Algorithms and Complexity: The Complexity of Theorem-Proving Procedures

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INTER-INSTITUTIONAL GRADUATE PROGRAM “ALGORITHMS, LOGIC AND DISCRETE MATHEMATICS”

- Presented at the 3rd Annual ACM SIGACT Symposium on the Theory of Computing (STOC, May 3-5, 1971)

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- In 1982, Cook received the Turing award. His citation reads:

  For his advancement of our understanding of the complexity of computation in a significant and profound way. His seminal paper, The Complexity of Theorem Proving Procedures, presented at the 1971 ACM SIGACT Symposium on the Theory of Computing, laid the foundations for the theory of NP-Completeness. The ensuing exploration of the boundaries and nature of NP-complete class of problems has been one of the most active and important research activities in computer science for the last decade.

Overview

Cook's Theorem

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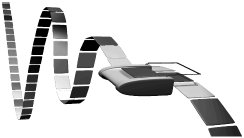
- NP-completeness.

Turing Machine

In his 1948 essay, "*Intelligent Machinery*", Turing wrote that his machine consisted of:

> *...an unlimited memory capacity obtained in the form of an infinite tape marked out into squares, on each of which a symbol could be printed. At any moment there is one symbol in the machine; it is called the scanned symbol. The machine can alter the scanned symbol, and its behavior is in part determined by that symbol, but the symbols on the tape elsewhere do not affect the behavior of the machine. However, the tape can be moved back and forth through the machine, this being one of the elementary operations of the machine. Any symbol on the tape may therefore eventually have an innings.*

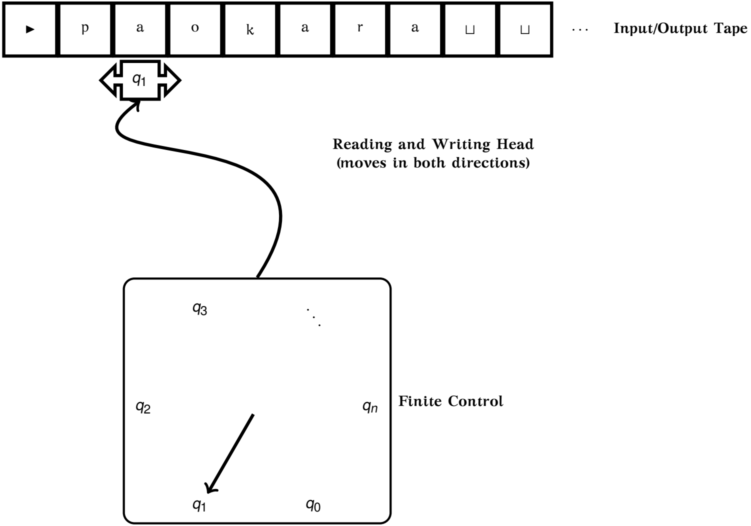


Formal Definition of Turing Machine

A *(one-tape) Turing machine* can be formally defined as a 6-tuple $M = \langle Q, \Gamma, \Sigma, \delta, q_0, F \rangle$ where

- Q is a finite, non-empty set of states;
- Γ is a finite, non-empty set of tape alphabet symbols;
- $\sqcup \in \Gamma$ is the *blank symbol* (the only symbol allowed to occur on the tape infinitely often at any step during the computation);
- $\blacktriangleright \in \Gamma$ is a special symbol, that defines the*begining* of the tape.
- $\Sigma \subseteq \Gamma \setminus \{b, \blacktriangleright\}$ is the set of input alphabet symbols, that is, the set of symbols allowed to appear in the initial tape contents;
- $q_0 \in Q$ is the *initial state*;
- $F \subseteq Q$ is the set of *final states* or *accepting states*. The initial tape contents is said to be accepted by M if it eventually halts in a state from F.
- $\delta : (Q \setminus F) \times \Gamma \cup \{\blacktriangleright\} \not\to Q \times \Gamma \cup \{\blacktriangleright\} \times \{L, R\}$ is a partial function called the *transition function*, where L is left shift, R is right shift and for any $q \in Q \setminus F$, $\delta(q, \blacktriangleright) = (\ldots, \blacktriangleright, R)$, if δ is defined on $(q, \blacktriangleright)$. If δ is not defined on the current state and the current tape symbol, then the machine halts (rejects).

Turing Machine



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- Every known and "unknown" models of notion of computation (calculability) are effectively equivalent.
- Every effective computation can be carried out by a Turing machine.

Church

Turing

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i.e. $\delta \subseteq ((Q \setminus F) \times \Gamma \cup \{\blacktriangleright\}) \times Q \times \Gamma \cup \{\blacktriangleright\} \times \{L, R\}$ where again for any $q, q' \in Q \setminus F$ and any $y \in \Gamma$, $(q, \blacktriangleright, q', y, L) \notin \delta$.

