IS, CLIQUE, MAX CUT

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MPLA, Algorithms and Complexity 2.

IS

- $G = (V, E)$, $I \subseteq V$
  The set I is independent if whenever $i, j \in I$ then there is no edge between i and j.

IS

Input: $G = (V, E)$, $K \in \mathbb{N}$

Question: Is there an independent set I with $|I| = K$?

- Independent Set is NP-complete.

IS

- $3 - SAT \leq^P IS$.
- Given an instance φ of 3-SAT with m clauses $C_i = (\alpha_{i1}, \alpha_{i2}, \alpha_{i3})$, we construct an instance of IS (G, K) where:

  $K = m$

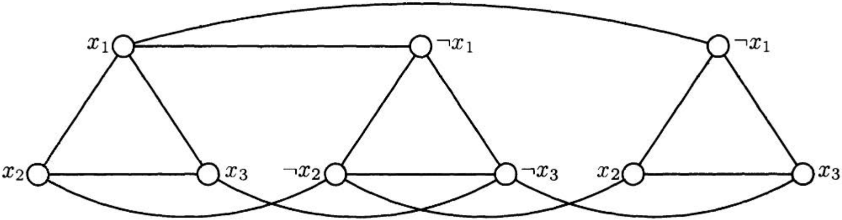
  $V = \{\alpha_{ij} : i = 1, ..., m; j = 1, 2, 3\}$

  $E = \{[\alpha_{ij}, \alpha_{ik}] : i = 1, ..., m; j \neq k\} \bigcup \{[\alpha_{ij}, \alpha_{lk}] : i \neq l, \alpha_{ij} = \neg\alpha_{lk}\}$

  (the first set defines the m triangles and the second group joins opposing literals).
- There is an independent set I of K nodes in $G \Leftrightarrow \varphi$ is satisfiable.

IS

$$(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$$



IS

$\Leftarrow$ Suppose that such I exists.
Then the true literals are just those which are labels of nodes of I because each triangle offers a node to I $(K = m)$
and there is an edge between the nodes x_i and $\neg x_i$ so they cannot both be in I

$\Rightarrow$ If a satisfying truth assignment exists,
then we identify a true literal in each clause,
and pick the node in the triangle of this clause labeled by this literal

4-DEGREE INDEPENDENT SET

- 4-DEGREE INDEPENDENT SET is NP-complete.

- Proposition 3SAT remains NP-complete even for expressions in which each variable is restricted to appear at most three times, and each literal at most twice.

- We construct the graph as in Independent Set.
- Each node in the graph has degree at most four.
- Complication: there are clauses now that contain just two literals.
- Such clauses are represented by a single edge joining the two literals.

CLIQUE

K-clique is a set of nodes that have all possible edges between them.

CLIQUE

Input: $G = (V, E)$, $K \in \mathbb{N}$

Question: Is there a set of K nodes that form a clique?

We reduce INDEPENDENT SET to CLIQUE by taking the complement of the graph.

NODE COVER

NODE COVER

Input: $G = (V, E)$, $K \in \mathbb{N}$

Question: Is there a set C with K or fewer nodes such each edge of G has at least one of its endpoints in C?

I is an independent set of a graph $G = (V, E) \Leftrightarrow V - I$ is a node cover of the same graph.

MAX CUT

- Cut : a partition of the nodes into two nonempty sets S and $V - S$.
- Size of cut : the number of edges between S and $V - S$.
- MIN CUT $\in$P
- MAX CUT is NP-complete.
  For the proof we will need the NAESAT problem.

MAX CUT

- NAESAT: in no clause are all literals equal in truth value (neither all true, nor all false)

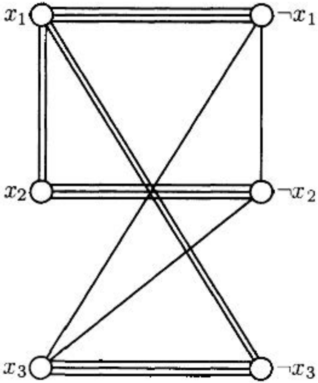
  For an instance φ with m clauses and n variables we construct a graph $G = (V, E)$ as follows:

  - $V = \{x_1, ..., x_n, \neg x_1, ..., \neg x_n\}$
  - If $C_i = (\alpha, \beta, \gamma)$ we add to E the three edges of the triangle $[\alpha, \beta, \gamma]$.

- If $C_i = (\alpha, \beta)$ we unite the nodes α, β with a double edge.
- If x_i or $\neg x_i$ occurres n_i times in the clauses then we add n_i copies of the edge $[x_i, \neg x_i]$

IS

$$(x_1 \vee x_2) \wedge (x_1 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \equiv$$
$$(x_1 \vee x_2 \vee x_2) \wedge (x_1 \vee \neg x_3 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$



MAX CUT

There is a truth assignment T that satisfies all clauses in the sense of NAESAT

$\Leftrightarrow$ the graph $G = (V, E)$ has a cut $(S, V - S)$ of size $5m$ or more.

- With no loss of generality we assume that all variables are separated from their negations.

  If both x_i and $\neg x_i$ are on the same side of the cut

  then we could change the side of one of them without decreasing the size of the cut

  because x_i and $\neg x_i$ together offers $2n_i$ edges to the cut.
- The literals in S are true and those in $V - S$ are false.

MAX CUT

- There are $3m$ edges that connect opposite literals
- The remaining $2m$ edges must be obtained from the triangles that correspond to the m clauses

  Each triangle offers at most two to the size of the cut, so all triangles must be split.
- That means that at least one of the literals in the triangle is false, and at least one true.

MAX BISECTION

MAX BISECTION

Input: $G = (V, E)$, $K \in \mathbb{N}$

Question: Is there a cut of size K or more such that $|S| = |V - S|$?

MAX BISECTION is NP-complete.

Proof: $MAXCUT \leq^P MAXBISECTION$

- We construct a new graph G' by adding $|V|$ disconnected new nodes.
- Every cut of G can be made into a bisection by appropriately splitting the new nodes between S and $V - S$.

BISECTION WIDTH

Input: $G = (V, E)$, $K \in \mathbb{N}$

Question: Is there a cut of size K or less such that $|S| = |V - S|$?

BISECTION WIDTH is NP-complete.

Proof: We observe that a graph $G = (V, E)$ where $|V| = 2n$ has a bisection of size K or more $\Leftrightarrow$ the complement of G has a bisection of size $n^2 - K$.