Chapters 4.8-4.12

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1 Axiomatisation

1.1 PA Axiomatisation

Definition 1.1 (4.44)**.** Given a set of agents A and a set of atoms P the following table presents the axiomatisation for **PA** , over the language $L_{K[]}$*(PA)*

all instances of propositional tautologies	
$K_a\ (\phi\to\psi)\to(K_a\phi\to K_a\ \psi)$	Distribution of knowledge
$K_a\phi\to\phi$	Truth
$K_a\phi\to K_aK_a\phi$	Positive introspection
$\neg K_a\phi\to K_a\neg K_a\phi$	Negative introspection
$[\phi]$ p $\leftrightarrow(\phi\to$ p $)$	Atomic permanence
$[\phi]\neg\psi\leftrightarrow(\phi\to\neg[\phi]\psi)$	Anouncement and negation
$[\phi](\psi\wedge\chi)\leftrightarrow[\phi]\psi\wedge[\phi]\chi$	Announcement and conjuction
$[\phi]K_a\psi\leftrightarrow(\phi\to K_a[\phi]\psi)$	Announcement and knowledge
$[\phi][\psi]\chi\leftrightarrow[\phi\wedge[\phi]\psi]\chi$	Announcement composition
From ϕ and $\phi\to\psi$, infer ψ	modus ponens
From ϕ infer $K_a\phi$	necessitation of K_a
From ϕ infer $[\psi]\phi$	necessitation of public announcement

Example 1 (4.45)**.** *We will prove that in* ***PA*** $\vdash[p]K_ap$ *.*

1 $p\to p$	*tautology*
2 $[p]p\leftrightarrow(p\to p)$	*atomic permanence*
3 $[p]p$	*1,2 PR*
4 $K_a[p]p$	*3, necessitation*
5 $p\to K_a[p]p$	*4, propositional*
6 $[\ p\ ]K_a\ p\leftrightarrow(\ p\to K_a[\ p\ ]\ p\ )$	*announcement and knowledge*
7 $[p]K_ap$	*5,6 PR*

Proposition 1 (4.46)**.** *Some properties of* ***PA*** *are:*
1. Substitution of equals
If $\vdash\psi\leftrightarrow\chi$*, then* $\vdash\phi(p/\psi)\leftrightarrow\phi(p/\chi)$
2. Partial functionality
$\vdash(\phi\to[\phi]\psi)\leftrightarrow[\phi]\psi$
3. Public announcement and implication
$\vdash[\phi](\psi\to\chi)\leftrightarrow([\phi]\psi\to[\phi]\chi)$

Example 2 (4.47)**.** *We will prove that the schema* $<\phi>\psi\rightarrow[\phi]\psi$ *is derivable in* ***PA*** *.*

1 $[\phi]\neg\psi\leftrightarrow(\phi\rightarrow\neg[\phi]\psi)$	*Announcement and negation*
2 $\neg[\phi]\neg\psi\leftrightarrow\neg(\phi\rightarrow\neg[\phi]\psi)$	*1, PR*
3 $<\phi>\psi\leftrightarrow\phi\wedge[\phi]\psi$	*2, <.> introduction and PR*
4 $<\phi>\psi\rightarrow[\phi]\psi$	*3, PR*

Theorem 1.1 (4.51)**.** *The axiomatisation* ***PA*** *(A,P) is sound and complete .*
Completeness will be proved in chapter 7. For soundness, it remains to show that the derivation rule $'$*necessitation of announcement*$'$ *, "from* ϕ *follows* $[\psi]\phi$*" is sound [4.52].*

Proof. Assume ϕ holds in every frame F. Let M,s be arbitary model and world. We need to show that for any ψ it holds that M,s$\models[\psi]\phi$. This is by definition equivalent to M,s$\models\psi$ implies M$|\psi$,s$\models\phi$ which is True because ϕ holds in all models and states, thus it holds in all $\psi-$states.

