$M \subseteq M'$

$M' = M|_{\phi}$

$M' = M|_{S'}$

$M = (S, \sim, V)$

$M' = (S', \sim', V')$

$S' \subseteq S$

$\sim' = \sim \cap (S' \times S')$

$V'(p) = V(p) \cap S'$

$$M' \subseteq M \;\&\; M, s \models \phi \Rightarrow M', s \models \phi$$

$\&$

$s \in M'$

$$\mathcal{L}_{KC[]}(A, P)$$

$$\phi := p \mid \neg\phi \mid \phi \wedge \phi \mid K_a \phi \mid C_B \phi \mid [\phi]\phi$$

$$\mathcal{L}^0_{KC[]}(A, P)$$

$$\phi := p \mid \neg p \mid \phi \wedge \phi \mid \phi \vee \phi \mid K_a \phi \mid C_B \phi \mid [\neg\phi]\phi$$

4.37. Formulas of $\Sigma^{\circ}_{K\subset[J]}(A,P)$ are preserved under submodels

Proof

$\left(\text{for all } \phi \in \Sigma^{\circ}_{K\subset[J]}(A,P)\right)$
$\left(\text{for all } M\right)\left(\text{for all } M' \subseteq M \text{ such that } M' = (S', \sim', V')\right)\left(\text{for all } s \in S'\right)$
$\left[M, s \models \phi \Rightarrow M', s \models \phi\right]$

We will use Induction on ϕ

$M = (S, \sim, V)$, $M' = (S', \sim', V')$, $M' \subseteq M$

$-\ \phi = p$ $\quad M, s \models p \Rightarrow s \in V(p) \overset{s \in S'}{\Rightarrow}$

$s \in V(p) \cap S' \Rightarrow s \in V'(p) \Rightarrow M', s \models p$

$-\ \phi = \phi_1 \wedge \phi_2$

$M, s \models \phi_1 \wedge \phi_2 \Rightarrow M, s \models \phi_1$ and $M, s \models \phi_2 \Rightarrow$

$M', s \models \phi_1$ and $M', s \models \phi_2$

$-\ \phi = \neg p$ $\quad -\ \phi = \phi_1 \vee \phi_2$

- $\phi = K_a \psi$. $M, s \models K_a \psi \iff (\forall t)[s \sim_a t \Rightarrow M, t \models \psi]$

$M', s \models K_a \psi$

- $\phi = C_B \psi$ as before

- $\phi = [\neg\phi_1]\phi_2$

towards a contradiction

$M, s \models [\neg\phi_1]\phi_2$

$M', s \not\models [\neg\phi_1]\phi_2$

$M, s \models \neg\phi_1 \Rightarrow$

$M', s \models \neg\phi_1$ and

$M|_{\neg\phi_1}, s \models \phi_2$

$M'|_{\neg\phi_1}, s \not\models \phi_2$

$M, s \models \phi_1 \Rightarrow$
$M', s \models \phi_1$

$M', s \not\models \phi_1$ $\quad$ $M'|_{\neg\phi_1}, s \not\models \phi_2$

$\Downarrow$ i.h.

$M, s \not\models \phi_1$ $\quad \overset{\text{i.h.}}{\Longrightarrow}$ $M'|_{\neg\phi_1}, s \not\models \phi_2$

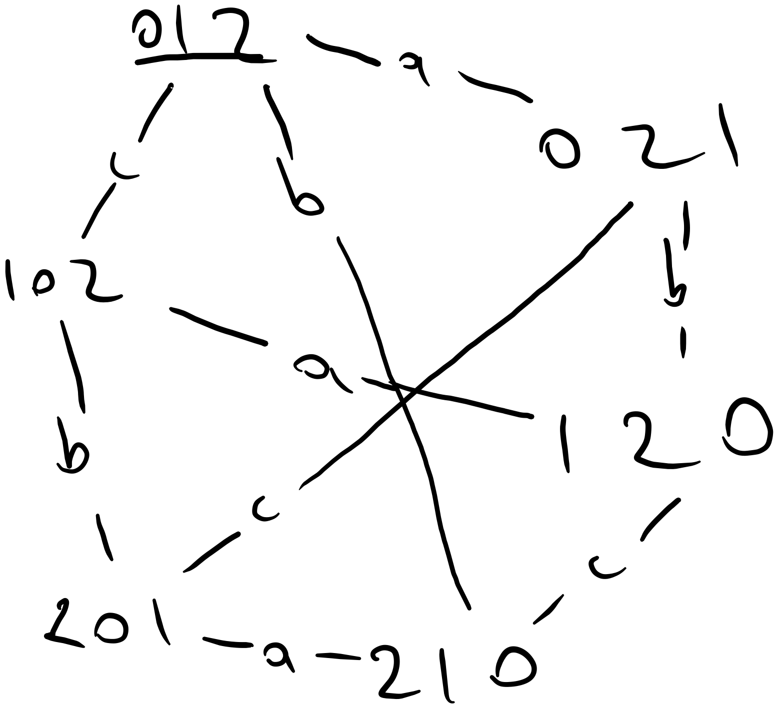
$\Downarrow$

$M|_{\neg\phi_1}, s \models \phi_2$

$[\phi]\phi$ is valid $\iff$ $\phi \to [\phi]\phi$ is valid

$[\neg K_a p]\neg K_a p$

$M' \subseteq M \Rightarrow \big(M, s \models \neg K_a p \Rightarrow M', s \models \neg K_a p\big)$



