Dynamic Epistemic Logic

Graduate Course

ALMA / Corelab
National Technical University of Athens

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Overview

Definability

Let ϕ be a modal formula and $\mathcal{F}$ a class of frames. We say that **ϕ defines $\mathcal{F}$** if for all frames F we have that

$$F \in \mathrm{F} \text{ if and only if } F \vDash \phi$$

.

Each one of the following properties is defined by a modal formula

Property	Modal formula
1. Reflexive: $\forall w\,(wRw)$	T: $\Box p \to p$
2. Symmetric: $\forall w \forall v\,(wRv \to vRw)$	B: $p \to \Box\Diamond p$
3. Serial: $\forall w \exists v\,(wRv)$	D: $\Box p \to \Diamond p$
4. Transitive: $\forall w \forall v \forall u\,(wRv \wedge vRu \to wRu)$	4: $\Box p \to \Box\Box p$
5. Euclidean: $\forall w \forall v \forall u\,(wRv \wedge wRu \to vRu)$	5: $\Diamond p \to \Box\Diamond p$

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5. Euclidean: $\forall w \forall v \forall u\,(wRv \wedge wRu \to vRu)$	5: $\Diamond p \to \Box\Diamond p$
6. Partially functional: $\forall w \forall v \forall u\,(wRv \wedge wRu \to v = u)$	DC: $\Diamond p \to \Box p$

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7. Functional: $\forall w \exists! v\,(wRv)$	D & DC: $\Diamond p \leftrightarrow \Box p$

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7. Functional: $\forall w \exists! v\,(wRv)$	D & DC: $\Diamond p \leftrightarrow \Box p$
8. Dense: $\forall w \forall v\,(wRv \to \exists u(wRu \wedge uRv))$	$4C$: $\Box\Box p \to \Box p$

The axiomatic system K

Axioms

- (P) all instances of propositional tautologies in the language $\mathcal{L}_\Box$
- (K) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

Rules

- (MP) from $\vdash \phi$ and $\vdash \phi \to \psi$ infer $\vdash \psi$
- (NC) from $\vdash \phi$ infer $\vdash \Box\phi$

How to define a system of modal logic Σ?

System of modal logic A set of formulas Σ is a system of modal logic iff it contains all propositional tautologies (PL) and is closed under modus ponens (MP) and uniform substitution (US).

Uniform substitution: from $\vdash \phi$, infer $\vdash \theta$, where θ is obtained from ϕ by uniformly replacing proposition variables in ϕ by arbitrary formulas.

We will usually just say ‘**logic**’ or sometimes ‘**system**’ instead of ‘system of modal logic’.

The **theorems** of a logic are just the formulas in it.
We write $\vdash_{\Sigma} A$ to mean that A is a theorem of Σ.

How to define a system of modal logic Σ?

Given a set of formulas Γ and a set of rules of inference R, define Σ to be the smallest system of modal logic containing Γ and closed under R.

Equivalently, given the definition of 'system of modal logic', the smallest set of formulas containing PL and Γ, and closed under R, modus ponens (MP), and uniform substitution (US).

Γ and R are sometimes called '**axioms**' of Σ.

More axioms...

$T: \Box p \to p \iff T_{\Diamond}: p \to \Diamond p$

$B: p \to \Box\Diamond p \iff \Diamond \Box p \to p$

$D: \Box p \to \Diamond p$

$4: \Box p \to \Box \Box p$

$5: \neg \Box p \to \Box \neg \Box p \iff \Diamond p \to \Box\Diamond p \iff \Diamond \Box p \to \Box p$

For example,

$\Box p \to p \iff$

$\neg p \to \neg \Box p \iff$

$\neg p \to \Diamond \neg p \iff$

$q \to \Diamond q$

More systems

T: $\Box p \rightarrow p \iff T_{\Diamond}$: $p \rightarrow \Diamond p$
B: $p \rightarrow \Box\Diamond p \iff \Diamond\Box p \rightarrow p$
D: $\Box p \rightarrow \Diamond p$
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Systems of modal logic

- K+T is called T
- K+B is called KB
- K+D is called KD
- K+4 is called K4

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- K+B is called KB
- K+D is called KD
- K+4 is called K4
- K+T+4 is called S4
- K+T+4+5 is called S5
- K+D+4+5 is called KD45

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T $\subseteq$ S4	Reminder: $\mathsf{K} + T \subseteq \mathsf{K} + T + 4$

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T ⊆ S4 Reminder: K + *T* ⊆ K + *T* + *4*

K4 ⊆ S4

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$\mathsf{T} \subseteq \mathsf{S4}$ Reminder: $\mathsf{K} + T \subseteq \mathsf{K} + T + 4$

$\mathsf{K4} \subseteq \mathsf{S4}$

$\mathsf{S4} \subseteq \mathsf{S5}$

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$\mathsf{T} \subseteq \mathsf{S4}$ Reminder: $\mathsf{K} + T \subseteq \mathsf{K} + T + 4$

$\mathsf{K4} \subseteq \mathsf{S4}$

$\mathsf{S4} \subseteq \mathsf{S5}$

$\mathsf{K} \subseteq \Sigma$ for every $\Sigma \in \{\mathsf{T}, \mathsf{KB}, \mathsf{KD}, \mathsf{K4}, \mathsf{S4}, \mathsf{S5}, \mathsf{KD45}\}$

- Some relationships among these systems are not so trivial.

