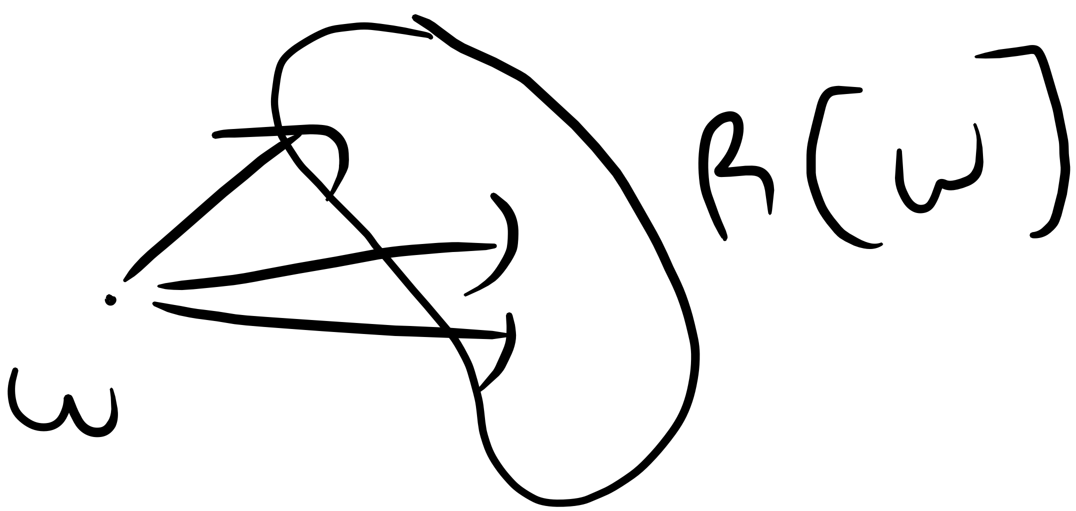
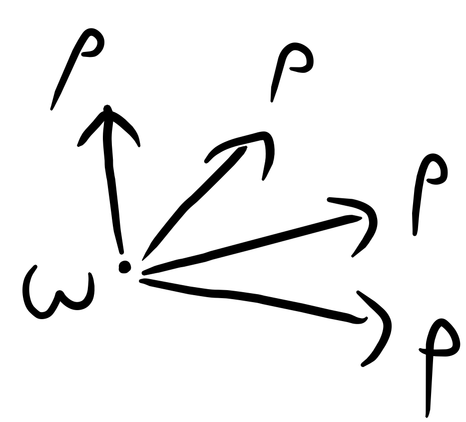
Notation

$$R[w] = \{u \mid wRu\}$$



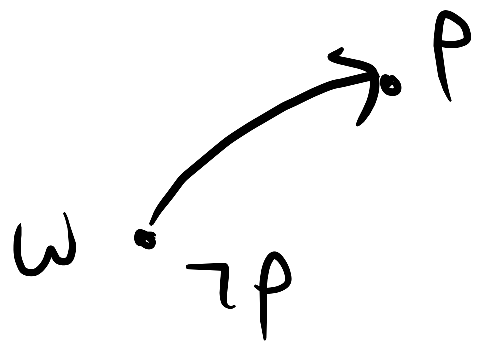
Validity: A formula ϕ is valid if for all W, for all w $M, w \models \phi$