□

1.2 PAC Axiomatisation

Definition 1.2 (4.53)**.** The axiomatisation **PAC** is defined in the following table.

all instances of propositional tautologies	
$K_a\ (\phi\rightarrow\ \psi)\rightarrow(\ K_a\phi\rightarrow K_a\ \psi)$	Distribution of knowledge
$K_a\phi\rightarrow\phi$	Truth
$K_a\phi\rightarrow K_aK_a\phi$	Positive introspection
$\neg K_a\phi\rightarrow K_a\neg\ K_a\phi$	Negative introspection
$[\phi]$ p $\leftrightarrow(\phi\rightarrow$ p)	Atomic permanence
$[\phi]\neg\psi\leftrightarrow(\phi\rightarrow\neg[\phi]\psi)$	Anouncement and negation
$[\phi](\psi\wedge\chi)\leftrightarrow[\phi]\psi\wedge\ [\phi]\chi$	Announcement and conjuction
$[\phi]K_a\psi\leftrightarrow(\phi\rightarrow K_a[\phi]\psi)$	Announcement and knowledge
$[\phi][\psi]\chi\leftrightarrow[\phi\wedge[\phi]\psi]\chi$	Announcement composition
$C_B(\phi\rightarrow\psi)\rightarrow(C_B\phi\rightarrow C_B\psi)$	Distribution of C_B over $\rightarrow$
$C_B\phi\rightarrow(\phi\wedge E_BC_B\phi)$	mix of common knowledge
$C_B(\phi\rightarrow E_B\phi)\rightarrow(\phi\rightarrow C_B\phi)$	induction of common knowledge
From ϕ and $\phi\rightarrow\psi$, infer ψ	modus ponens
From ϕ infer $K_a\phi$	necessitation of K_a
From ϕ infer $C_B\phi$	necessitation of common knowledge
From ϕ infer $[\psi]\phi$	necessitation of public announcement
From $\chi\rightarrow[\phi]\psi$ and $\chi\wedge\phi\rightarrow E_B\chi$, infer $\chi\rightarrow[\phi]C_B\psi$	announcement and common knowledge

Note 1.1. *Induction of common knowledge* is derivable in **PAC** *minus* that axiom. We mention it because we like to see **PAC** as an *extension* of **S5C**.

Example 3 (4.54)**.** *We will prove that after the announcement that some atomic proposition is false, then this is commonly known. Formally,* $\vdash [\neg p]C_A\neg p$.

. $\neg p \to\neg(\neg p\to p)$	*tautology*
2. $[\neg p]\leftrightarrow(\neg p\to p)$	*atomic permanence*
3. $\neg p\to\neg[\neg p]p$	*1,2 PR*
4. $[\neg p]\neg p\leftrightarrow(\neg p\to\neg[\neg p]p)$	*anouncement and negation*
5. $[\neg p]\neg p$	*3,4,PR*
6. $T\to[\neg p]\neg p$	*5,weakening*
7. T	*tautology*
8. $K_a T$	*7,necessitation*
9. $T\wedge\neg p\to K_a T$	*8,weakening*
10. $T\wedge\neg p\to E_A T$	*9 for all a in A*
11. $T\to[\neg p]C_A\neg p$	*10,6 anouncement and common knowledge*
12. $[\neg p]C_A\neg p$	*11,PR*

Example 4 (4.58)**.** *Show that* $\vdash [\phi]\psi$ *iff* $\vdash [\phi]C_B\psi$.

Proof. 1. $[\phi]\psi$	assumption
2. T	tautology
3. $\mathrm{T}\to[\phi]\psi$	1,2 weakening
4. $K_a\mathrm{T}$	2, necessitation
5. $E_B\mathrm{T}$	4, for all a in B
6. $\mathrm{T}\wedge\phi\to E_B\mathrm{T}$	5,weakening
7. $\mathrm{T}\to[\phi]C_B\psi$	3,6 anouncement and common knowledge
8. $[\phi]C_B\psi$	2,7 PR

1. $[\phi]C_B\phi$	assumption
2. $C_B\psi\to\psi$	mix, PR
3. $[\phi]C_B\psi\to[\phi]\psi$	2,announcement and implication
4. $[\phi]\psi$	1,3, modus ponens

□

Theorem 1.2 (4.59)**.** *The axiomatisation* ***PAC*** *is sound and complete .*

2 Logic Puzzles with anouncements

2.1 Muddy children (4.10)

The Puzzle: A group of children has been playing outside and are called back into the house by their father. The children gather round him. As one may imagine, some of them have become dirty from the play and in particular: they may have mud on their forehead. Children can only see whether other children are muddy, and not if there is any mud on their own forehead. All this is commonly known, and the children are, obviously, perfect logicians. Father now says: “At least one of you has mud on his or her forehead.” And then: “Will those who know whether they are muddy please step forward. ” If nobody steps forward, father keeps repeating the request. Prove that, if m of n children are muddy, the muddy children will step forward after father has made his request m times.