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KD ⊆ T

Proof: We prove that $\vdash_{\mathrm{T}} D$.

1. $\Box p \to p$ — Axiom T
2. $p \to \Diamond p$ — Axiom $T_{\Diamond}$
3. $\Box p \to \Diamond p$ — 1,2, Prop. reasoning, MP

The axiomatic system S5

The following systems are equivalent:

$$S5 = KT5 = KT45 = KT4B = KDB4 = KDB5$$

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Proof sketch: $KT5 \subseteq KT45 \subseteq \mathbf{KT4B} \subseteq \mathbf{KDB4} \subseteq KDB5 \subseteq KT5$
For example, $KT4B \subseteq KDB4$, as $\vdash_{KDB4} T$:

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For example, KT4B ⊆ KDB4, as $\vdash_{KDB4} T$:

1. $\Box p \rightarrow \Box\Box p$ Axiom *4*

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For example, KT4B ⊆ KDB4, as $\vdash_{KDB4} T$:

1. $\Box p \to \Box\Box p$ — Axiom *4*
2. $\Box\Box p \to \Diamond\Box p$ — Axiom *D*, US

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For example, KT4B ⊆ KDB4, as $\vdash_{KDB4} T$:

1. $\Box p \rightarrow \Box\Box p$ — Axiom *4*
2. $\Box\Box p \rightarrow \Diamond\Box p$ — Axiom *D*, US
3. $\Diamond\Box p \rightarrow p$ — Axiom *B*

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2. $\Box\Box p \to \Diamond\Box p$ — Axiom *D*, US
3. $\Diamond\Box p \to p$ — Axiom *B*
4. $\Box p \to p$ — 1,2,3, Prop. reasoning, MP

Classes of Kripke structures (reminder)

We define the following classes of frames:

1. $\mathcal{K}$ = the class of all frames
2. $\mathcal{KD}$ = the class of serial frames
3. $\mathcal{T}$ = the class of reflexive frames
4. $\mathcal{K}4$ = the class of transitive frames
5. $\mathcal{KB}$ = the class of symmetric frames
6. $\mathcal{KD}45$ = the class of serial, transitive and euclidean frames
7. $\mathcal{S}5$ = the class of reflexive, transitive and symmetric frames = the class of frames where the accessibility relation is an equivalence relation

- Think of a frame as a model $M = \langle S, R, V \rangle$ without the valuation V. A frame is the underlying graph of a model.
- We say that a formula ϕ is valid with respect to a class of frames $\mathcal{F}$, symb. $\mathcal{F} \vDash \phi$, if ϕ is valid on every frame F in $\mathcal{F}$.

Definition of soundness

An axiomatic system Σ is **sound** with respect to a class $\mathcal{F}$ of frames if every formula provable from Σ is valid with respect to $\mathcal{F}$.

Definition of completeness

An axiomatic system Σ is **complete** with respect to a class $\mathcal{F}$ of frames if every formula that is valid with respect to $\mathcal{F}$ is provable from Σ.

We think of an axiom system as **characterizing a class of frames** exactly if it provides a sound and complete axiomatization of that class.

K is a sound and complete axiomatization with respect to $\mathcal{K}$ (the class of all frames).

T is a sound and complete axiomatization with respect to $\mathcal{T}$ (the class of reflexive frames).

KB is a sound and complete axiomatization with respect to $\mathcal{KB}$ (the class of symmetric frames).

KD is a sound and complete axiomatization with respect to $\mathcal{KD}$ (the class of serial frames).

K4 is a sound and complete axiomatization with respect to $\mathcal{K}4$ (the class of transitive frames).

KD45 is a sound and complete axiomatization with respect to $\mathcal{KD}45$ (the class of serial, transitive and euclidean frames).

S5 is a sound and complete axiomatization with respect to $\mathcal{S}5$ (the class of reflexive, transitive and symmetric frames).

Soundeness

K is a sound axiomatization with respect to $\mathcal{K}$ (the class of all frames).

Proof: Every K-provable formula is valid in $\mathcal{K}$. Let $F \in \mathcal{K}$.

1. If ϕ is a propositional tautology, then $F \vDash \phi$.
2. If $\mathcal{K} \vDash \phi$ and $\mathcal{K} \vDash \phi \to \psi$, then for any frame $F \in \mathcal{K}$ it holds that $F \vDash \psi$.
3. $F \vDash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$.
4. If $\mathcal{K} \vDash \phi$, then for any frame $F \in \mathcal{K}$ it holds that $F \vDash \Box\phi$.

Useful definition (reminder)

Definition

1. A set of formulas Σ is **consistent** iff there is no ϕ such that both $\vdash_\Sigma \phi$ and $\vdash_\Sigma \neg\phi$ hold.

2. A formula ψ is Σ-consistent iff $\Sigma \cup \{\psi\}$ is consistent.

Fact 1

A formula ψ is Σ-consistent iff $\nvdash_\Sigma \neg\psi$.

Useful proposition (reminder)

Proposition
Σ is complete with respect to a class of frames $\mathcal{F}$ iff every Σ-consistent formula is satisfiable on some frame $F \in \mathcal{F}$.

