itself.
- Strong NP-Completeness: A problem that remains NP-Complete even for input of size at most $p(n)$ is called strongly NP-Complete.