Note 2.1 (*A classic example of unsuccessful update*). From the ”announcement” that nobody knows wherether he or she is muddy, they may learn that they are muddy. The announcement here corresponds to the ”public truthful *event*” of nobody stepping forward. That merely reveals the true nature of the dynamic objects we are considering: information changing actions.
Father’s request should be seen as the signal synchronising the information change.

Let’s get dirty. We will first take a look at a special case and try to reason informally about the children’s knowledge.

Suppose we have three children Anne (a) , Bill (b) and Cath(c), and that Anne and Bill are dirty, while Cath is clean. The puzzle suggests that after two requests from the Father, Anne and Bill know that they are muddy and will step forward.

But why is that?

We adopt Bill's perspective. Bill sees that Anne is muddy and Cath is not. He hears the anouncement of the Father that at least one child is muddy. Now if he was not muddy, Anne would see 2 non-muddy children and therefore know that she was muddy. Father now asks the children to step forward and none steps forward. Because of that Bill concludes that Anne didn't know that she is muddy, therefore, the tentative hypothesis that Bill had is incorrect, leaving him with one option, that he is indeed muddy. The same goes for Anne. Now the second time Father repeats his request, both Anne and Bill step forward.

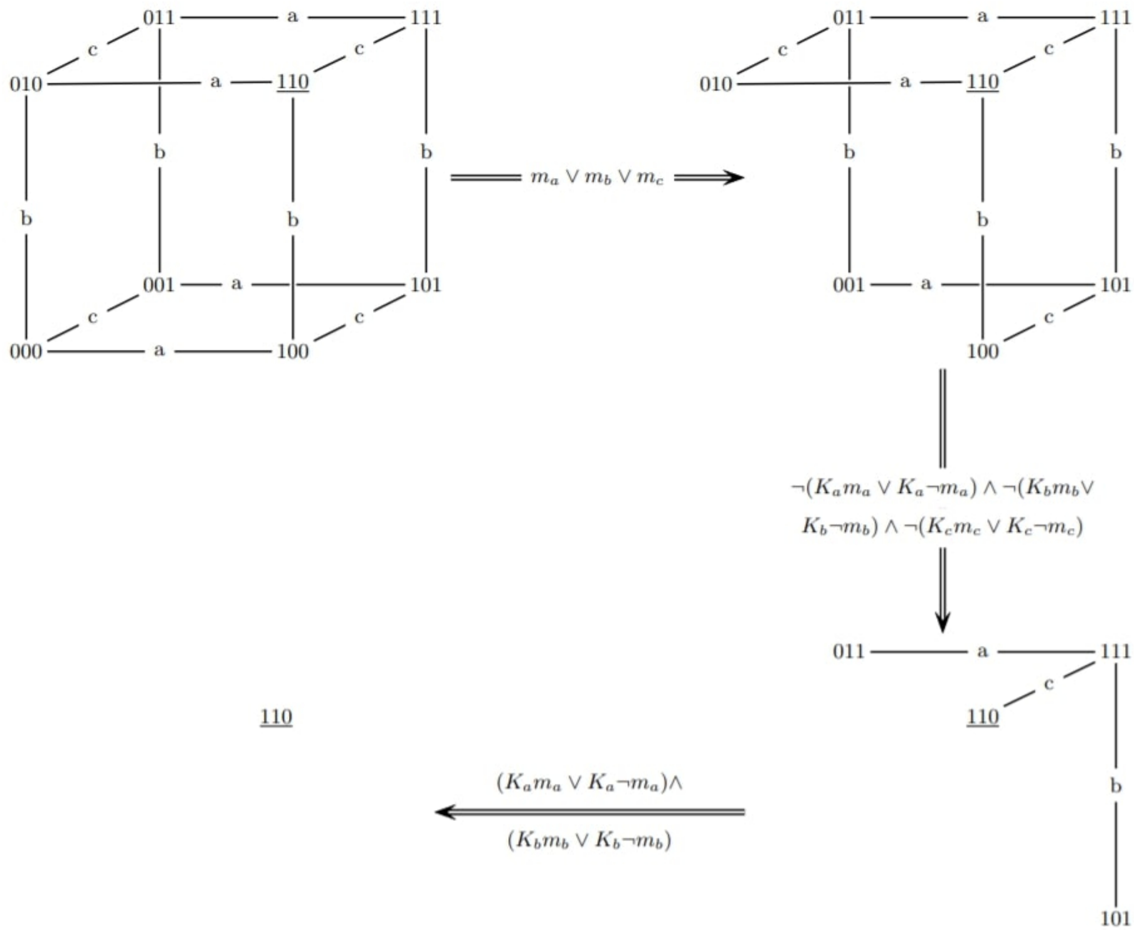
Possible world semantics formalisation. We can represent the initial situation of the three muddy children

