2.3.3 Axiomatization (S5C)

Axioms

all instances of prop. tautologies in L_{KC}

$K_a(\phi \to \psi) \to (K_a\phi \to K_a\psi)$ $\quad a \in A$

$K_a\phi \to \phi$ $\quad \phi, \psi \in L_{KC}$

$K_a\phi \to K_aK_a\phi$ $\quad B \subseteq A$

$\neg K_a\phi \to K_a\neg K_a\phi$

$C_B(\phi \to \psi) \to (C_B\phi \to C_B\psi)$ distr of comm. know.

$C_B\phi \to (\phi \wedge E_BC_B\phi)$ mix

$C_B(\phi \to E_B\phi) \to (\phi \to C_B\phi)$ ind. of comm. know.

Rules

$\phi, \phi \to \psi \vdash \psi$ modus ponens

if $\vdash \phi$ then $\vdash K_a\phi$ nec. of K_a

if $\vdash \phi$ then $\vdash C_B\phi$ nec. of comm. know.

Soundness of induction axiom

$$C_\beta(\phi \to E_\beta \phi) \to (\phi \to C_\beta \phi)$$

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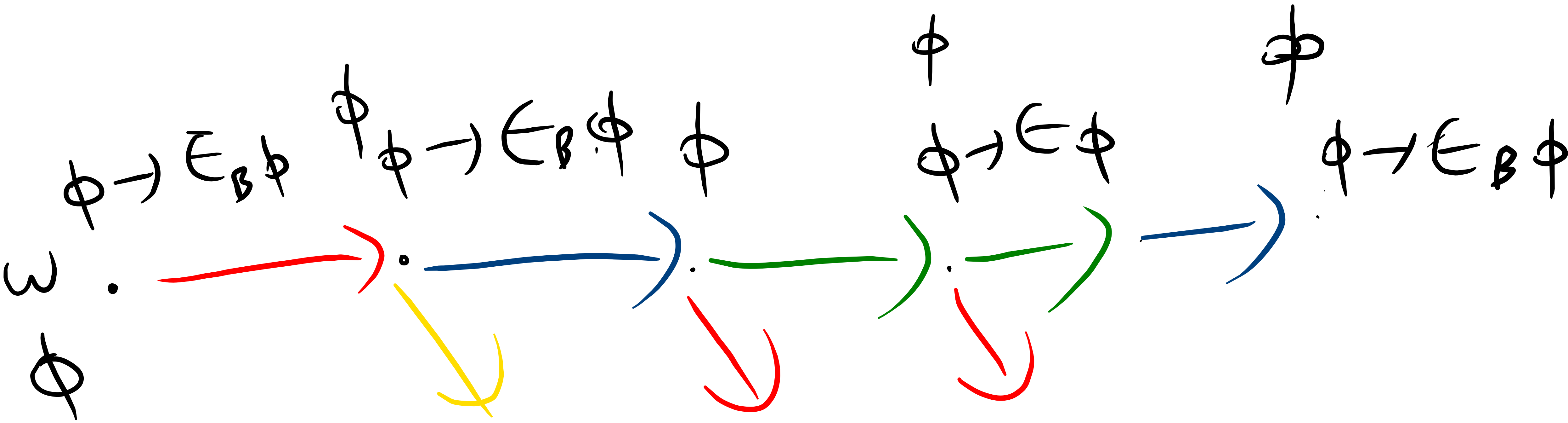
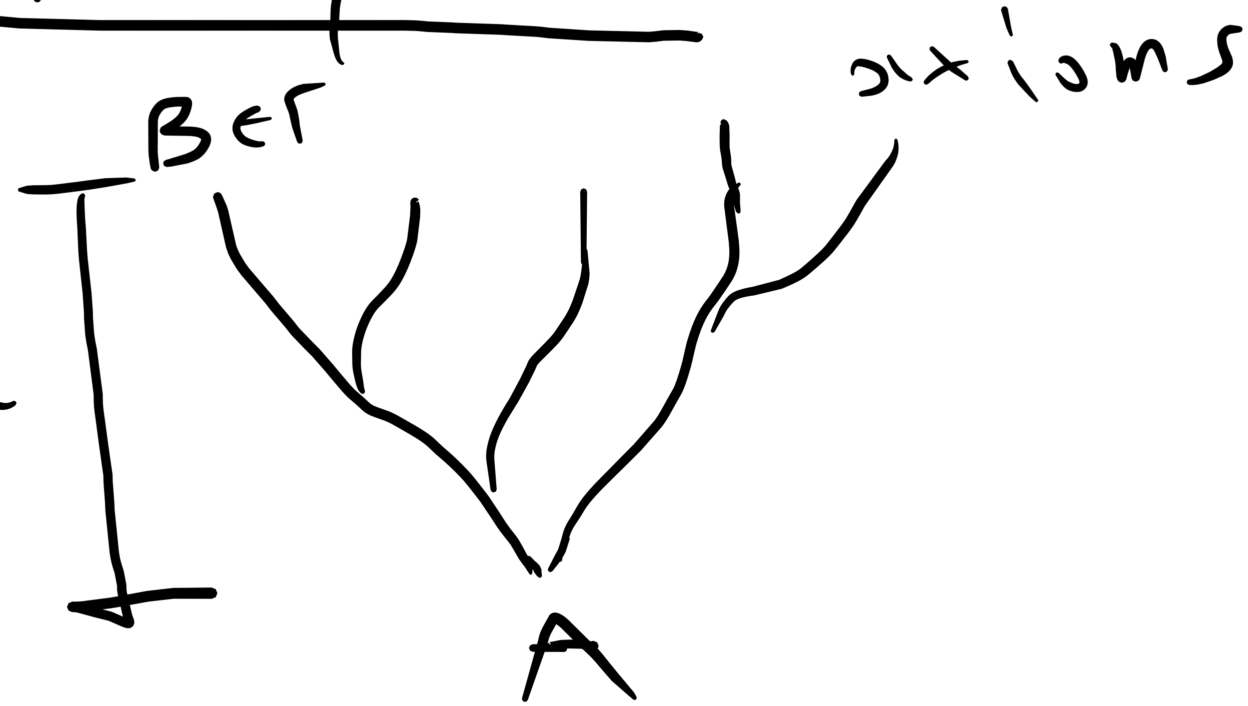


