"Dynamic Epistemic Logic"

Hans van Ditmarsch
Wiebe van der Hoek
Barteld Kooi

Johann van Benthem

Workload

- presentations from the book
- Assignment & oral Exams

Dynamic Epistemic Logic

Classical Propositional Logic

Truth: if it rains then I do not play

$$r \longrightarrow \neg p$$

Modal Logic:

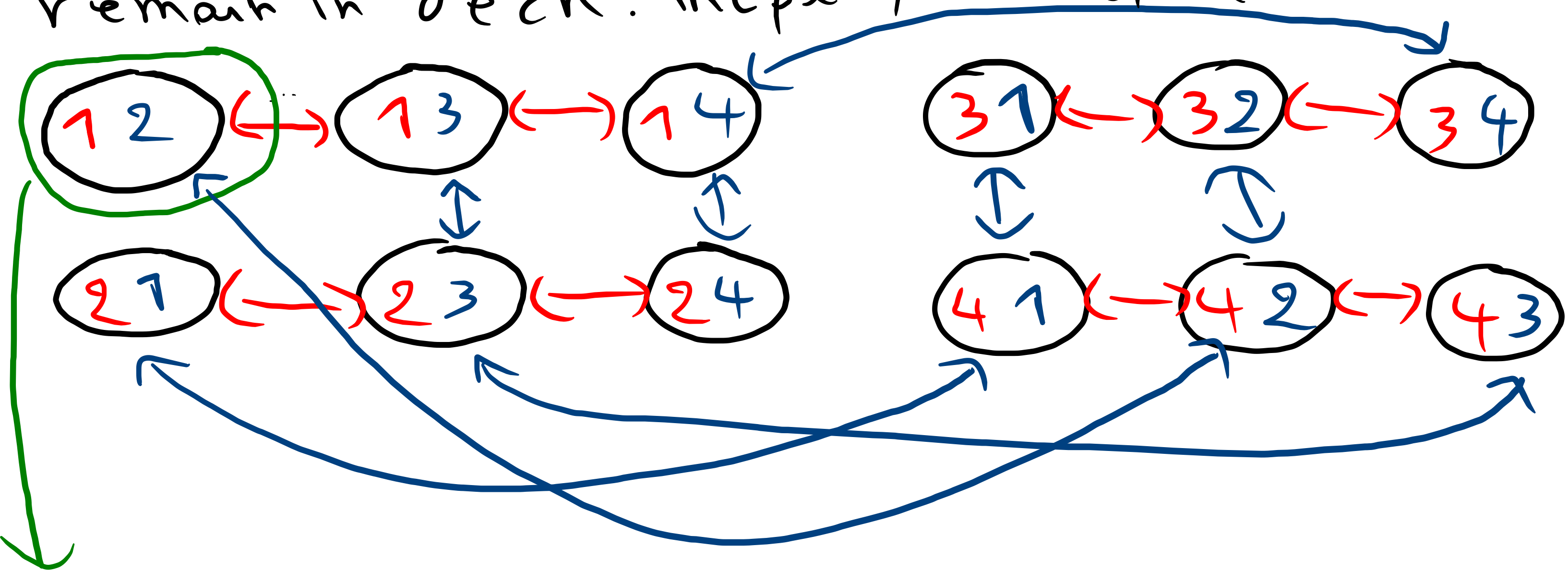
$\Box\phi$: "I know ϕ", "I believe ϕ",

.... "always ϕ", "necessarily ϕ"