with a cube. Each of the three children can be muddy or not. P=$\{m_a, m_b, m_c\}$, where m_a stands for "Anne is muddy". Label the vertices of the cube xyz, with x,y,z$\in\{0,1\}$,where x=0 means Anne is not muddy etc. Thus vertex 110 represents the state were Anne and Bill are muddy and Cath is not.

Assume 110 is the actual state. In this state although it is true that everybody knows there is at least one muddy child, this is *not common knowledge* (for instance we have 110$\sim_a$010$\sim_b$000).

Let *muddy* be the formula $m_a \vee m_b \vee m_c$ and *knowmuddy* be the formula $(K_a m_a \vee K_a \neg m_a) \vee (K_b m_b \vee K_b \neg m_b) \vee (K_c m_c \vee K_c \neg m_c)$
When Father announces muddy, the state 000 ends up being deleted from the model because Cube,000$\models\neg$muddy.
As a result, in all states after the announcement, muddy holds. Formally Cube|muddy,110$\models C_{abc}$muddy.
Therefore muddy is a succesful update in (Cube,110).

The epistemic model we have acquired after the announcement muddy has the following special feature: Every state where only one child is muddy is indistinquisable for two children, meaning that if it was actually the case that only one child is muddy, then that child would know it after the update (for instance Cube|muddy,010$\models K_b m_b$).
Now, after Father's request, no children step forward. This corresponds to the anouncement $\neg$knowmuddy.
In the model Anne knows that if she is not muddy then Bill knows that he is muddy , so after the anouncement that Bill doesn't know whether he is muddy she stops considering 010 a possible world, leaving her with only state 110 where she knows that she is muddy (similarly for Bill). Therefore after announcement of $\neg$ knowmuddy it becomes false, making it an unsuccessful update.
Cube|muddy,110$\models$<$\neg$knowmuddy>knowmuddy


011
a
111
010
c
110
b
001
101
000
100
$m_a \vee m_b \vee m_c$
$\neg(K_a m_a \vee K_a \neg m_a) \wedge \neg(K_b m_b \vee$
$K_b \neg m_b) \wedge \neg(K_c m_c \vee K_c \neg m_c)$
$(K_a m_a \vee K_a \neg m_a) \wedge$
$(K_b m_b \vee K_b \neg m_b)$

Example 5 (*A semantical proof for the general case*)**.** *We want to prove that for n children were k of them are muddy, the k−th time Father repeats his request , the muddy children step forward.*
For k≥1 that is equivalent to the statement that after k−1 truthful announcements of ¬knowmuddy , all muddy children know that they are muddy.
We are going to prove by induction on k the following statement:
For k≥2 , a state s with exactly k muddy children and a muddy child b,
$M_{mc}|muddy|\neg knowmuddy^{k-2},s \models <\neg knowmuddy> K_b m_b$

Case k=2. Let s be a 2−state and b a muddy child. To prove $M_{mc}|muddy,s \models <\neg knowmmuddy> K_b m_b$ *we need to prove that* $M_{mc}|muddy,s \models \neg knowmmuddy$ *and that* $M_{mc}|muddy|\neg knowmuddy,s \models K_b m_b$

Note 2.2. Every child confuses the actual world with exactly one other.

