Logic = set of "true" formulas

How do we describe them?

<u>Semantical way</u>

$\models \phi$

valid formulas

Examples

$\models \phi \vee \neg \phi$

$\models \Box(\phi \to \psi) \to (\Box \phi \to \Box \psi)$

<u>Syntactical way</u>

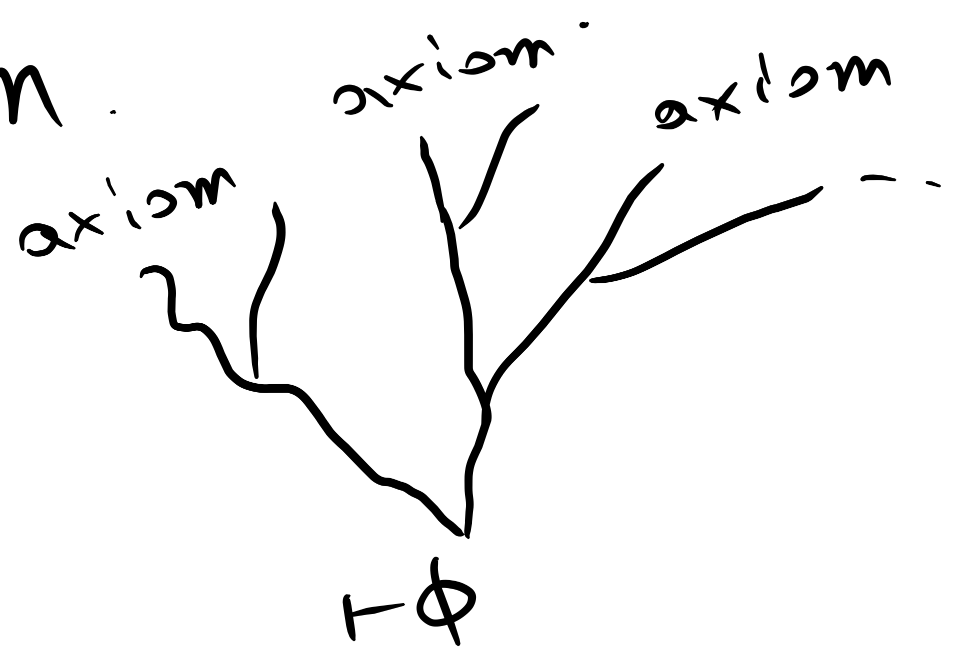
$\vdash \phi$

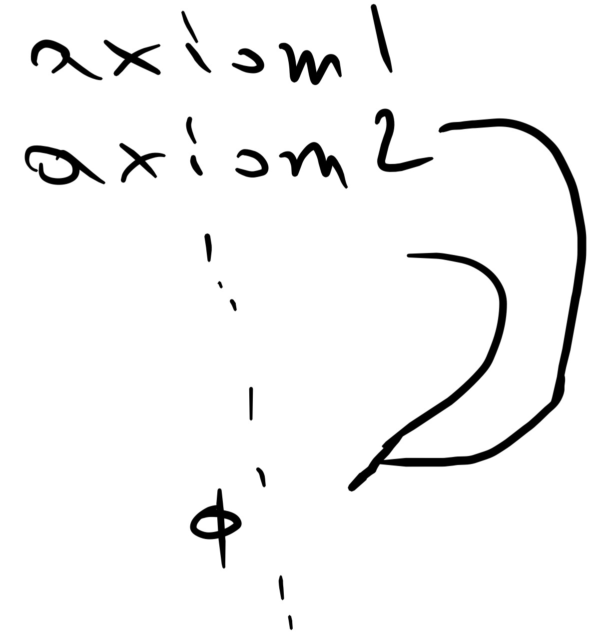
provable formulas

Semantics: We stated properties and we proved them in the meta-language

$$\models \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$$

Syntax: We have an axiomatic system and we have to prove everything using the rules and axioms of this system.





All men are mortal
Socrates is a man

Socrates is mortal

All A are B
C is A

C is B

System K

Axioms

(P) all instances of propositional tautologies in the language $L_\Box$

(K) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

Rules

(MP) from $\vdash \phi$ and $\vdash \phi \to \psi$ infer $\vdash \psi$

(NC) from $\vdash \phi$ infer $\vdash \Box\phi$

Definition (Provability of Formulas).