Proof: 1. ($\Leftarrow$) We argue by contraposition. Suppose Σ is not complete with respect to $\mathcal{F}$. Then there is a formula ϕ such that $\mathcal{F} \vDash \phi$ but $\nvdash_{\Sigma} \phi$. The formula $\neg\phi$ is Σ-consistent, but not satisfiable on any frame in $\mathcal{F}$.

($\Rightarrow$) We argue by contraposition. Suppose there is a Σ-consistent formula ϕ that is not satisfiable on any frame in $\mathcal{F}$. Then, $\neg\phi$ is valid with respect to $\mathcal{F}$. But $\nvdash_{\Sigma} \neg\phi$, since ϕ is Σ-consistent. So Σ is not complete with respect to $\mathcal{F}$.

Completeness

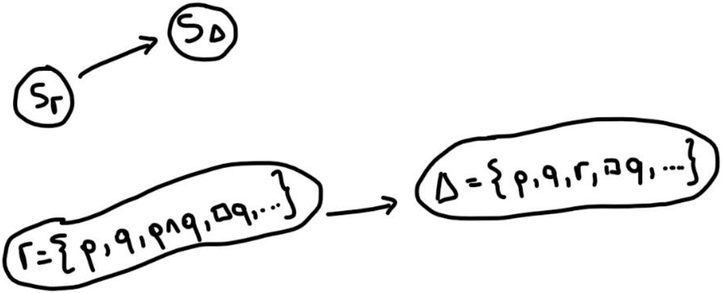
K is a complete axiomatization with respect to $\mathcal{K}$ (the class of all frames).

Every K-consistent formula is satisfiable on some frame $F \in \mathcal{K}$.

The main idea

1. We are going to build a model $\mathcal{M}^c$ such that every K-consistent formula is satisfiable on $\mathcal{M}^c$.
2. Each *state* of the model $\mathcal{M}^c$ corresponds to a *maximal K-consistent set of formulas*. And conversely, every *maximal K-consistent set of formulas* corresponds to a *state* in $\mathcal{M}^c$. There is a one-to-one correspodence between the set of states and the set of maximal K-consistent sets.
3. A K-consistent formula ϕ is satisfiable on every state that corresponds to a maximal K-consistent set containing ϕ.
4. This model is called the **canonical** model.

The canonical model



Useful definition (reminder)

Definition

A set Γ of formulas is a **maximal consistent** set if it is consistent and for every $\phi \notin \Gamma$, the set $\Gamma \cup \{\phi\}$ is inconsistent.

Useful theorem (reminder)

Deduction Theorem
If $\Gamma \cup \{\phi\} \vdash \psi$, then $\Gamma \vdash \phi \rightarrow \psi$.

The converse of the Deduction Theorem also holds and it is essentially an application of Modus Ponens.

Lemma

Every K-consistent set Γ of formulas can be extended to a K-maximal consistent set Γ^+.

In addition, if Γ^+ is a K-maximal consistent set, then it satisfies the following properties:

1. for every formula ϕ, exactly one of ϕ and $\neg\phi$ is in Γ^+,
2. $\phi \wedge \psi \in \Gamma^+$ iff $\phi \in \Gamma^+$ and $\psi \in \Gamma^+$,
3. if $\phi \in \Gamma^+$ and $\phi \to \psi \in \Gamma^+$, then $\psi \in \Gamma^+$,
4. if ϕ is K provable, then $\phi \in \Gamma^+$.

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Proof:

- Let Γ be a consistent set and $\phi \in \mathcal{L}_{\Box}$.

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- Let Γ be a consistent set and $\phi \in \mathcal{L}_\Box$.
- Then, (at least) one of $\Gamma \cup \{\phi\}$ or $\Gamma \cup \{\neg\phi\}$ is consistent.

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- Then, (at least) one of $\Gamma \cup \{\phi\}$ or $\Gamma \cup \{\neg\phi\}$ is consistent.
- Assume that $\Gamma \cup \{\neg\phi\}$ is inconsistent.

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- Assume that $\Gamma \cup \{\neg\phi\}$ is inconsistent.
- Then, $\vdash_{\Gamma} \phi$ (by Fact 1).

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- Assume that $\Gamma \cup \{\neg\phi\}$ is inconsistent.
- Then, $\vdash_{\Gamma} \phi$ (by Fact 1).
- So there is a proof of ϕ from Γ (and Γ is consistent).

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Proof:

- Let Γ be a consistent set and $\phi \in \mathcal{L}_\Box$.
- Then, (at least) one of $\Gamma \cup \{\phi\}$ or $\Gamma \cup \{\neg\phi\}$ is consistent.
- Assume that $\Gamma \cup \{\neg\phi\}$ is inconsistent.
- Then, $\vdash_\Gamma \phi$ (by Fact 1).
- So there is a proof of ϕ from Γ (and Γ is consistent).
- Therefore, $\Gamma \cup \{\phi\}$ is consistent.