For the first, any child a confuses the 2−state s, either with a 1−state (if it is muddy) or with a 3−state (if it is not muddy). In both of these worlds muddy holds, so they remain after the update with muddy. So after the update with muddy we have ¬knowmuddy.
For the second, let b be a muddy child and let $s \sim_b s_b$*. Then* s_b *is a 1−state and let a be the other muddy child. Then* $M_{mc}|muddy,s_b \models K_a m_a$ *and from this* $M_{mc}|muddy,s_b \not\models \neg knowmuddy$*. Therefore after update with ¬knowmuddy only s state remains and b knows that he is muddy.*
Inductive hypothesis. Assume the statement holds for j≤k.
Inductive step. Let s be a k+1 − state, b a muddy child. We want to prove that
$M_{mc}|muddy|\neg knowmuddy^{k-1},s \models <\neg knowmuddy> K_b m_b$ *so that* $M_{mc}|muddy|\neg knowmuddy^{k-1},s \models \neg knowmuddy$ *and that*
$M_{mc}|muddy|\neg knowmuddy^{k},s \models K_b m_b$.
Assume child a. Then a confuses s either with a k−state or with a k+2 − state. If the latter is the case then the world remains in the model. If the first is the case , let this world be s_a*. Then* s_a *is a k−state so the induction hypothesis holds and it remains also in the model. So* $M_{mc}|muddy|\neg knowmuddy^{k-1},s \models \neg knowmuddy$ *holds.*
Assume now a muddy child b. Then it confuses $s \sim_b s_b$ *with the second being a k−state. Then by the inductive hypothesis if a is another muddy child then* $M_{mc}|muddy|\neg knowmuddy^{k-2},s_b \models <\neg knowmuddy> K_a m_a$ *and* $M_{mc}|muddy|\neg knowmuddy^{k-1},s_b \not\models \neg knowmuddy$*. And finally after the update for the k−th time with ¬knowmuddy b knows that he is muddy.*

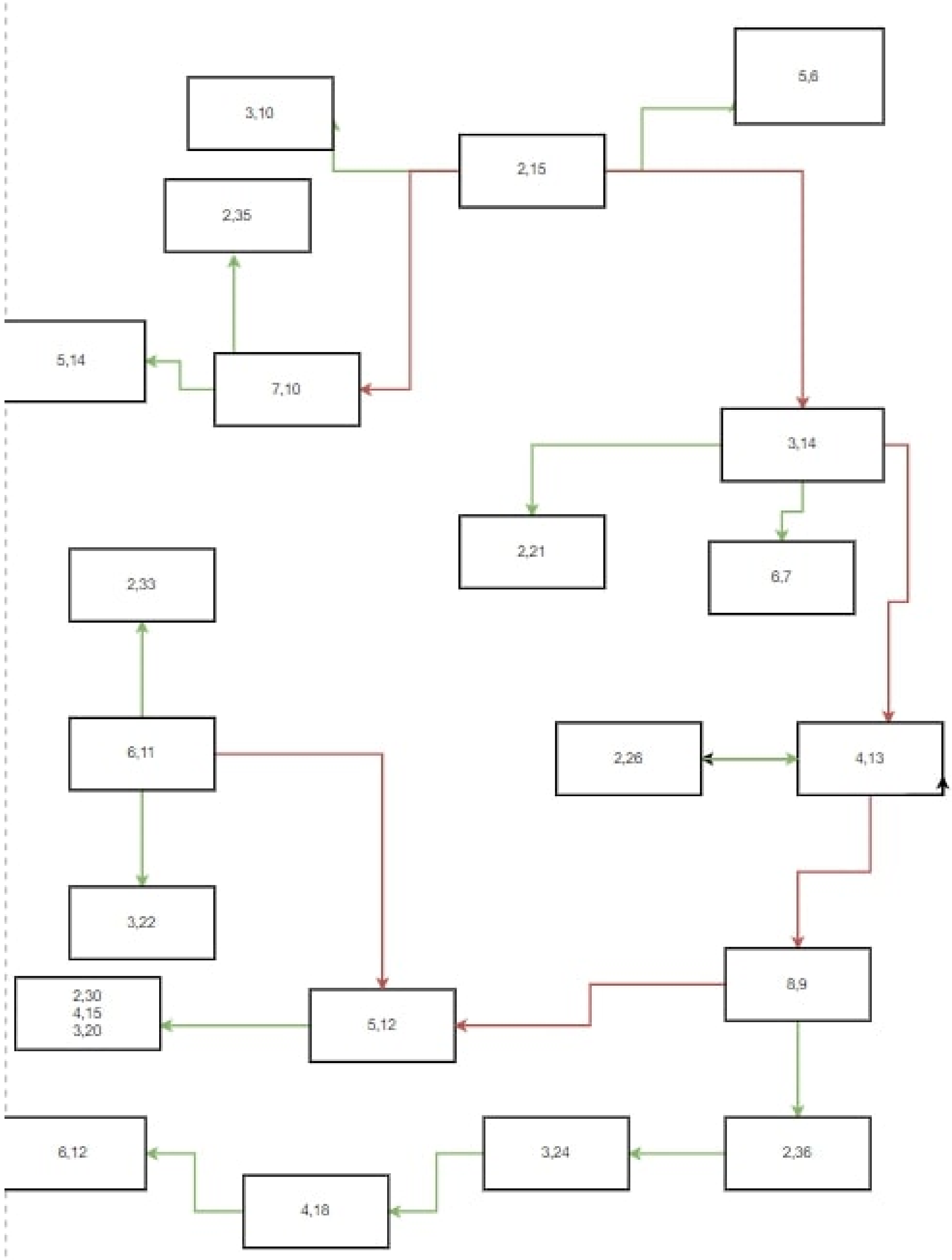
2.2 Sum and Product (4.11)