Let A be an axiomatic system that consists of some axioms and rules. A formula ϕ is provable in A if there is a finite sequence of formulas, $\phi_1, \ldots, \phi_n$ such that

- $\phi_n = \phi$
- for each i
  - either ϕ_i is an instance of an axiom of A
  - or ϕ_i is obtained by applying a rule of A to some $\phi_{j_1}, \ldots, \phi_{j_k}$ for $j_1, \ldots, j_k < i$.

All instances of propositional tautologies in the language $L_\Box$

$p \to p \quad \rightsquigarrow \quad \Box\Diamond p \to \Box\Diamond p$

$p \to (q \to p) \quad \rightsquigarrow \quad \Box\phi \to (\Diamond\psi \to \Box\phi)$

Instead of axiom (P), which contains all instances of prop tautologies, we could have used any axiomatic system for propositional logic, e.g.

(A1) $\phi \to (\psi \to \phi)$

(A2) $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$

(A3) $(\neg\phi \to \neg\psi) \to (\psi \to \phi)$

But we do not do it, in order to save space :)

Exercise 2.18 (p. 27)

1. From $\vdash \phi \to \chi$ $\vdash \chi \to \psi$ infer $\vdash \phi \to \psi$

(1)	$\phi \to \chi$	hypothesis
(2)	$\chi \to \psi$	hypothesis
(3)	$(\phi \to \chi) \to ((\chi \to \psi) \to (\phi \to \psi))$	(P)
(4)	$(\chi \to \psi) \to (\phi \to \psi)$	(MP) (1), (3)
(5)	$\phi \to \psi$	(MP) (2), (4)

$$\dfrac{\overset{(\text{hyp})}{(2)} \qquad \dfrac{\overset{(\text{hyp})}{(1)} \qquad \overset{(\text{P})}{(3)}}{(4)}\ (\text{MP})}{(5)}\ (\text{MP})$$

2. From $\vdash \phi \to \psi$ infer $\vdash \Box\phi \to \Box\psi$

(1) $\vdash \phi \to \psi$ hypothesis

(2) $\vdash \Box(\phi \to \psi)$ (NC) (1)

(3) $\vdash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ (K)

(4) $\vdash \Box\phi \to \Box\psi$ (MP) (2), (3)

3. $\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi$

(1) $\phi \wedge \psi \to \phi$ (P)

(2) $\phi \wedge \psi \to \psi$ (P)

(3) $\Box(\phi \wedge \psi \to \phi)$ (NC) (1)

(4) $\Box(\phi \wedge \psi \to \psi)$ (NC) (2)

(5) $\Box(\phi \wedge \psi \to \phi) \to (\Box(\phi \wedge \psi) \to \Box\phi)$ (K)

(6) $\Box(\phi \wedge \psi) \to \Box\phi$ (MP) (3), (5)

(7) $\Box(\phi \wedge \psi \to \psi) \to (\Box(\phi \wedge \psi) \to \Box\psi)$ (K)

(8) $\Box(\phi \wedge \psi) \to \Box\psi$ (MP) (4), (7)

(9) $\Box(\phi \wedge \psi) \to (\Box\phi \wedge \Box\psi)$ (6), (8), Prop. reasoning.

$$\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$$

(1) $\phi \to (\psi \to (\phi \wedge \psi))$ (P)

(2) $\Box(\phi \to (\psi \to (\phi \wedge \psi)))$ (NC), (1)

(3) $\Box(\phi \to (\psi \to (\phi \wedge \psi))) \to$
$\Box\phi \to (\Box(\psi \to (\phi \wedge \psi)))$ (K)

(4) $\Box\phi \to \Box(\psi \to (\phi \wedge \psi))$ MP (2), (3)

(5) $\Box(\psi \to \phi \wedge \psi) \to (\Box\psi \to \Box(\phi \wedge \psi))$ (K)

(6) $\Box\phi \to (\Box\psi \to \Box(\phi \wedge \psi))$ (4), (5), prop reasoning.

(7) $(\Box\phi \wedge \Box\psi) \to \Box(\phi \wedge \psi)$ (6), prop reasoning.

Try Ex. 2.18 and 2.20 from the book