Belief Revision and Dynamic Doxastic Logic

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Outline

Possible World Semantics for Belief Revision

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- A proposition is a set of possible worlds.
- There is a one to one correspondence between belief sets and propositions.
- Each belief set can be represented by the proposition that consists of those possible worlds that contain the belief set in question.

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- It is assumed that $\bigcap \varnothing = \mathcal{K}_{\perp}$
- It is convenient to represent sets of possible worlds as geometrical surfaces.

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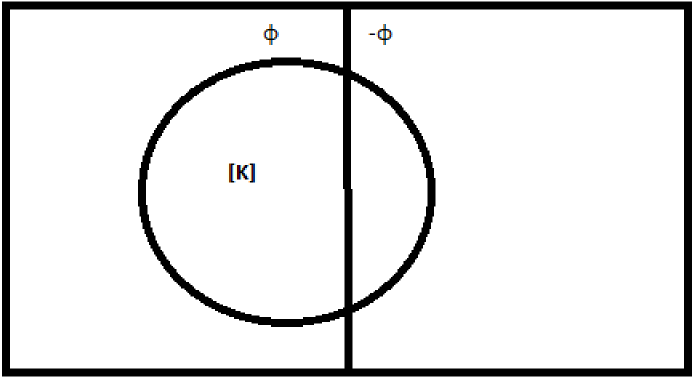
- To expand by ϕ, simply remove all worlds that do not contain ϕ from $\mathcal{K}$.

Expansion



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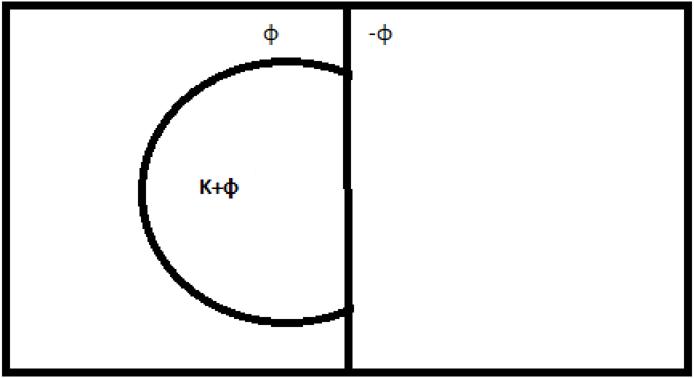

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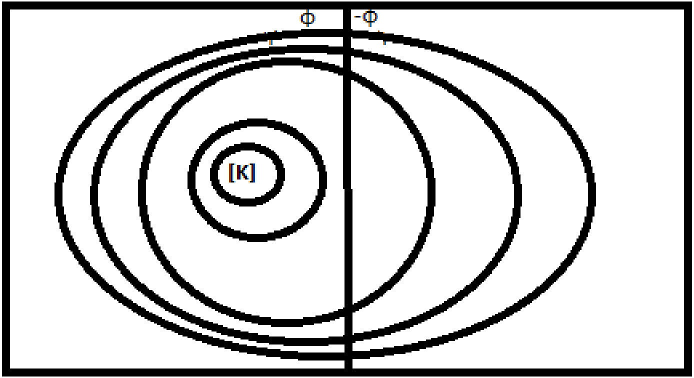
- The equivalent for entrenchment here, is what is called a system of spheres.
- Intuitively, we have spheres surrounding $[\mathcal{K}]$, and when we have to give up some of our beliefs, we fall back to the closest sphere that we can.

Contraction



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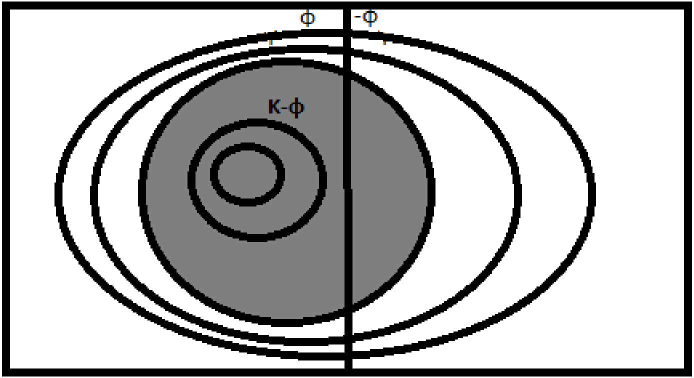
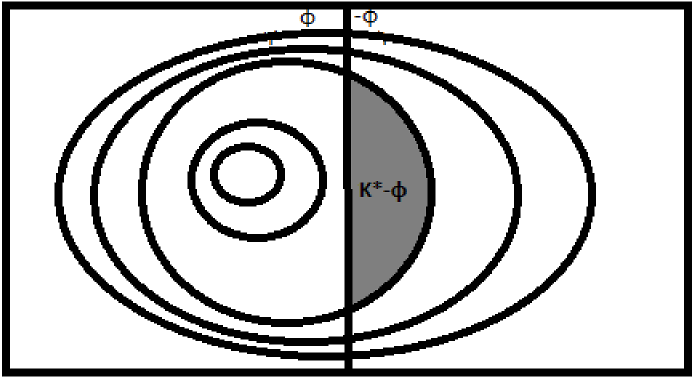


Revision

Revision can be easily defined in the same way.In the following figure we show revision of $\mathcal{K}$ with $\neg\phi$.