A says to Mr.P and Mr.S : "I have chosen two natural numbers x and y such that 1<x<y and x+y<100. Now I am going to give to Mr.S their sum s=x+y only and to Mr.P their product p=x∗y only. The content of these anouncements remains a secret." He acts accordingly. The following conversation between Mr.S and Mr.P then takes place:

i)Mr.P: *I don't know the numbers*
ii)Mr.S: *I knew that*
iii)Mr.P: *Now I know the numbers*
iv)Mr.S: *Now I know them too*

Determine x and y.

Arithmetic point of view: The given bounds of the numbers suggest that we seek a pair of natural numbers (x,y) s.t. $2 \leq x < y \leq 98$.
What does the first anouncement suggest? In which initial states does P know the numbers? Well, P confuses pairs that have the same product, so for him to know initial what the numbers are p must uniquely expressible as a product of such x and y. That means P would know the numbers iff (x,y)=(p,q) or (p.p^2) for p,q primes. So these pairs are excluded.
What does the second announcement suggest? S confuses pairs with the same sum, therefore for him to know that P doesn't know the numbers means that for every pair that S thinks is possible, it's product can be expressed in two valid ways. That means, S sees a number that can't be expressed as sum of two primes or a prime and it's square. For example, the numbers can't be (3,6) because S would see 9 and he would consider it possible that the pair is (2,7) is which case P would know the numbers. A state in which it holds is (2,15).
After the second announcement P claims that he knows the numbers. This means that all pairs except one that he thought possible, their sum is expressible as a sum of two primes or a prime and it's square. This doesn't hold for example for (2,15) as because $30 = 5 * 6$ and 11 can't be written that way. It holds for (2,9) because the only other possibility for P is (3,6) and in this state it doesn't hold that S knows that P doesn't knows because S confuses it with (2,7) where P knows.
For the final announcement, that S knows the numbers too, this suggests that for the sum there is a unique pair after the first three announcements P knows the numbers. For example (2,9) because althought after the 2 announcements P knows the numbers, this is the same for the state (3,8).
The pair that satisfies these conditions can be seen to be (4,13).


3,10
2,15
5,6
2,35
5,14
7,10
3,14
2,21
6,7
2,33
6,11
2,26
4,13
3,22
2,30
4,15
3,20
5,12
8,9
6,12
3,24
2,36
4,18

PAC semantics formalisation

First we need to determine the set of atomic propositions. Define $I=\{(x,y)|1<x<y \text{ and } x+y\leq 100\}$. For each $(i,j)\in I$ let x_i and y_j stand for the propositions x=i and y=j . For the agents we have {S,P}.

The proposition "S knows that the numbers are 4 and 13" is represented with $K_S(x_4 \wedge y_{13})$. The proposition " Sum knows the numbers" is described by $K_S(\mathrm{x,y})=\vee_{i,jinI} K_S(x_i,y_j)$. Similarly we define $K_P(\mathrm{x,y})=\vee_{i,jinI} K_P(x_i,y_j)$.