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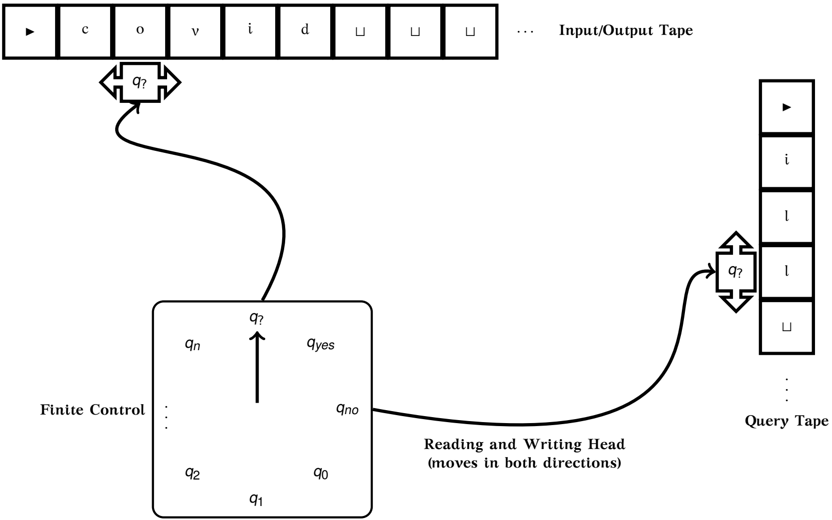
Query Machine

A *query machine* is a multitape Turing machine with a distinguished tape called the *query tape*, and three distinguished states called the *query state* ($q_?$), *yes state* (q_{yes}), and *no state* (q_{no}), respectively. If M is a query machine and T is a set of strings, then a T-computation of M is a computation of M in which initially M is in the initial state and has an input string w on its input tape, and each time M assumes the query state there is a string u on the query tape, and the next state M assumes is the yes state if $u \in T$ and the no state if $u \notin T$. We think of an "*oracle*", which knows T, placing M in the yes state or no state.

Query Machine



Polynomial-time Turing reduction

Definition

A set S of strings is P-reducible (P for polynomial) to a set T of strings iff there is some query machine M and a polynomial $Q(n)$ such that for each input string w, the T-computation of M with input w halts within $Q(|w|)$ steps ($|w|$ is the length of w) and ends in an accepting state iff $w \in S$.

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Cook's Theorem will give evidence that {tautologies} is a difficult set to recognize, since many apparently difficult problems can be reduced to determining tautologyhood.

Special Set of Strings

1. The *subgraph problem* is the problem given two finite undirected graphs, determine whether the first is isomorphic to a subgraph of the second. A graph G can be represented by a string $\overline{G}$ on the alphabet $\{0, 1, *\}$ by listing the successive rows of its adjacency matrix, separated by $*s$. We let {subgraph pairs} denote the set of strings $\overline{G_1} * * \overline{G_2}$ such that G_1 is isomorphic to a subgraph of G_2.

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5. The set D_3 consists of those tautologies in disjunctive normal form in which each disjunct has at most three conjuncts (each of which is an atom or negation of an atom).

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Corollary

Each of the sets in definitions 1)–5) is P-reducible to $\{$DNF tautologies$\}$.

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- $\neg A(w)$ is easily put in DNF (using De Morgan's laws), and $\neg A(w)$ is a tautology iff $w \notin S$.
- Since the whole construction can be carried out in time bounded by a polynomial in $|w|$, the theorem will be proved.

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Notice that since the computation has at most $T = Q(n)$ steps, no tape square beyond T is scanned.

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- $S_{s,t}$ is true iff at time t square number s is scanned by the tape head, where $s, t \in [T]$.

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B is a conjunction $B_1 \wedge B_2 \wedge \cdots \wedge B_T$, where B_t asserts that at time t one and only one square is scanned:

$$B_t = (S_{1,t} \vee S_{2,t} \vee \cdots \vee S_{T,t}) \wedge \left[\bigwedge_{1 \leq i < j \leq T} (\neg S_{i,t} \vee \neg S_{j,t}) \right]$$

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For $s \in [T]$ and $t \in [T_j]$ $C_{s,t}$ asserts that at square s and time t there is one and only one symbol. C is the conjunction of all the $C_{s,t}$.

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E asserts the initial conditions are satisfied:

$$E = Q_1^0 \wedge S_{1,1} \wedge P_{1,1}^{i_1} \wedge P_{2,1}^{i_2} \wedge \cdots \wedge P_{n,1}^{i_n} \wedge P_{n+1,1}^{1} \wedge \cdots \wedge P_{T,1}^{1}$$

where $w = \sigma_{i_1} \cdots \sigma_{i_n}$, q_0 is the initial state and $\sigma_1 = \sqcup$ is the blank symbol.

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(or states:

(nondeterminism)

$$G^t_{i,j} = \bigwedge_{s=1}^{T} \left(\neg Q^i_t \vee \neg S_{s,t} \vee \neg P^j_{s,t} \vee Q^k_{t+1} \right)$$

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sort of NP-Completeness

Theorem

The following sets are P-reducible to each other in pairs (and hence each has the same polynomial degree of difficulty): {*tautologies*}, {*DNF tautologies*}, D_3, {*subgraph pairs*}.