Revision



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Theorem 1

Sphere based contraction corresponds exactly to transitively relational partial meet contraction.

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- A straightforward way would be to translate $\phi \in \mathcal{K}$ to $K\phi$ or $B\phi$.
- Gaining new beliefs should correspond to eliminating access to worlds.
- Giving up beliefs should allow the agent to consider worlds possible that were previously inaccessible.

Consecutive Numbers

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- Now suppose that someone publicly announces ϕ.
- This means that A can rule out (0,1) and conclude that $\neg\phi$.
- A after revising with ϕ, will have $\neg\phi \in \mathcal{K}^a \circledast \phi$, therefore violating postulate *success*.

Effect on Third Parties

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- Assume also that it is common knowledge that nether A nor B know whether p
- Suppose that A learns that in fact p.
- A revises with p. But now B also learns something. He learns that A know whether p!

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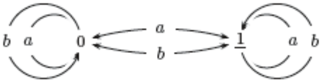

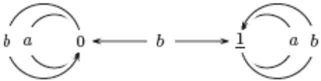

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- We need a language that states explicitly what kind of event made A change her mind, and therefore making precise what b notices about this.

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- We end up with $\mathcal{K}_\perp$ just by revising a consistent belief set with a consistent formula!

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- What would happen if our agent were to learn ϕ?
- By the *success* postulate and positive introspection we have $B\phi \in \mathcal{K} \circledast \phi$, and by *preservation* we have that $\neg B\phi \in \mathcal{K} \circledast \phi$, giving us again the inconsistent belief set!

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- By positive introspection, we have $B\phi \in \mathcal{K}_2$ and by negative introspection we have $\neg B\phi \in \mathcal{K}_1$.
- Since $\mathcal{K}_1 \subset \mathcal{K}_2$, $\neg B\phi \in \mathcal{K}_2$. $\mathcal{K}_2$ is incosistent!
- This even violates axiom D.

Dynamic Doxastic Logic

Segerberg

"AGM is not really logic, it's a theory about theories."

Dynamic Doxastic Logic

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- Bp is a formula in DDL, while $p \in \mathcal{K}$ of AGM is not in the same language as p.
- $B\neg Bp$ can be expressed in DDL, while $((p \notin \mathcal{K}) \in \mathcal{K}$ is not well formed in AGM.
- Operators $[\oplus], [\ominus], [\circledast]$ with the intended reading of for instance $[\oplus\phi]\psi$, after expansion with ϕ, ψ holds.

Dynamic Doxastic Logic

Definition 2

$\phi := p \mid \neg\phi \mid \phi \wedge \phi \mid B_a\phi \mid [\alpha]\phi$
$\alpha := \oplus_a\phi \mid \ominus_a\phi \mid \circledast_a\phi$

- In this language a postulate like success for revision becomes simply $[\circledast_a\phi]\phi$.
- We can even express interesting properties such as: Although i believes that ϕ and ψ are equivalent, he does not believe that j revision with either of them has the same effect.
- $B_i((\phi \leftrightarrow \psi) \wedge [\circledast_j\phi]B_j\chi \wedge [\circledast_j\psi]\neg B_j\chi)$

Dynamic Doxastic Logic

- If we want to stay close to the AGM postulates we will have to restrict our language.

Definition 3

$\phi_0 := \phi \mid \neg\phi_0 \mid \phi_0 \wedge \phi_0$
$\phi := \phi_0 \mid \neg\phi \mid \phi \wedge \phi \mid B\phi_0 \mid [\oplus\phi_0]\phi \mid [\ominus\phi_0]\phi$

- We assume that all the modal operators are normal modal operators(they satisfy Necessitation and Modus Ponens), and we also assume that $[\oplus], [\ominus]$ have the properties of objective persistence, functionality and idempotence.

Dynamic Doxastic Logic

$\psi \leftrightarrow [\odot\phi]\psi$	*objective persistence*
$< \odot\phi > \psi \rightarrow [\odot\phi]\psi$	*partial functionality*
$[\odot\phi] \rightarrow [\odot\phi][\odot\phi]\psi$	*idempotence*

Doxastic Dynamic Logic

Some of the AGM postulates in this language are give below:

1. $\vdash \phi \leftrightarrow \psi \implies \vdash [\odot\phi]\chi \leftrightarrow [\odot\psi]\chi$ — *congruence*
2. $B\phi \rightarrow (\chi \leftrightarrow [\oplus\phi]\chi)$ — *modal*$(\mathcal{K} \oplus 4)$
3. $[\oplus\phi]B\psi \leftrightarrow B(\phi \rightarrow \psi)$ — *Ramsey Expansion*
4. $[\ominus\phi]B\chi \rightarrow B\chi$ — *modal*$(\mathcal{K} \ominus 2)$
5. $B\phi \rightarrow (B\chi \rightarrow [\ominus\phi][\oplus\phi]\chi)$ — *modal recovery*

Dynamic Doxastic Logic

Leitgeb and Segerberg(2007)

"We predict that the two research programmes of DDL and DEL will merge in the long run into the single logical endeavor of DBC:Dynamic Logics of Belief Change."

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