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If Γ^+ is a K-maximal consistent set, then it satisfies the following properties:

- for every formula ϕ, exactly one of ϕ and $\neg\phi$ is in Γ^+,

Proof: Let $\phi_0, \phi_1, \phi_2, \ldots$ be an enumeration of formulas in $\mathcal{L}_\Box$. We define the set Γ^+ as follows:

$$\Gamma_0 = \Gamma$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\phi_n\}, & \text{if this is K-consistent} \\ \Gamma_n \cup \{\neg\phi_n\}, & \text{otherwise} \end{cases}$$

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$$\Gamma^+ = \bigcup_{n \geq 0} \Gamma_n$$

If Γ^+ is a K-maximal consistent set, then it satisfies the following properties:

- for every formula ϕ, exactly one of ϕ and $\neg\phi$ is in Γ^+,

Proof: Let Γ^+ be a maximal consistent set and $\phi \in \mathcal{L}_\Box$. Then,

- either $\Gamma^+ \cup \{\phi\}$ is consistent and so $\phi \in \Gamma^+$, since Γ^+ is maximal,
- or $\Gamma^+ \cup \{\phi\}$ is inconsistent, which implies that $\Gamma^+ \cup \{\neg\phi\}$ is consistent and so $\neg\phi \in \Gamma^+$, since Γ^+ is maximal.

Lemma

In addition, if Γ^+ is a K-maximal consistent set, then it satisfies the following properties:

- $\phi \wedge \psi \in \Gamma^+$ iff $\phi \in \Gamma^+$ and $\psi \in \Gamma^+$,

Proof: ($\Rightarrow$) Let Γ^+ be a maximal consistent set and $\phi \wedge \psi \in \Gamma^+$. Then,

- $\phi \in \Gamma^+$. For otherwise, we would have $\neg\phi \in \Gamma^+$ and Γ^+ would be inconsistent.
- $\psi \in \Gamma^+$ for the same reason.

($\Leftarrow$) Similarly.

Lemma

In addition, if Γ^+ is a K-maximal consistent set, then it satisfies the following properties:

- if $\phi \in \Gamma^+$ and $\phi \rightarrow \psi \in \Gamma^+$, then $\psi \in \Gamma^+$,

Proof: Let Γ^+ be a maximal consistent set and $\phi \in \Gamma^+$ and $\phi \rightarrow \psi \in \Gamma^+$. Then,

- $\vdash_{\Gamma^+} \psi$, since K is closed under Modus Ponens. So it holds that $\Gamma^+ \cup \{\psi\}$ is consistent and $\psi \in \Gamma^+$.

Lemma

In addition, if Γ^+ is a K-maximal consistent set, then it satisfies the following properties:

- if ϕ is K provable, then $\phi \in \Gamma^+$.

Proof: Let Γ^+ be a maximal consistent set and ϕ is K provable. Then,

- $\vdash_{\Gamma^+} \phi$. So it holds that $\Gamma^+ \cup \{\phi\}$ is consistent and $\phi \in \Gamma^+$.

Proof of completeness

Every K-consistent formula is satisfiable on some frame $F \in \mathcal{K}$.

Proof:
We construct a special model $\mathcal{M}^c$ in which every K-consistent formula is satisfiable!

$\mathcal{M}^c$ is called the *canonical* model.

$\mathcal{M}^c$ has a state s_Γ corresponding to every maximal consistent set Γ.

Proof of completeness

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Proof:
$\mathcal{M}^c$ has a state s_Γ corresponding to every maximal consistent set Γ.

We will show that:

$$\mathcal{M}^c, s_\Gamma \models \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

It suffices to prove $(*)$. Why?

Proof of completeness

We define the canonical model $\mathcal{M}^c$ for K to be the triple (W^c, R^c, V^c), where:

- $W^c = \{s_\Gamma \mid \Gamma \text{ is a maximal consistent set}\}$

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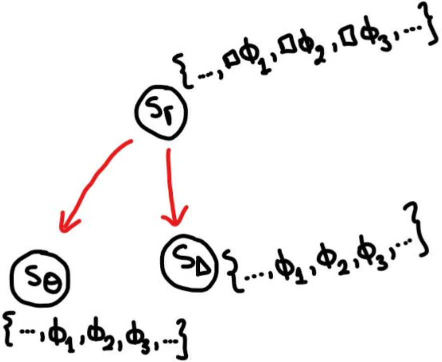
- $W^c = \{s_\Gamma \mid \Gamma \text{ is a maximal consistent set}\}$
- $s_\Gamma R^c s_\Delta \iff$ if $\Box\phi \in \Gamma$, then $\phi \in \Delta$, for every $\phi \in \mathcal{L}_\Box$



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We define $\Gamma_\Box = \{\phi \mid \Box\phi \in \Gamma\}$. The definition of R^c becomes:

- $s_\Gamma R^c s_\Delta \iff \Gamma_\Box \subseteq \Delta$ or
- $R^c = \{(s_\Gamma, s_\Delta) \mid \Gamma_\Box \subseteq \Delta\}$

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- $W^c = \{s_\Gamma \mid \Gamma \text{ is a maximal consistent set}\}$
- $s_\Gamma R^c s_\Delta \iff$ if $\Box\phi \in \Gamma$, then $\phi \in \Delta$, for every $\phi \in \mathcal{L}_\Box$
- $V^c(p) = \{s_\Gamma \mid p \in \Gamma\}$

Proof of completeness

We show by induction on the structure of ϕ that for all Γ, we have that:

$$\mathcal{M}^c, s_\Gamma \vDash \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

Proof of completeness

$$\text{For all } \Gamma,\ \mathcal{M}^c, s_\Gamma \vDash \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

- If ϕ is a propositional variable, then from the definition of V^c, it holds that $s_\Gamma \vDash p$ iff $p \in \Gamma$.