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- D_2 consisting of all DNF tautologies with at most two conjuncts per disjunct, is in $\mathscr{L}_*$ (Davis-Putnam procedure). Hence D_2 cannot be added to the list in theorem 2 (unless all sets in the list are in $\mathscr{L}_*$).

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- {DNF tautologies} is actually coNP-complete.

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Then A is a tautology if and only if A' is a tautology where

$$A' = (P \wedge R_3 \wedge \cdots \wedge R_s) \vee (\neg P \wedge R_1 \wedge R_2) \vee B_2 \vee B_3 \vee \cdots \vee B_k,$$

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Clearly the entire process is bounded in time by a polynomial in the length of A.

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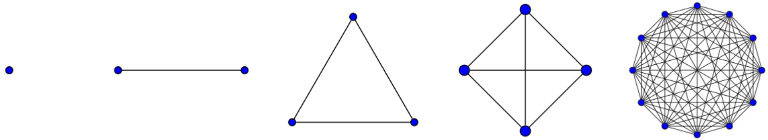

K_1 K_2 K_3 K_4 K_{12}

Proof of the 2nd Theorem

Let G_2 be the graph with vertices $\left\{u_{ij}\right\}$, $i \in [k]$, $j \in [3]$, such that u_{ij} is connected by an edge to u_{rs} iff $i \neq r$ and the two literals $\left(R_{ij}, R_{rs}\right)$ do not form an opposite pair (that is they are neither of the form $(P, \neg P)$ nor of the form $(\neg P, P)$).

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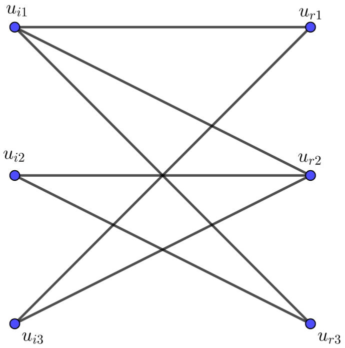
$C_i = R_{i1} \wedge R_{i2} \wedge R_{i3} = R \wedge \neg P \wedge X \qquad C_r = R_{r1} \wedge R_{r2} \wedge R_{r3} = P \wedge R \wedge \neg X$



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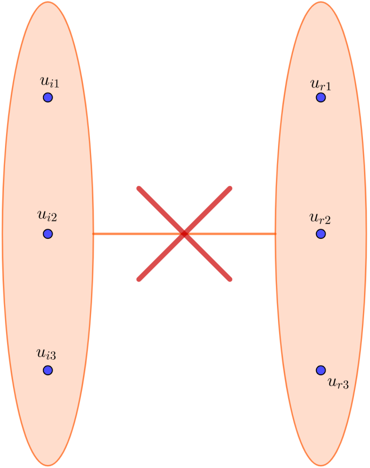

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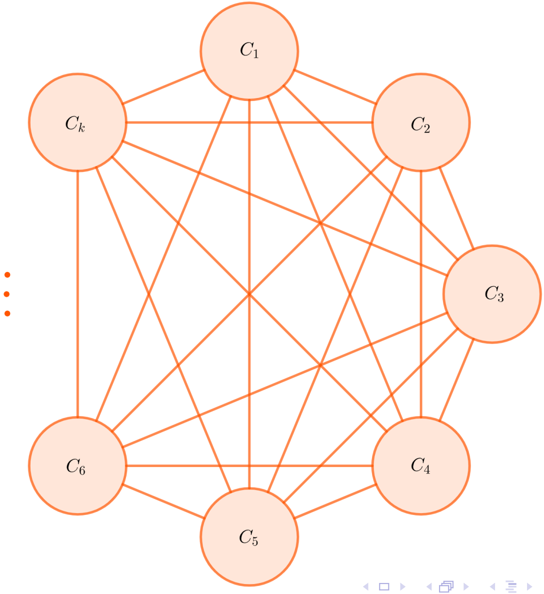

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This completes the proof of the 2nd theorem. $\square$

Discussion ☕

Theorem 1 and its corollary give strong evidence that it is not easy to determine whether a given proposition formula is a tautology, even if the formula is in normal disjunctive form. Theorems 1 and 2 together suggest that it is fruitless to search for a polynomial decision procedure for the subgraph problem, since success would bring polynomial decision procedures to many other apparently intractible problems. Of course the same remark applies to any combinatorial problem to which {tautologies} is P-reducible.

Furthermore, the theorems suggest that {tautologies} is a good candidate for an interesting set not in $\mathscr{L}_*$, and I feel it is worth spending considerable effort trying to prove this conjecture. Such a proof would be a major breakthrough in complexity theory.

In view of the apparent complexity of {DNF tautologies}, it is interesting to examine the Davis-Putnam procedure. This procedure was designed to determine whether a given formula in conjunctive normal form is satisfiable, but of course the "dual" procedure determines whether a given formula in disjunctive normal form is a tautology. I have not yet been able to find a series of examples showing the procedure (treated sympathetically to avoid certain pitfalls) must require more than polynomial time. Nor have I found an interesting upper bound for the time required.

The Legend Himself

An interview conducted with Cook by Bruce Kapron for the ACM on February 25, 2016.

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