$0_a \wedge \neg K_b 0_a$

$$\begin{tikzcd} \underline{012} \arrow[r, dash, "a"] \arrow[d, Rightarrow, "1_b \wedge \neg K_a 1_b"] & 021 \\ 012 \end{tikzcd}$$

Axiomatization

$\vdash [p] K_a p$

$[\phi] K_a \psi \leftrightarrow (\phi \to K_a [\phi] \psi)$

1. $p \to p$
2. $[p] p \leftrightarrow (p \to p)$
3. $[p] p$
4. $K_a [p] p$
5. $p \to K_a [p] p$
6. $(p \to K_a [p] p) \leftrightarrow [p] K_a p$
7. $[p] K_a p$

$\vdash \langle\phi\rangle\psi \longrightarrow [\phi]\psi \mid \vdash \neg\langle\phi\rangle\neg\psi \to \langle\phi\rangle\psi$

1. $[\phi]\neg\psi \leftrightarrow \phi \longrightarrow \neg[\phi]\psi$
2. $\neg[\phi]\neg\psi \leftrightarrow \neg(\phi \longrightarrow \neg[\phi]\psi)$
3. $\langle\phi\rangle\psi \leftrightarrow \phi \wedge [\phi]\psi$
4. $\langle\phi\rangle\psi \longrightarrow [\phi]\psi$

if $\vdash \psi \leftrightarrow \chi$ then $\vdash \phi(p/\psi) \leftrightarrow \phi(p/\chi)$

by induction in ϕ

- $\phi = p$ or - $\phi = \phi_1 \wedge \phi_2$

i.h. $\vdash \psi \leftrightarrow \chi \Rightarrow \vdash \phi_1(p/\psi) \leftrightarrow \phi_1(p/\chi)$

$\vdash \psi \leftrightarrow \chi \Rightarrow \vdash \phi_2(p/\psi) \leftrightarrow \phi_2(p/\chi)$

$$\phi(p/\psi) = (\phi_1 \wedge \phi_2)(p/\psi) =$$

$$\phi_1(p/\psi) \wedge \phi_1(p/\psi)$$

- $\phi = K_a \phi_1$ $\quad\quad$ $\vdash \psi \leftrightarrow \chi \implies \vdash \phi_1(p/\psi) \leftrightarrow \phi_1(p/\chi)$

$$\phi_1(p/\psi) = K_a \phi_1(p/\psi)$$

$$\phi_1(p/\chi) = K_a \phi_1(p/\chi)$$

$$\frac{\vdash \phi}{\vdash K_a \phi}$$

$$\vdash \phi_1(p/\psi) \to \phi_1(p/\chi)$$

$$\vdash K_a(\phi_1(p/\psi) \to \phi_1(p/\chi))$$

$$\vdash K_a(\phi_1(p/\psi) \to \phi_1(p/\chi)) \to (K_a \phi_1(p/\psi) \to K_a \phi_1(p/\chi))$$

- $\phi = [\phi_1]\phi_2$

$\vdash \psi \leftrightarrow \chi$

$\vdash \phi_1(p/\psi) \leftrightarrow \phi_1(p/\chi)$

$\vdash \phi_2(p/\psi) \leftrightarrow \phi_2(p/\chi)$

$$\phi(p/\psi) = [\phi_1(p/\psi)]\,\phi_2(p/\psi)$$

$$\phi(p/\chi) = [\phi_1(p/\chi)]\,\phi_2(p/\chi)$$

$\vdash \psi \leftrightarrow \chi$

$\vdash \phi_1(p/\psi) \leftrightarrow \phi_1(p/\chi)$

$\vdash \phi_2(p/\psi) \leftrightarrow \phi_2(p/\chi)$

$$\left([\phi_1)\rho'\right)(p/\psi) = \left(\phi_1(p/\psi)\right]\rho'$$

$$[\phi_1)\rho' \leftrightarrow (\phi_1 \rightarrow \rho')$$

$$\phi_1[p/\psi)\rho' \leftrightarrow \left(\phi_1(p/\psi] \rightarrow \rho'\right)$$

$$\phi_1(p/\chi]\rho' \leftrightarrow \left(\phi_1(p/\chi] \rightarrow \rho'\right)$$

$$\vdash \psi \leftrightarrow \chi$$

$$\vdash \phi_1(p/\psi) \leftrightarrow \phi_1(p/\chi)$$

$$\vdash \phi_2(p/\psi) \leftrightarrow \phi_2(p/\chi)$$

$$(\phi_1 \to \rho')(\rho'/\phi_2) \equiv (\phi_1 \to \phi_2)$$

$$\phi' < (\phi_1)\, \phi_2$$

$$\vdash \psi (\to) \chi \Rightarrow \vdash \phi'(\rho'/\psi) (\to) \phi'(\rho'/\chi)$$

$$\phi' = (\phi_1)\, \rho'$$

$$\vdash \phi_2(\rho/\psi) (\to) \phi_2(\rho/\chi) \Rightarrow \vdash ((\phi_1)\, \rho')(\rho'/\phi_2(\rho/\psi))$$

$$\Updownarrow$$

$$((\phi_1)\, \rho')(\rho'/\phi_2(\rho/\chi))$$

$\Phi_1(\rho/4$

$$\phi = \left(\phi_1\right] \phi_2$$

$$\phi(p/4) = \left(\phi_1(p/4)\right] \phi_2(p/4)$$