Recall that we defined $V^c(p) = \{s_\Gamma \mid p \in \Gamma\}$.

Proof of completeness

$$\text{For all } \Gamma,\ \mathcal{M}^c, s_\Gamma \models \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

- $\phi = \neg\psi$

Let $s_\Gamma \in W^c$.

Proof of completeness

$$\text{For all } \Gamma,\ \mathcal{M}^c, s_\Gamma \vDash \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

- $\phi = \neg\psi$

Let $s_\Gamma \in W^c$.

By inductive hypothesis, $\mathcal{M}^c, s_\Gamma \vDash \psi$ iff $\psi \in \Gamma$.

Proof of completeness

For all Γ, $\mathcal{M}^c, s_\Gamma \vDash \phi$ iff $\phi \in \Gamma$. $\quad (*)$

- $\phi = \neg\psi$

Let $s_\Gamma \in W^c$.

By inductive hypothesis, $\mathcal{M}^c, s_\Gamma \vDash \psi$ iff $\psi \in \Gamma$.

Equivalently, $\mathcal{M}^c, s_\Gamma \nvDash \psi$ iff $\psi \notin \Gamma$.

Proof of completeness

$$\text{For all } \Gamma,\ \mathcal{M}^c, s_\Gamma \models \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

- $\phi = \neg\psi$

Let $s_\Gamma \in W^c$.

By inductive hypothesis, $\mathcal{M}^c, s_\Gamma \models \psi$ iff $\psi \in \Gamma$.

Equivalently, $\mathcal{M}^c, s_\Gamma \not\models \psi$ iff $\psi \notin \Gamma$.

By the definition of truth, $\mathcal{M}^c, s_\Gamma \models \neg\psi$ iff $\psi \notin \Gamma$.

Proof of completeness

$$\text{For all } \Gamma,\ \mathcal{M}^c, s_\Gamma \models \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

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By the definition of truth, $\mathcal{M}^c, s_\Gamma \models \neg\psi$ iff $\psi \notin \Gamma$.

Since Γ is maximal consistent, $\mathcal{M}^c, s_\Gamma \models \neg\psi$ iff $\neg\psi \in \Gamma$ by the Lemma.

Proof of completeness

For all Γ, $\mathcal{M}^c, s_\Gamma \vDash \phi$ iff $\phi \in \Gamma$. $(*)$

- $\phi = \neg\psi$

Let $s_\Gamma \in W^c$.

By inductive hypothesis, $\mathcal{M}^c, s_\Gamma \vDash \psi$ iff $\psi \in \Gamma$.

Equivalently, $\mathcal{M}^c, s_\Gamma \nvDash \psi$ iff $\psi \notin \Gamma$.

By the definition of truth, $\mathcal{M}^c, s_\Gamma \vDash \neg\psi$ iff $\psi \notin \Gamma$.

Since Γ is maximal consistent, $\mathcal{M}^c, s_\Gamma \vDash \neg\psi$ iff $\neg\psi \in \Gamma$ by the Lemma.

So, $\mathcal{M}^c, s_\Gamma \vDash \phi$ iff $\phi \in \Gamma$.

Proof of completeness

$$\text{For all } \Gamma,\ \mathcal{M}^c, s_\Gamma \vDash \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

- $\phi = \psi_1 \wedge \psi_2$

Let $s_\Gamma \in W^c$.

Proof of completeness

$$\text{For all } \Gamma,\ \mathcal{M}^c, s_\Gamma \models \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

- $\phi = \psi_1 \wedge \psi_2$

Let $s_\Gamma \in W^c$.
By inductive hypothesis, $\mathcal{M}^c, s_\Gamma \models \psi_1$ iff $\psi_1 \in \Gamma$ and $\mathcal{M}^c, s_\Gamma \models \psi_2$ iff $\psi_2 \in \Gamma$.

For all Γ, $\mathcal{M}^c, s_\Gamma \vDash \phi$ iff $\phi \in \Gamma$. $(*)$

- $\phi = \psi_1 \wedge \psi_2$

Let $s_\Gamma \in W^c$.
By inductive hypothesis, $\mathcal{M}^c, s_\Gamma \vDash \psi_1$ iff $\psi_1 \in \Gamma$ and $\mathcal{M}^c, s_\Gamma \vDash \psi_2$ iff $\psi_2 \in \Gamma$.

$\mathcal{M}^c, s_\Gamma \vDash \psi_1 \wedge \psi_2$

$\Leftrightarrow \mathcal{M}^c, s_\Gamma \vDash \psi_1$ and $\mathcal{M}^c, s_\Gamma \vDash \psi_2$ (by the definition of truth)

$\Leftrightarrow \psi_1 \in \Gamma$ and $\psi_2 \in \Gamma$ (by inductive hypothesis)

$\Leftrightarrow \psi_1 \wedge \psi_2 \in \Gamma$ (by Lemma)

So, $\mathcal{M}^c, s_\Gamma \vDash \phi$ iff $\phi \in \Gamma$.