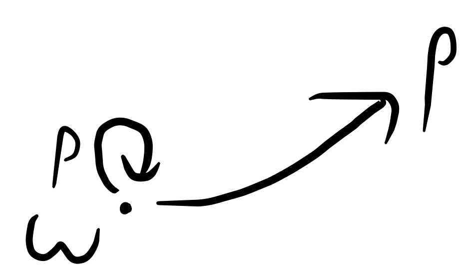
Ex $\overset{?}{\models} \Box p \to p$



$M, w \models \Box p \to p$

$M, w \models \Box p \implies M, w \models p$

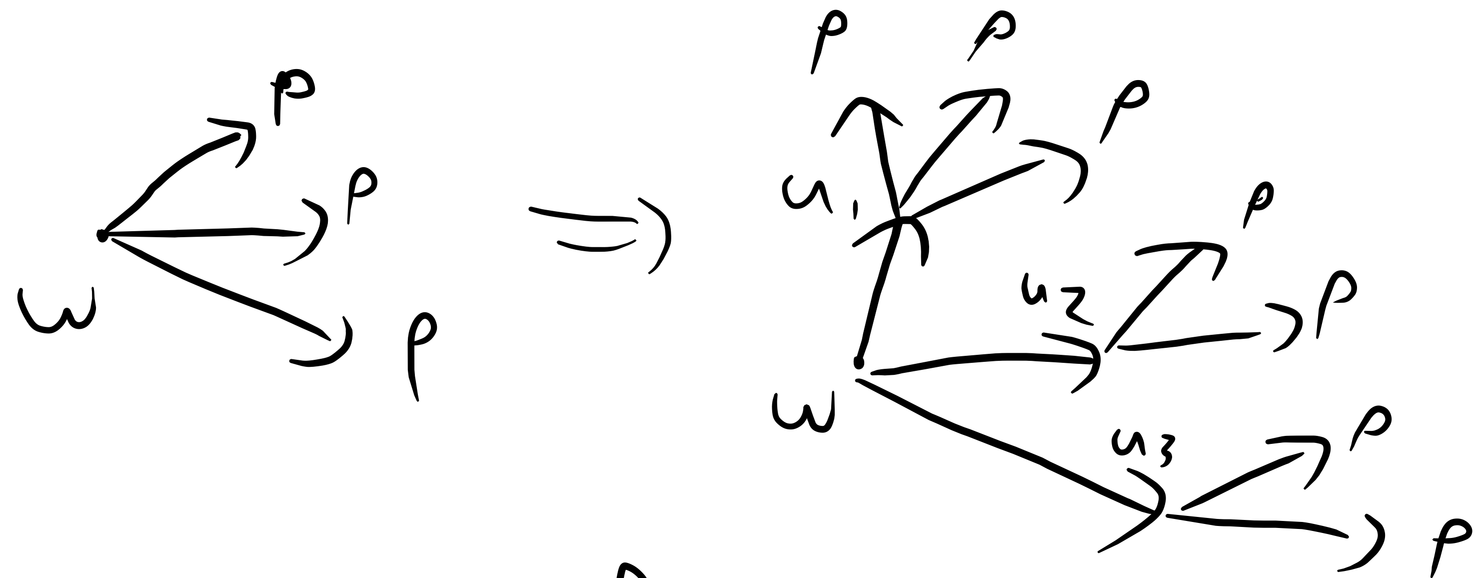


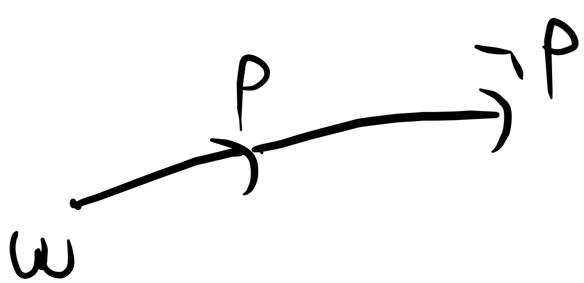


If M is reflexive then for all w
$M, w \models \Box p \to p$

$\overset{?}{\models} \Box p \to \Box\Box p$

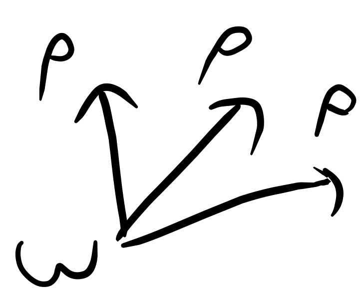
$M,w \models \Box p \implies M,w \models \Box\Box p$

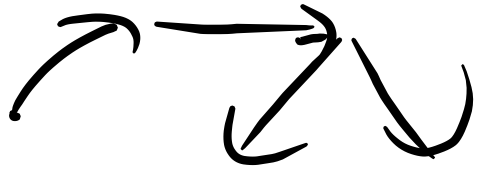




If M is transitive then for all w

$M,w \models \Box p \to \Box\Box p$



Frame:  $\langle S, R \rangle$

Def 2.13 (p.24)

We define the following classes of frames:

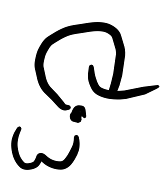
① K = {the class of all frames}

② KD = {the class of serial frames}

$F = \langle S, R \rangle$ is serial iff $(\forall w)(\exists u)[wRu]$

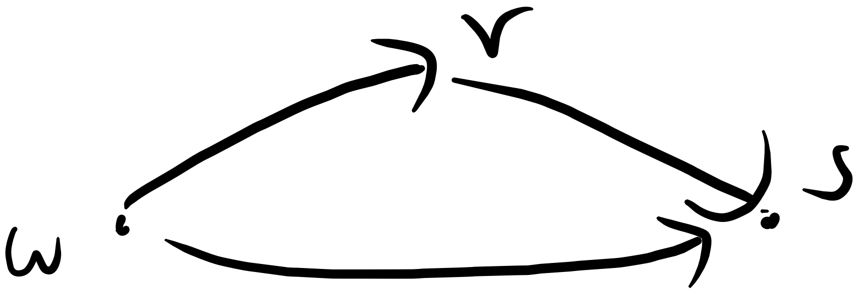
③ T = {the class of reflexive frames}

$F = \langle S, R \rangle$ is reflexive iff $(\forall w)[wRw]$



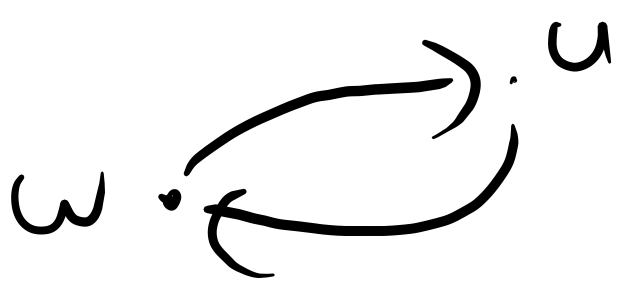
④ K4 = {the class of transitive frames}

$F = \langle S, R \rangle$ is transitive iff $(\forall w)(\forall r)(\forall s)[wRr \wedge rRs \Rightarrow wRs]$



⑤ KB = {the class of symmetric frames}

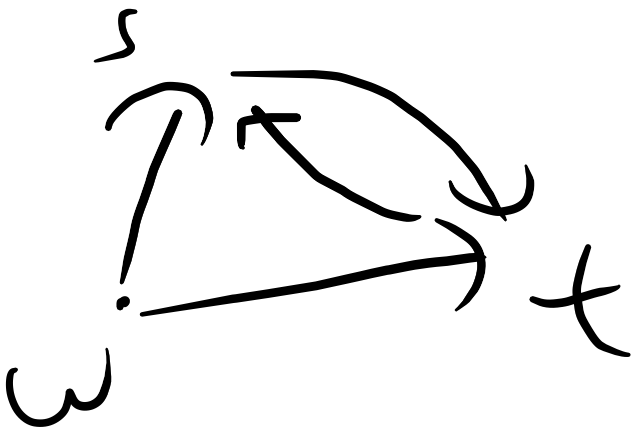
$F = \langle S, R \rangle$ is symmetric iff $(\forall w)(\forall u)[wRu \Rightarrow uRw]$



⑥ K_{D45} = {the class of all euclidean frames}

$F = \langle S, R \rangle$ is euclidean iff

$$(\forall w)(\forall s)(\forall t)(wRs \wedge wRt \Rightarrow sRt)$$



⑦ $S5$ = {the class of all frames where the accessibility relation is an equivalence relation}