$\boxed{\text{Example}}$ 4 cards: 1, 2, 3, 4

two players: A B

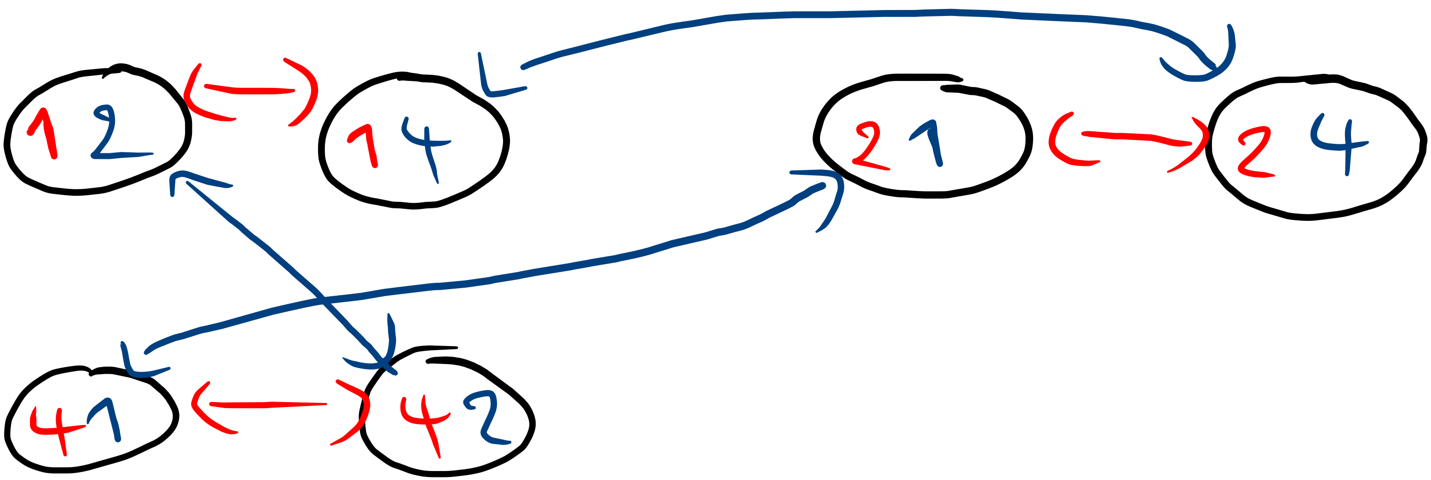
each player takes a card and the remaining two cards remain in deck. The players keep their cards secret.



A: I know that I have card 1

A: I know that B has 2, 3 or 4

Dynamic: Actions. Player B reveals a card from the deck. It is card 3.



A: I know that B has 2 or 4

Basic modal language: $\mathcal{L}_{\Box}$

Let P a countable set of atomic propositions. The language $\mathcal{L}_{\Box}$ is defined using the following BNF:

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_a \phi$$

where $a \in A$ and A is a set of Agents and $p \in P$.

$\Box_a \phi$ = "agent a believes ϕ"

$\Diamond_a \phi = \neg \Box_a \neg \phi$ = "agent a considers ϕ possible"

Examples

$\Box_a \Box_b \phi$ = "Agent a believes that agent b belies ϕ"

$\Diamond_a \phi \longrightarrow \Box_b \psi$

$\Diamond_a \Diamond_b (\phi \to \psi) \wedge \Box_a \Box_b \Box_c (\phi \wedge \Diamond_d \psi)$

a: Anne b: Bill c: Cath

p: "Anne has a sister"

q: "Anne has a brother"

(1) If Anne has a sister, she knows it.

$$p \to \Box_a p$$

(2) Bill knows that Anne has a sister or Bill knows that Anne does not have a sister.

$$\Box_b p \lor \Box_b \neg p$$

$$\cancel{\Box_b (p \lor \neg p)}$$

(3) Anne considers it possible that Bill does not know that Anne has a sister.

$$\Diamond_a \neg \Box_b p$$

Anne knows that p does not hold: $\Box_a \neg p$

Anne does not know that p holds: $\neg \Box_a p$

$$\Box_a \neg p \not\Leftrightarrow \neg \Box_a p$$

$\phi =$ "$P = NP$"

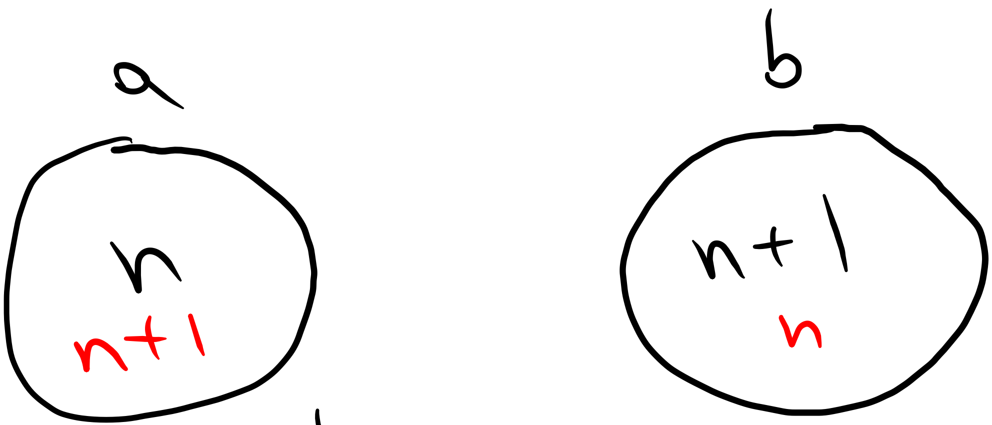
I know that $P \neq NP$ $\quad \Box \neg \phi$

I don't know that $P = NP$ $\quad \neg \Box \phi$

Example 2.4 (DEL, p.14)

Consecutive numbers:

a: Anne b: Bill. They see a number on each other's head.