Proof of completeness

$$\text{For all } \Gamma,\ \mathcal{M}^c, s_\Gamma \vDash \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

- $\phi = \Box\psi$

Let $s_\Gamma \in W^c$.

Proof of completeness

$$\text{For all } \Gamma,\ \mathcal{M}^c, s_\Gamma \vDash \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

- $\phi = \Box\psi$

Let $s_\Gamma \in W^c$.
By inductive hypothesis, $\mathcal{M}^c, s_\Gamma \vDash \psi$ iff $\psi \in \Gamma$.

($\Leftarrow$) Let $\phi \in \Gamma$. Since $\Box\psi \in \Gamma$, for every s_Δ such that $s_\Gamma R^c s_\Delta$, we have that $\psi \in \Delta$ (by the definition of R^c).

Proof of completeness

$$\text{For all } \Gamma,\ \mathcal{M}^c, s_\Gamma \vDash \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

- $\phi = \Box\psi$

Let $s_\Gamma \in W^c$.
By inductive hypothesis, $\mathcal{M}^c, s_\Gamma \vDash \psi$ iff $\psi \in \Gamma$.

($\Leftarrow$) Let $\phi \in \Gamma$. Since $\Box\psi \in \Gamma$, for every s_Δ such that $s_\Gamma R^c s_\Delta$, we have that $\psi \in \Delta$ (by the definition of R^c).

So, for every s_Δ such that $s_\Gamma R^c s_\Delta$, it holds that $\mathcal{M}^c, s_\Delta \vDash \psi$ (by inductive hypothesis).

Proof of completeness

$$\text{For all } \Gamma,\ \mathcal{M}^c, s_\Gamma \vDash \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

- $\phi = \Box\psi$

Let $s_\Gamma \in W^c$.
By inductive hypothesis, $\mathcal{M}^c, s_\Gamma \vDash \psi$ iff $\psi \in \Gamma$.

($\Leftarrow$) Let $\phi \in \Gamma$. Since $\Box\psi \in \Gamma$, for every s_Δ such that $s_\Gamma R^c s_\Delta$, we have that $\psi \in \Delta$ (by the definition of R^c).

So, for every s_Δ such that $s_\Gamma R^c s_\Delta$, it holds that $\mathcal{M}^c, s_\Delta \vDash \psi$ (by inductive hypothesis).

This means that $\mathcal{M}^c, s_\Gamma \vDash \Box\psi$ (by the definition of truth).

Proof of completeness

$$\text{For all } \Gamma,\ \mathcal{M}^c, s_\Gamma \vDash \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

- $\phi = \Box\psi$

Let $s_\Gamma \in W^c$.
By inductive hypothesis, $\mathcal{M}^c, s_\Gamma \vDash \psi$ iff $\psi \in \Gamma$.

($\Leftarrow$) Let $\phi \in \Gamma$. Since $\Box\psi \in \Gamma$, for every s_Δ such that $s_\Gamma R^c s_\Delta$, we have that $\psi \in \Delta$ (by the definition of R^c).

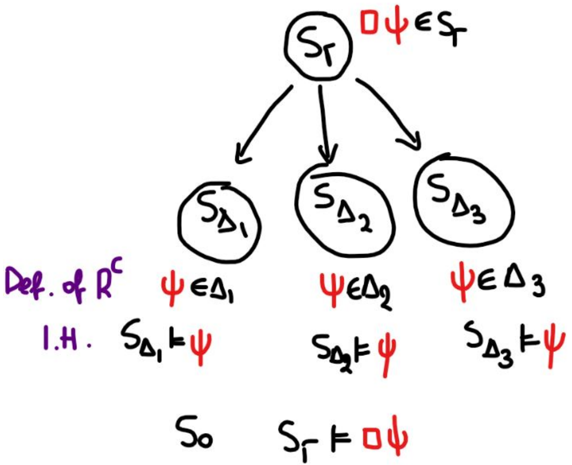
So, for every s_Δ such that $s_\Gamma R^c s_\Delta$, it holds that $\mathcal{M}^c, s_\Delta \vDash \psi$ (by inductive hypothesis).

This means that $\mathcal{M}^c, s_\Gamma \vDash \Box\psi$ (by the definition of truth).

So, $\mathcal{M}^c, s_\Gamma \vDash \phi$.

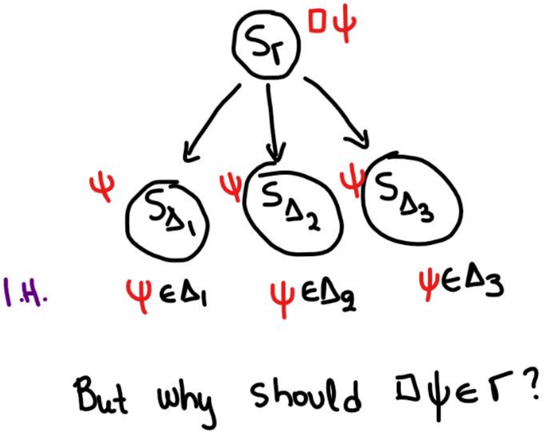
Intuition

It is not hard to prove that if "all maximal consistent sets accessible from Γ contain ψ", then s_Γ satisfies $\Box\psi$.