Reflexive $\Rightarrow$ Serial OK

Reflex. & Euclidean $\Rightarrow$ Symmetry

$(\forall w, s, r)(wRs \wedge wRr \Rightarrow sRr)$

$(\forall w)\ [wRw]$

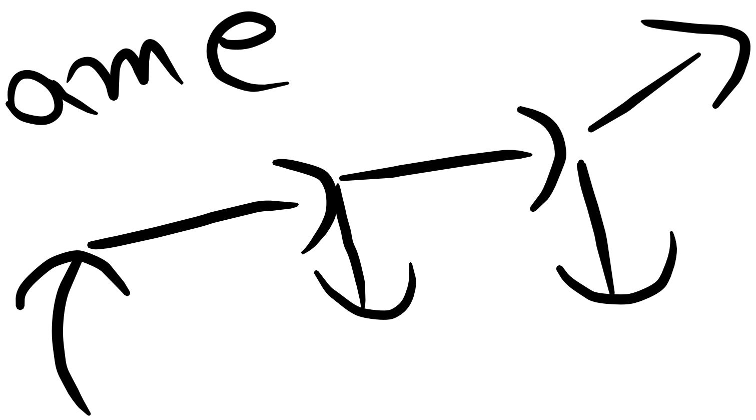
$wRs \wedge wRw \Rightarrow sRw$

symmetry

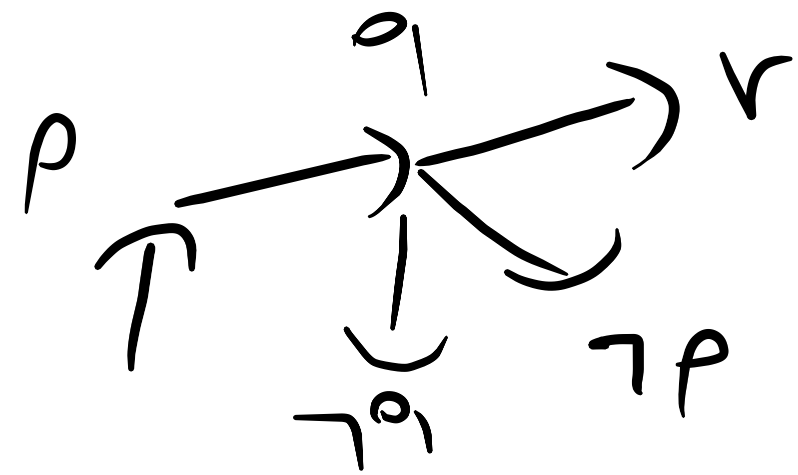
Validity on Frames

A formula ϕ is valid on a fram $F = \langle S, R \rangle$ iff for all $M = \langle S, R, V \rangle$ it holds $M \models \phi$

Frame



Model



<u>Theorem</u>: The formula $\Box p \to p$ <u>defines</u> reflexivity. That means that $\Box p \to p$ is valid on a frame F iff F is reflexive.

<u>Proof</u>

($\Leftarrow$) Let $F = \langle S, R \rangle$ such that R is reflexive. Let $M = \langle S, R, V \rangle$ for some V.

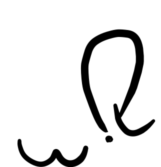
Let $w \in S$

$M, w \models \Box p \to p$ iff

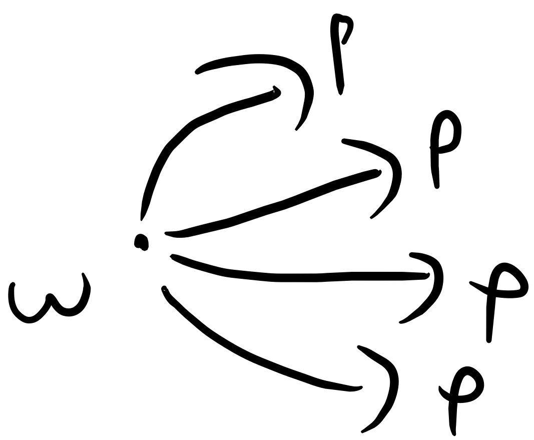
$M, w \models \Box p \Rightarrow M, w \models p$

$(\forall u \in R(w)) [M, u \models p] \Rightarrow M, w \models p$

it holds since $w \in R(w)$



($\Rightarrow$) Let $F=\langle S,R\rangle$ such that $\Box p\to p$ is valid on F. And assume tha R is not reflexive. Then there exists a $w\in R$ such that $w\notin R[w]$. We define $V(p)=R[w]$ and $V(q)=\emptyset$ for $q\neq p$. Then we have the model $M=\langle S,R,V\rangle$ and it hols



$M,w\vDash\Box p$ and

$M,w\nvDash p$. So itholds

$M,w\nvDash\Box p\to p$

which is absurd, since $\Box p\to p$ is valid on F.

Theorem The formula $\Box p \to \Box\Box p$ defines transitivity. That means that for all frames F, $\Box p \to \Box\Box p$ is valid on F iff F is transitive.