Length of a proof

$\Gamma \vdash^{i} A$



$B \in \Gamma \qquad \Gamma \vdash^{0} B$

C is axiom instance $\qquad \Gamma \vdash^{0} C$

$$\frac{\Gamma \vdash^{i} A \qquad \Gamma \vdash^{j} A \to B}{\Gamma \vdash^{\max\{i,j\}+1} B}$$

$$\frac{\vdash^{i} A}{\vdash^{i+1} K_a A}$$

Deduction Theorem $\Gamma \subseteq L_{KC}$, $A, B \in L_{KC}$

$\Gamma, A \vdash B \iff \Gamma \vdash A \to B$

Proof

$\Leftarrow$ obvious

$\Rightarrow$ Let n be such $\Gamma, A \vdash^n B$. We proceed by generalised induction on n.

- if B is an axiom

$\Gamma \vdash B$

$\Gamma \vdash B \to (A \to B)$ prop. taut

$\Gamma \vdash A \to B$

– If B is a result of modus ponens.
Then there is a C such that

$\Gamma, A \vdash^{j} C \to B \quad \Gamma, A \vdash^{K} C$ for

$j, K < i$. By induction hypothesis we have

$\Gamma \vdash A \to (C \to B) \quad \Gamma \vdash A \to C$

By prop. reasoning we get

$$\Gamma \vdash A \to B$$

- if B is a result of a necessitation rule. Then $B = *D$, where $* \in \{K_a, C_B\}$. Then we have that D is a theorem. So

$$\Gamma \vdash D$$

$$\Gamma \vdash *D$$

$$\Gamma \vdash A \to *D \qquad \text{by prop. reas.}$$

$$\Gamma \vdash A \to B$$

Exercise 2.37

1. $\vdash C_B \phi \leftrightarrow C_B C_B \phi$ (positive introspection of common Knowledge)

$\vdash C_B C_B \phi \to C_B \phi$ mix ✓

$\vdash C_B \phi \to E_B C_B \phi$ mix

$\vdash C_B (C_B \phi \to E_B C_B \phi)$ nec of comm. Know

$\vdash C_B (C_B \phi \to E_B C_B \phi) \to (C_B \phi \to C_B C_B \phi)$

$\vdash C_B \to C_B C_B \phi$

3. $\vdash C_B\phi \leftrightarrow K_a C_B \phi$

$C_B\phi \to E_B C_B \phi$ mix (1)

$E_B C_B \to K_a C_B \phi$ prop. taut. (2)

$C_B\phi \to K_a C_B \phi$ (1), (2) prop reas

$K_a C_B\phi \to C_B \phi$ (T)

6. $\vdash C_B \phi \to E_B C_B \phi$ mix

$\vdash E_B C_B \phi \to E_{B'} C_B \phi$ $B' \subseteq B$

$\vdash C_B \phi \to E_{B'} C_B \phi$ prop. reas.

$\vdash C_{B'}(C_B \phi \to E_{B'} C_B \phi)$ (1), nec (2)

$\vdash C_{B'}(C_B \phi \to E_{B'} C_B \phi) \to (C_B \phi \to C_{B'} C_B \phi)$ (3)

$\vdash C_B \phi \to C_{B'} C_B \phi$ mp (2), (3)

$\vdash C_B \phi \to \phi$ mix

$\vdash C_{B'}(C_B \phi \to \phi)$

$\vdash C_{B'}(C_B \phi \to \phi) \to (C_{B'} C_B \phi \to C_{B'} \phi)$

$\vdash C_{B'} C_B \phi \to C_{B'} \phi$

$\vdash C_B \phi \to C_{B'} \phi$

5. $\vdash C_B\phi \to K_{a_1}K_{a_2}\dots K_{a_n}\phi$

(1) $\vdash C_B\phi \to E_B C_B\phi$ mix

(2) $\vdash E_B C_B\phi \to K_{a_1} C_B\phi$ prop. reas

(3) $\vdash C_B\phi \to K_{a_1} C_B\phi$

$\vdash C_B\phi \to K_{a_2} C_B\phi$

$\vdash K_{a_1}(C_B\phi \to K_{a_2} C_B\phi)$

using distr of K_{a_1}

$\vdash C_B\phi \to K_{a_1}K_{a_2} C_B\phi$

$\vdots$

$\vdash C_B \to \phi$

Show cases 2 and 4 from Ex. 2.37.

- <u>Hints</u> first show that

$K_a(\phi \to \psi) \wedge \hat{K}_a\phi \to \hat{K}_a\psi$ (*)

using the contrapositive of the distribution rule for K_a

Then show

$\hat{K}_a C_B\phi \to C_B\phi$ using (*), axiom mix and axiom $\phi \to K_a\hat{K}_a\phi$ which holds in S5C