Intuition for the converse

It is a little harder to prove that if $\Box\psi$ is true on s_Γ, then $\Box\psi$ belongs to the maximal consistent set Γ.



Proof of completeness

$$\text{For all } \Gamma,\ \mathcal{M}^c, s_\Gamma \vDash \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

- $\phi = \Box\psi$

Let $s_\Gamma \in W^c$. We are going to show that $\phi \in \Gamma$.

$$\text{For all } \Gamma,\ \mathcal{M}^c, s_\Gamma \vDash \phi \text{ iff } \phi \in \Gamma. \qquad (*)$$

- $\phi = \Box\psi$

Let $s_\Gamma \in W^c$. We are going to show that $\phi \in \Gamma$.
By inductive hypothesis, $\mathcal{M}^c, s_\Gamma \vDash \psi$ iff $\psi \in \Gamma$.

($\Rightarrow$) Let $\mathcal{M}^c, s_\Gamma \vDash \phi$.

Proof of completeness

($\Rightarrow$) $\mathcal{M}^c, s_\Gamma \models \Box\psi$. We are going to show that $\Box\psi \in \Gamma$.

We are going to prove the following facts:

1. The set $\Gamma_\Box \cup \{\neg\psi\}$ is inconsistent.
2. A finite subset $\{\phi_1, ..., \phi_k, \neg\psi\}$ of $\Gamma_\Box \cup \{\neg\psi\}$ is inconsistent.
3. The set $\{\Box\phi_1, ..., \Box\phi_k, \neg\Box\psi\}$ is inconsistent.
4. $\Box\psi \in \Gamma$.

Recall that $\Gamma_\Box = \{\phi \mid \Box\phi \in \Gamma\}$.

Proof of completeness

Fact 1. The set $\Gamma_{\Box} \cup \{\neg\psi\}$ is inconsistent.

Proof of Fact 1: Suppose that $\Gamma_{\Box} \cup \{\neg\psi\}$ is consistent.

- Then, it can be extended to a maximal consistent set, let's say Θ.
- Since, $\Gamma_{\Box} \subseteq \Theta$, we have that $s_{\Gamma} R^c s_{\Theta}$, by definition of R^c.
- It holds that $\neg\psi \in \Theta$, so by inductive hypothesis $\mathcal{M}^c, s_{\Theta} \models \neg\psi$.
- Therefore, $\mathcal{M}^c, s_{\Gamma} \models \neg \Box \psi$.

Contradiction!

Recall that we know $\mathcal{M}^c, s_{\Gamma} \models \Box\psi$ and we are going to show that $\Box\psi \in \Gamma$.

Proof of completeness

Fact 2. A finite subset $\{\phi_1, ..., \phi_k, \neg\psi\}$ of $\Gamma_{\Box} \cup \{\neg\psi\}$ is inconsistent.

Proof of Fact 2: Since every proof is finite, for any inconsistent set, there is a finite subset of that set which is inconsistent.

Proof of completeness

Fact 3. The set $\{\Box\phi_1, ..., \Box\phi_k, \neg\Box\psi\}$ is inconsistent.

Proof of Fact 3:

1. Since $\{\phi_1, ..., \phi_k\} \cup \{\neg\psi\}$ is inconsistent, it holds that $\{\phi_1, ..., \phi_k\} \vdash \psi$. By the Deduction Theorem, $\vdash (\phi_1 \rightarrow (\phi_2 \rightarrow (....(\phi_k \rightarrow \psi)....))$.

Proof of completeness

Fact 3. The set $\{\Box\phi_1, ..., \Box\phi_k, \neg\Box\psi\}$ is inconsistent.

Proof of Fact 3:

1. Since $\{\phi_1, ..., \phi_k\} \cup \{\neg\psi\}$ is inconsistent, it holds that $\{\phi_1, ..., \phi_k\} \vdash \psi$. By the Deduction Theorem, $\vdash (\phi_1 \to (\phi_2 \to (....(\phi_k \to \psi)....))$.
2. By the Necessitation Rule, we have that $\vdash \Box(\phi_1 \to (\phi_2 \to (....(\phi_k \to \psi)....))$.

Proof of completeness

Fact 3. The set $\{\Box\phi_1, ..., \Box\phi_k, \neg\Box\psi\}$ is inconsistent.

Proof of Fact 3:

1. Since $\{\phi_1, ..., \phi_k\} \cup \{\neg\psi\}$ is inconsistent, it holds that $\{\phi_1, ..., \phi_k\} \vdash \psi$. By the Deduction Theorem, $\vdash (\phi_1 \to (\phi_2 \to (....(\phi_k \to \psi)....))$.
2. By the Necessitation Rule, we have that $\vdash \Box(\phi_1 \to (\phi_2 \to (....(\phi_k \to \psi)....))$.
3. By the axiom (K) and propositional reasoning we have that $\vdash \Box(\phi_1 \to (\phi_2 \to (....(\phi_k \to \psi)....)) \to (\Box\phi_1 \to (\Box\phi_2 \to (....(\Box\phi_k \to \Box\psi)....))$.