Proof Let $F = \langle S, R \rangle$ be a frame

$(\Leftarrow)$ Let $M = \langle S, R, V \rangle$ for some V and let $w \in S$. Then

$M, w \models \Box p \implies M, w \models \Box\Box p$

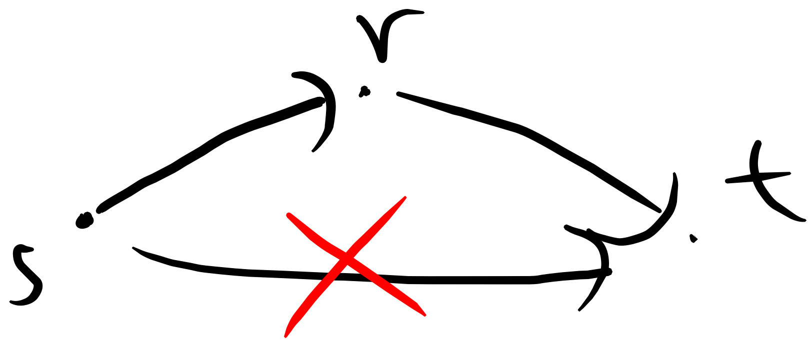
$(\forall u \in R[w])[M, u \models p] \implies (\forall r \in R[w])[M, r \models \Box p]$

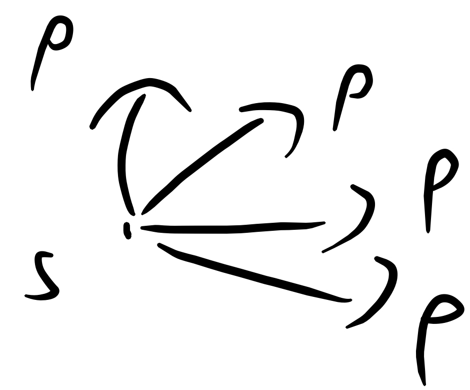
$(\forall u \in R[w])[M, u \models p] \implies (\forall r \in R[w])(\forall s \in R[r])[M, s \models p]$ $(*)$

$r \in R[w]$ and $s \in R[r] \implies s \in R[w]$

Hence $M, s \models p$. Thus $(*)$ holds.

($\Rightarrow$) Assume that $\Box p \to \Box\Box p$ is valid on F. And assume that F is not transitive. Hence, there are s, r and t such that sRr and rRt and $s \not R t$





We define $V(p) = R[s]$

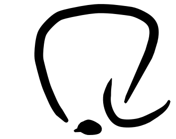
Hence we have the model $M = (S, R, V)$

We have $M, s \models \Box p$

but $M, t \not\models p$, which means $M, s \not\models \Box\Box p$

So, $M, s \not\models \Box p \to \Box\Box p$. Which is absurd, sine $\Box p \to \Box\Box p$ is valid on $F = (S, R)$.

(T) $\Box p \to p$ defines reflexivity.



(B) $p \to \Box \Diamond p$ defines symmetry

(D) $\Box p \to \Diamond p$ defines seriality

$\Box \bot \to \bot$

where $\bot = p \wedge \neg p$ for some p

(4) $\Box p \to \Box \Box p$ defines transitivity

(5) $\Diamond p \to \Box \Diamond p$ defines euclideaners

$\neg \Box p \to \Box \neg \Box p$

Exercise

Show the cases for (B), (D) and (5) in the previous slide.

Remark.

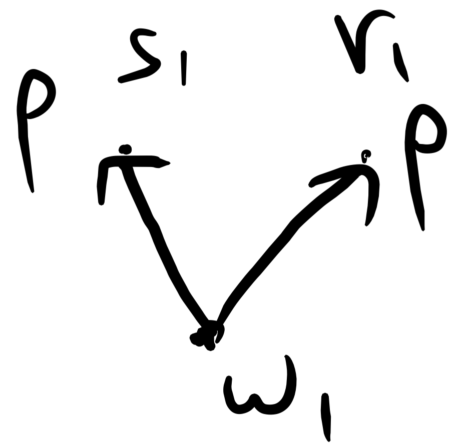
$\Box p \to \Box\Box p$ = "if I know p then I know that I know p"

positive introspection

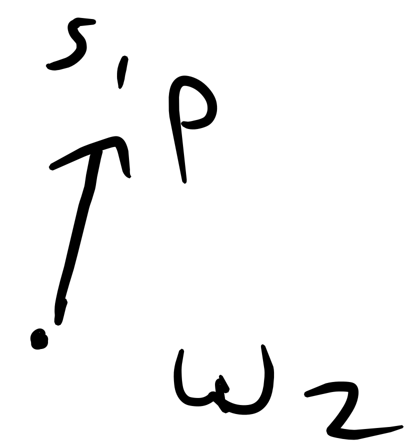
$\neg\Box p \to \Box\neg\Box p$ = "if I don't know p then I know that I do not know p"