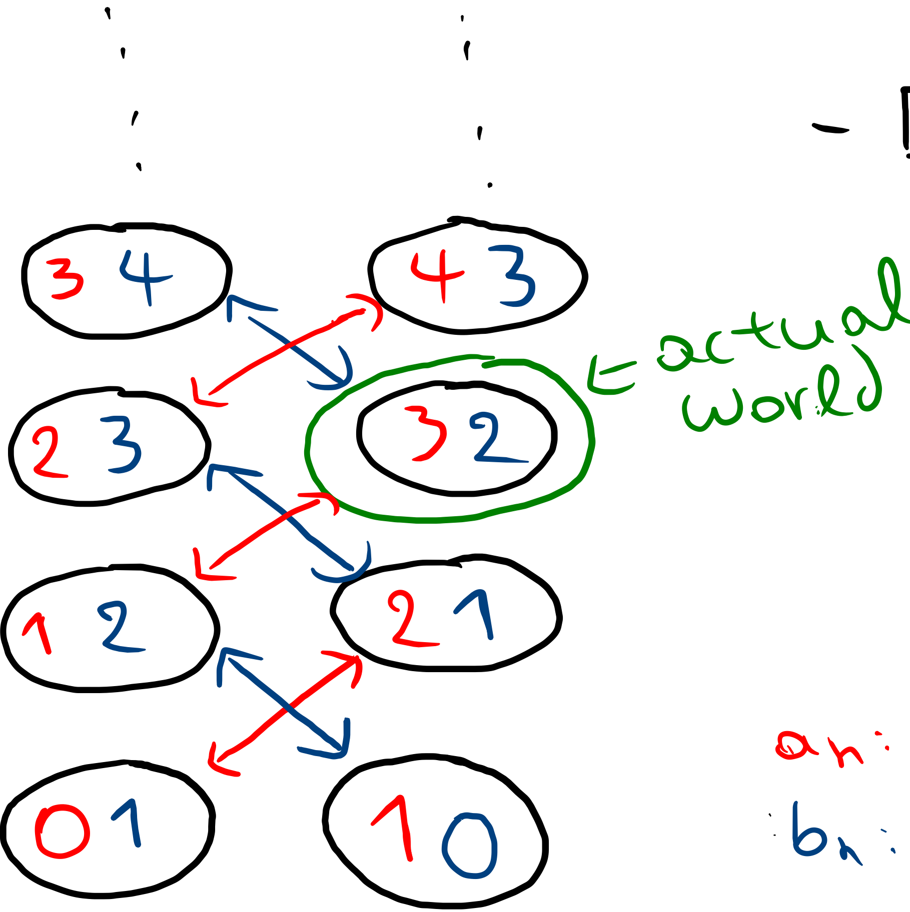
$n \in \mathbb{N}$

The numbers are consecute natural numbers.

a_n: Anne has number n on her head

b_n: Bill ——— " ——— his —"—

a: Anne b: Bill



- $\Box_a b_2$
- $\Box_a(a_1 \vee a_3)$
- $\Box_a \Box_b (b_0 \vee b_2 \vee b_4)$

a_n: Anne has n

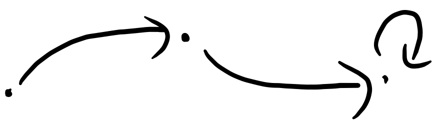
b_n: Bill has n

When we write $\Box\phi$, we assume that there is only one agent.

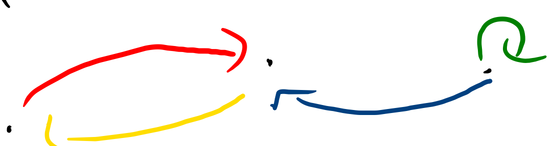
Semantics

Definition (Frame)

A frame is a directed graph (S, R) where S is a set of possible worlds and $R \subseteq S \times S$ is an accessibility relation.