Proof of completeness

Fact 3. The set $\{\Box\phi_1, ..., \Box\phi_k, \neg\Box\psi\}$ is inconsistent.

Proof of Fact 3:

1. Since $\{\phi_1, ..., \phi_k\} \cup \{\neg\psi\}$ is inconsistent, it holds that $\{\phi_1, ..., \phi_k\} \vdash \psi$. By the Deduction Theorem, $\vdash (\phi_1 \to (\phi_2 \to (....(\phi_k \to \psi)....))$.
2. By the Necessitation Rule, we have that $\vdash \Box(\phi_1 \to (\phi_2 \to (....(\phi_k \to \psi)....))$.
3. By the axiom (K) and propositional reasoning we have that $\vdash \Box(\phi_1 \to (\phi_2 \to (....(\phi_k \to \psi)....)) \to (\Box\phi_1 \to (\Box\phi_2 \to (....(\Box\phi_k \to \Box\psi)....))$.
4. By 2, 3 and Modus Ponens we have that $\vdash (\Box\phi_1 \to (\Box\phi_2 \to (....(\Box\phi_k \to \Box\psi)....))$.

Proof of completeness

Fact 3. The set $\{\Box\phi_1, ..., \Box\phi_k, \neg\Box\psi\}$ is inconsistent.

Proof of Fact 3:

1. Since $\{\phi_1, ..., \phi_k\} \cup \{\neg\psi\}$ is inconsistent, it holds that $\{\phi_1, ..., \phi_k\} \vdash \psi$. By the Deduction Theorem, $\vdash (\phi_1 \to (\phi_2 \to (....(\phi_k \to \psi)....))$.
2. By the Necessitation Rule, we have that $\vdash \Box(\phi_1 \to (\phi_2 \to (....(\phi_k \to \psi)....))$.
3. By the axiom (K) and propositional reasoning we have that $\vdash \Box(\phi_1 \to (\phi_2 \to (....(\phi_k \to \psi)....)) \to (\Box\phi_1 \to (\Box\phi_2 \to (....(\Box\phi_k \to \Box\psi)....))$.
4. By 2, 3 and Modus Ponens we have that $\vdash (\Box\phi_1 \to (\Box\phi_2 \to (....(\Box\phi_k \to \Box\psi)....))$.
5. So, it holds that $\{\Box\phi_1, ..., \Box\phi_k\} \vdash \Box\psi$ which means that $\{\Box\phi_1, ..., \Box\phi_k, \neg\Box\psi\}$ is inconsistent.

Proof of completeness

Fact 4. $\Box\psi \in \Gamma$.

Proof of Fact 4:

- Since $\phi_1, \ldots, \phi_k \in \Gamma_\Box$, we have that $\Box\phi_1, \ldots, \Box\phi_k \in \Gamma$ (by definition of $\Gamma_\Box$).

Proof of completeness

Fact 4. $\Box\psi \in \Gamma$.

Proof of Fact 4:

- Since $\phi_1, \ldots, \phi_k \in \Gamma_\Box$, we have that $\Box\phi_1, \ldots, \Box\phi_k \in \Gamma$ (by definition of $\Gamma_\Box$).
- Since Γ is consistent, $\neg\Box\psi \notin \Gamma$ (by Fact 3).

Proof of completeness

Fact 4. $\Box\psi \in \Gamma$.

Proof of Fact 4:

- Since $\phi_1, \ldots, \phi_k \in \Gamma_\Box$, we have that $\Box\phi_1, \ldots, \Box\phi_k \in \Gamma$ (by definition of $\Gamma_\Box$).
- Since Γ is consistent, $\neg\Box\psi \notin \Gamma$ (by Fact 3).
- But since Γ is maximal, exactly one of $\Box\psi$ and $\neg\Box\psi$ must be in Γ (by Lemma).
- So, $\Box\psi \in \Gamma$.

K4 is sound and complete with respect to $\mathcal{K}4$ (the class of transitive frames).

Proof: **Soundness**: Easy.

Completeness:

- We define the canonical model $\mathcal{M}^c$ for K4 as before but now W^c is the set of K4-maximal consistent sets of formulas.
- Every K4-consistent formula ϕ is satisfiable on the canonical model $\mathcal{M}^c$ for K4.
- $\mathcal{M}^c$ is a transitive model, i.e. R^c is transitive.

$\mathcal{M}^c$ is a transitive model, i.e. R^c is transitive.

Proof: Let $s_\Gamma, s_\Delta, s_\Theta \in W^c$ such that $s_\Gamma R^c s_\Delta$ and $s_\Delta R^c s_\Theta$. We are going to show that $s_\Gamma R^c s_\Theta$.

We are going to show that $\Box\phi \in \Gamma$ implies $\phi \in \Theta$. Then, by the definition of R^c, we have that $s_\Gamma R^c s_\Theta$.

Let $\Box\phi \in \Gamma$. Also, $\Box\phi \rightarrow \Box\Box\phi \in \Gamma$, since Γ is K4-maximal consistent. So, by Modus Ponens, $\Box\Box\phi \in \Gamma$.

Since $s_\Gamma R^c s_\Delta$, we have that $\Box\phi \in \Delta$. Finally, since $s_\Delta R^c s_\Theta$, we have that $\phi \in \Theta$.