negative introspection

Bisimulation



M_1

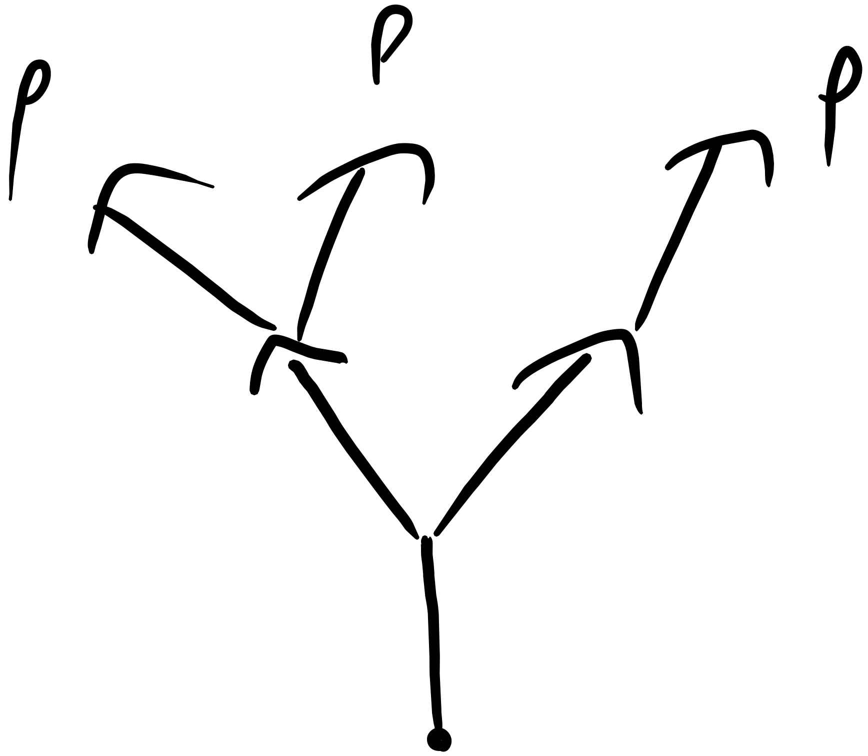


M_2

$$(\forall \phi)(M_1, w_1 \vDash \phi \iff M_2, w_2 \vDash \phi)$$

$\Box p$ $\qquad\qquad\qquad$ $\Box p$

$\Diamond p$ $\qquad\qquad\qquad$ $\Diamond p$

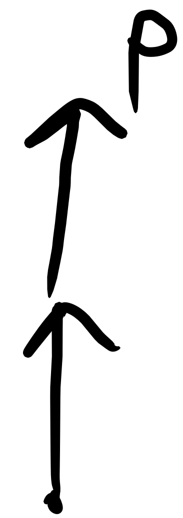


w_1

M_1

$\Box\Box p$

$\Diamond\Diamond p$



w_2

M_2

$\Box\Box p$

$\Diamond\Diamond p$

Bisimulation (Def 2.14, p. 24)

Let $M=(S,R,V)$ and $M'=(S',R',V')$. A non-empty relation $B \subseteq S \times S'$ is called a **bisimulation** iff for all $s \in S$ and $s' \in S'$, $(s,s') \in B$ iff

$$\begin{tikzcd}
M & \underset{p}{s} \arrow[r, "R"] \arrow[d, "B"'] & t \arrow[d, dashed, "B"] \\
M' & \underset{p}{s'} \arrow[r, dashed, "R'"'] & t'
\end{tikzcd}$$

forth

$(t,t') \in B$

$$\begin{tikzcd} s \arrow[r, dashed, "R"] \arrow[d, "B"'] & t \arrow[d, dashed, "B"] \\ s' \arrow[r, "R'"'] & t' \end{tikzcd}$$

back

$(t, t') \in B$

atoms for all $p \in P$,

$$s \in V(p) \iff s' \in V'(p)$$

Forth: for all $t \in S$

$$sRt \implies (\exists t' \in S')[s'R't' \text{ and } (t,t') \in B]$$

back: for all $t' \in S'$

$$s'R't' \implies (\exists t \in S)[sRt \text{ and } (t,t') \in B]$$

Forth preserves ignorance formulas

$$\begin{tikzcd}
s \ \neg\Box\phi \arrow[r] \arrow[d, "B"'] & t \ \neg\phi \arrow[d, dashed, "B"] \\
s' \arrow[r, dashed] & t' \ \neg\phi
\end{tikzcd}$$

back preserves knowledge formulas

$$\begin{tikzcd}
s \ \Box\phi \arrow[r, dashed] \arrow[d, "B"'] & t \ \phi \arrow[d, dashed, "B"] \\
s' \arrow[r] & t' \ \phi
\end{tikzcd}$$

If there is a bisimulation between M and M', linking s and s' we write $(M,s) \underline{\leftrightarrow} (M,s')$

Then we call (M,s) and (M,s') bisimilar.