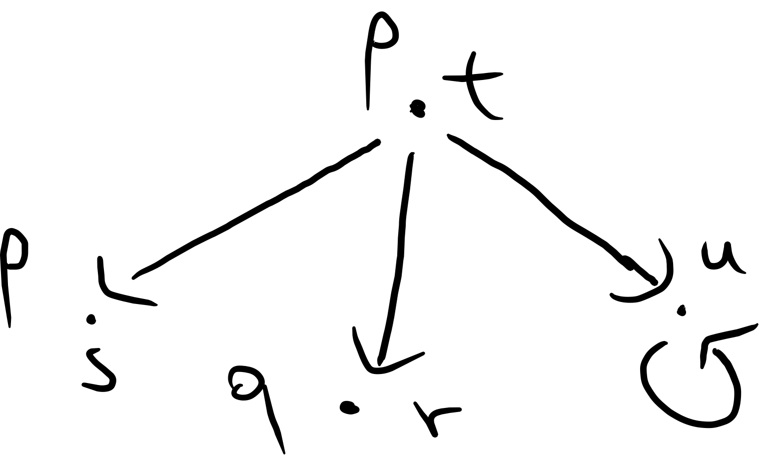
If we have more than one agent then R would be a function from A to $S \times S$.



If s, t are related via R we write sRt

Kripke Model

A Kripke model for $L_\Box$ is a triple $M = \langle S, R, V \rangle$ where $\langle S, R \rangle$ is a frame and V is function from P to 2^S (powerset of S).



$$V(p) = \{s, t\}$$

$$V(q) = \{r\}$$

Truth ($\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box\phi$)

Let $\mathcal{M} = \langle S, R, V \rangle$ be a model and let $s \in S$. We say that a formula ϕ holds in $\mathcal{M}, s$ if

- $\phi = p$, then $\mathcal{M}, s \models p \iff s \in V(p)$
- $\phi = \neg\psi$, then $\mathcal{M}, s \models \neg\psi \iff \mathcal{M}, s \not\models \psi \iff$ it does not hold that $\mathcal{M}, s \models \psi$
- $\phi = \phi_1 \wedge \phi_2$, then $\mathcal{M}, s \models \phi_1 \wedge \phi_2 \iff (\mathcal{M}, s \models \phi_1 \text{ and } \mathcal{M}, s \models \phi_2)$
- $\phi = \Box\psi$, then $\mathcal{M}, s \models \Box\psi \iff (\forall t)[sRt \Rightarrow \mathcal{M}, t \models \psi]$

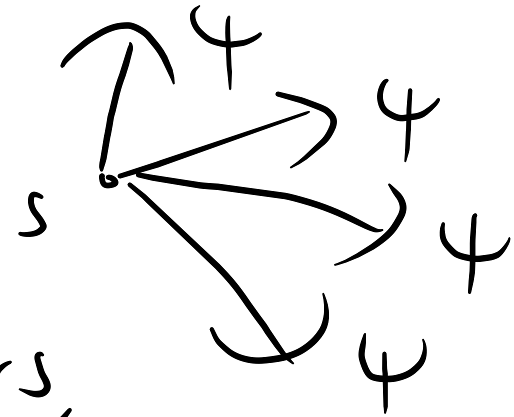
What about $\Diamond$?

$\phi = \Diamond\psi = \neg\Box\neg\psi$

$$\mathcal{M}, s \models \Diamond\psi \iff \mathcal{M}, s \models \neg\Box\neg\psi \iff$$
$$\neg(\forall t)[sRt \Rightarrow \mathcal{M}, t \models \neg\psi] \iff$$
$$(\exists t)[sRt \text{ and } \mathcal{M}, t \models \psi]$$

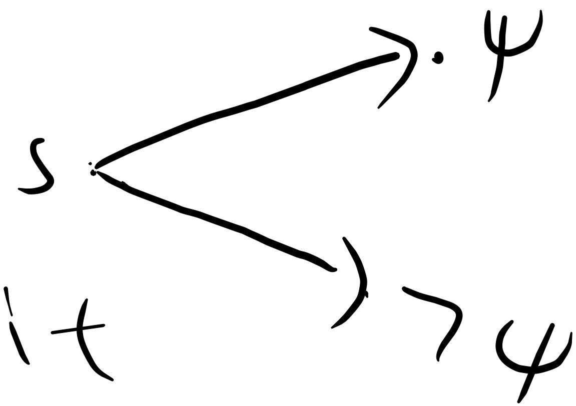
$\mathcal{M}, s \models \Box \varphi$

in ALL neighbours, it holds φ

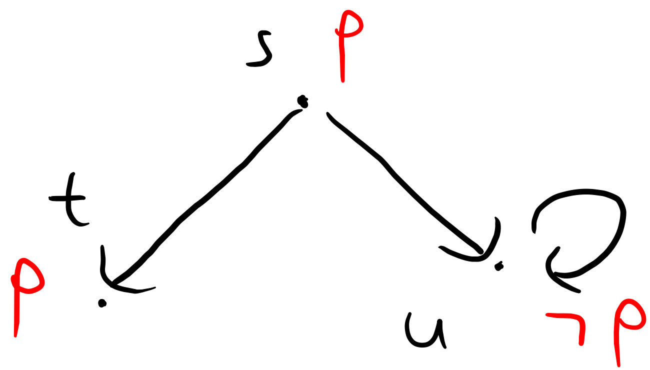


$\mathcal{M}, s \models \Diamond \varphi$

in SOME neighbour, it holds φ



$M = \langle S, R, V \rangle$ $\quad S = \{s, t, u\}$



$R = \{(s,t), (s,u), (u,u)\}$

$V(p) = \{s, t\}$

only one agent

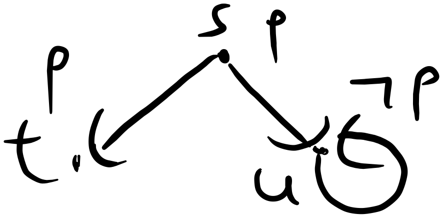
$M, s \models p \iff s \in V(p)$

$M, u \models \Box \neg p \iff (\forall x)[uRx \Rightarrow M, x \models \neg p]$

$\iff (\forall x)[uRx \Rightarrow M, x \not\models p]$ (*)

It holds that u is the only world accessible from u and also $u \notin V(p)$.

So, $M, u \not\models p$ and (*) holds



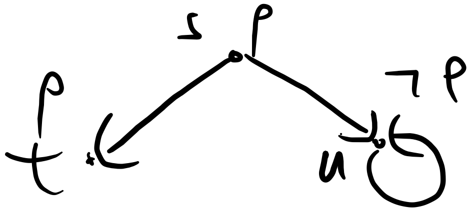
$M, s \models \Diamond p \iff$
$(\exists w)[sRw \text{ and } M, w \models p]$

it holds if we take $w = t$.

$M, t \models \Box \neg p \iff (\forall w)[tRw \Rightarrow M, w \models \neg p]$

since there is no world accessible from it it holds.

Since t has no neighbours, it holds that for all ϕ, $M, t \models \Box \phi$



$M,t \models \Box(p \wedge \neg p) \iff$

$(\forall w)(tRw \implies M,w \models p \wedge \neg p)$

$M,t \models p \wedge \neg p \iff (M,t \models p$ and

$M,t \models \neg p) \iff (t \in V(p)$ and $t \notin V(p))$

So $M,t \models p \wedge \neg p$ does not hold

$M,s \overset{?}{\models} \Diamond\Diamond p \iff (\exists w_1)[sRw_1$ and $M,w_1 \models \Diamond p] \iff$

$(\exists w_1)[sRw_1$ and $(\exists w_2)[w_1Rw_2$ and $M,w_2 \models p]$

So $M,s \models \Diamond\Diamond p$ does not hold

$M,s \models \Diamond\Diamond\neg p$ holds

A formula ϕ is called satisfiable if there is a model $M = \langle S, R, V \rangle$ and a world $s \in S$ such that $M, s \models \phi$.

<u>Example</u>: $\phi = p$

$M_1 = \langle S_1, R_1, V_1 \rangle$

$S_1 = \{w\}$, $R_1 = \emptyset$,

$V_1(p) = \{w\}$

$V_1(q) = \emptyset$ for $q \neq p$

Example of satisfiable formula

- $\phi = \Diamond p$ $\quad w^{\neg p} \longrightarrow u^{p}$

- $\phi = \Box(p \wedge \neg p)$ $\quad M' = (S', R', V')$

  w $\quad S' = \{w\}$, $R' = \emptyset$, $V(p) = \emptyset$ for all p

$M', w \models \Box(p \wedge \neg p) \iff (\forall u)(wRu \Rightarrow M', u \models p \wedge \neg p]$

This does not imply that $M, w \models p \wedge \neg p$

The duality between $\Box$ and $\Diamond$ is similar to the duality between $\forall$ and $\exists$

$$\Diamond \phi \iff \neg \Box \neg \phi$$

$$(\exists x) \phi \iff \neg (\forall x) \neg \phi$$