So bisimilarity is between worlds of models and not between models.

$(M,s) \equiv_{\Sigma_\Box} (M',s')$ iff for all ϕ it hols $M,s \models \phi \iff M',s' \models \phi$

<u>Thm 2.15</u> For all $M = (S,R,V)$ and $M' = (S',R',V')$ and $s \in S$, $s' \in S'$ it holds $(M,s) \underline{\leftrightarrow} (M',s') \Rightarrow (M,s) \equiv_{\Sigma_\Box} (M',s')$

<u>Proof</u> Let M, M', s, s' be such that $(M,s) \underline{\leftrightarrow} (M',s')$ and let $\phi \in \Sigma_\Box$. We show that $M,s \models \phi \iff M',s' \models \phi$ by induction on the construction of ϕ.

- $\phi = p$. By definition of bisimilarity we have $\mathcal{M}, s \models \phi \iff \mathcal{M}, s \models p \iff p \in V(s) \iff$
$p \in V(s') \iff \mathcal{M}', s' \models p \iff \mathcal{M}', s' \models \phi$

- $\phi = \neg\psi$, where we have that
$\mathcal{M}, s \models \psi \iff \mathcal{M}', s' \models \psi$ (i.h.)

$$\mathcal{M}, s \models \phi \iff \mathcal{M}, s \models \neg\psi \iff \mathcal{M}, s \not\models \psi \overset{\text{(i.h.)}}{\iff} \mathcal{M}', s' \not\models \psi \iff$$
$$\mathcal{M}', s' \models \neg\psi \iff \mathcal{M}', s' \models \phi$$

- $\phi = \phi_1 \wedge \phi_2$ where $M, s \models \phi_i \overset{\text{i.h.}}{\iff} M', s' \models \phi_i$
$i \in \{1, 2\}$.

$M, s \models \phi \iff M, s \models \phi_1 \wedge \phi_2 \iff (M, s \models \phi_1$ and $M, s \models \phi_2) \overset{\text{i.h.}}{\iff} (M', s' \models \phi_1$ and $M', s' \models \phi_2) \iff$

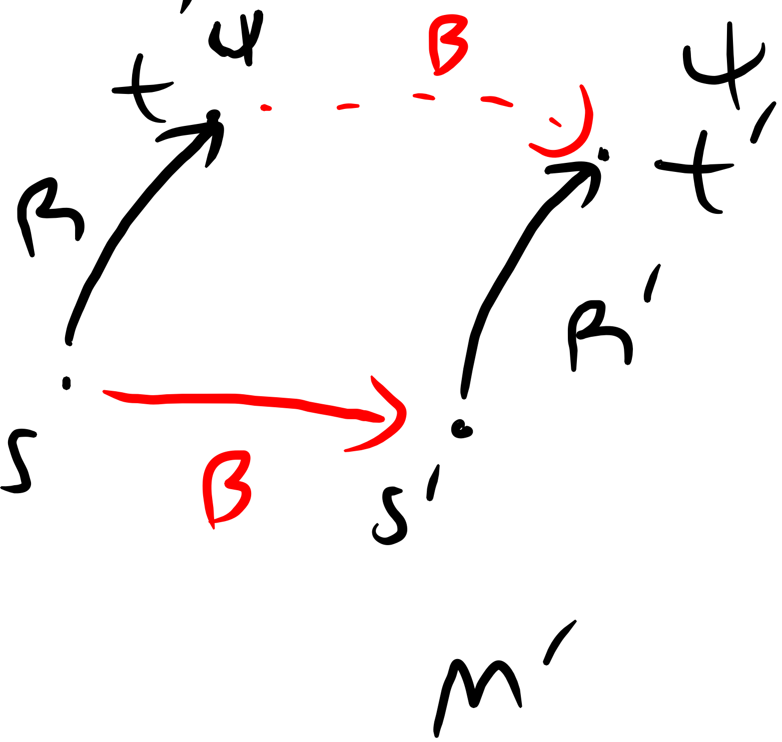
$M', s' \models \phi_1 \wedge \phi_2 \iff M', s' \models \phi$

- $\phi = \Box\psi$ where

for all t $\left(\left((M,t) \underline{\leftrightarrow} (M,t')\right) \Rightarrow \left(M,t \models \psi \Leftrightarrow M',t' \models \psi\right)\right)$ (i.h.)

($\Rightarrow$)

$M,s \models \phi \Rightarrow M,s \models \Box\psi \Rightarrow (\forall t \in R[s])(M,t \models \psi)$



$(M,t) \underline{\leftrightarrow} (M',t')$

$\uparrow$ back

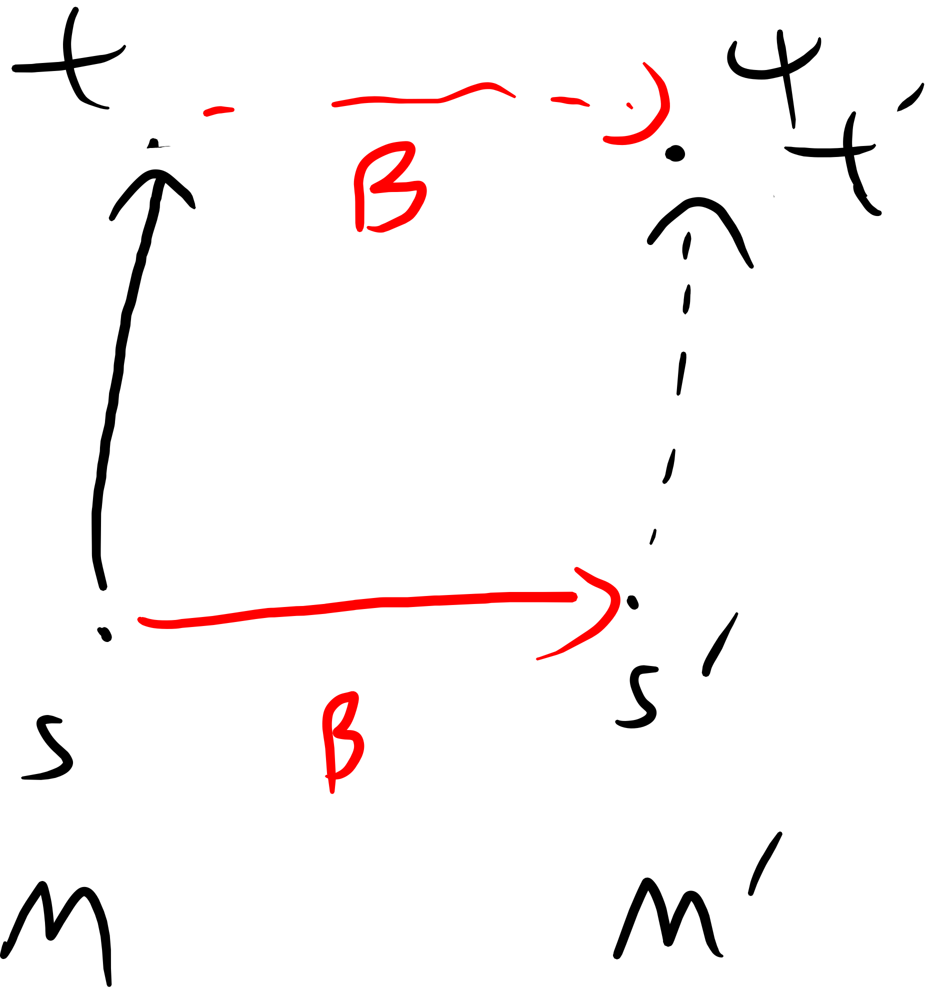
$\Downarrow$ i.h.

$M',t' \models \psi$

Hence for arbitrary t' with $t' \in R[s']$ it holds $M',t' \models \psi$. Hence $M,s' \models \Box\psi$

($\Leftarrow$)

$M', s' \models \phi \Rightarrow M', s' \models \Box\psi \Rightarrow (\forall t' \in R(s'))$
$(M', t' \models \psi)$



Let t be an arbitrary member of $R(s)$. Then by Forth, there is a $t' \in R(s')$ such that $(t, t') \in B$. It also holds that $M', t' \models \psi$ hence by i.h. it holds $M, t \models \psi$. Hence $M, s \models \Box\psi$