Now the announcements correspond to the following epistemic formulas : i) $\neg K_P(\mathrm{x,y})$

ii) $K_S \neg K_P(\mathrm{x,y})$

iii) $K_P(\mathrm{x,y})$

iv) $K_S(\mathrm{x,y})$

Note 2.3. Because of the truth axiom the second announcement subsumes the first.

The fact that the pair (4,13) is a solution to the problem is described by the truthfulness of

$SP_{x,y},(4,13) \models <K_S \neg K_P(\mathrm{x,y})><K_P(\mathrm{x,y})><K_S(\mathrm{x,y})>\mathrm{T}$

which says that anouncements ii,iii,iv can be done truthfully in that order.

To express that (4,13) is the only solution we require the model validity

$SP_{x,y} \models [K_S \neg K_P(\mathrm{x,y})][K_P(\mathrm{x,y})][K_S(\mathrm{x,y})](x_4 \wedge y_{13})$

For the respective possible world semantics graph induced by the problem, the fact that state (4,13) is the only one with this property can be justified in this way:

Node (4,13) is the only node of the graph for which the following hold

i) If we follow an s−edge then we can follow it with a p−edge to a different node.

ii) If we follow a p−edge to a different node , then we can follow an s−edge to a different node from where there isn′t a p−edge to a different node.

iii) If we follow an s−edge to a different node then ii doesn′t hold there.

The announcement ii can easily be seen to be an unsuccessful update: after ii P knows the numbers therefore it can no longer be true that S knows that P doesn't know the numbers. That is, ii is false after it's announcement, ergo it is an unsuccessful update.

Seeing the four announcements as one via the composition of announcements we see that it is an unsuccessful update in the initial state, because it is equivalent to ii∧[ii]iii∧[iii]iv and after it's announcement S knows that P knows the numbers so ii is false, hence the conjuction is false

2.3 Russian Cards(4.12)

From a pack of seven known cards 0,1,2,3,4,5,6 Anne and Bill each draw three cards and Cath gets the remaining card. How can Anne and Bill openly inform each other about their cards, without Cath learning for any of their cards who holds it?
Without loss of generality we are going to assume that Anne holds {0,1,2} , Bill {3,4,5} and Cath {6}.

Note 2.4 (Formal structure of card deal games). . We have encountered again a game where some agents draw from given stack of cards (In Example 4.2 three players each draw one card), so now it's a good time to formalise them.
Given a stack of known cards and some players, each player blindly draw some cards from the stack. In a state where cards are dealt that way , but no game actions of whatever kind have been taken it is commonly known what the cards are, that they are all different, how many cards each player holds, and that players only know their own cards. From the last it follows that two card deals are the same for an agent , if he holds the same cards and if all the players hold the same number of cards in both deals. This induces an equivalence relation on deals.
The general perspective is that we see a card deal d as a function that assigns card Q to players A. The size #d of a card deal lists for each player how many cards they hold. Two card deals d,e are indistinquisable for a player a if #d=#e and d^{-1}(a)=e^{-1}(a). For the deal in example we write 012.345.6 . We represent facts like $'$a holds card q$'$ as q_a . For a given deal d, it's epistemic model consists of all deals that assign the same number of cards to each players, that is a deals with size #d.
We name our epistemic model *Russian* and we see that it consists of 140 states. The description δ^d of a card deal d sums up the valuation, e.g.,
$\delta^{012.345.6}$=$0_a \wedge 1_a \wedge .... \wedge \neg 0_b \wedge ... \wedge 6_c$
For the restriction to the hand of one player we write δ^d_a. More informaly for our case write 012_a
After a sequence of announcements that is a solution to the Russian Cards problem, it should hold that Anne knows Bill$'$s cards, that Bill knows Anne$'$s cards and Cath does not know any of their cards. Formally we have the following definition:

Definition 2.1 (4.68). Given a card deal d, in the epistemic state (D,d) where the problem is solved it must hold that :

$$\text{aknowsbs}=\wedge_{\text{e}\in\ \text{D(D)}}(\delta^e_b \rightarrow K_a \delta^e_b)$$
$$\text{bknowsas}=\wedge_{\text{e}\in\ \text{D(D)}}(\delta^e_a \rightarrow K_a \delta^e_a)$$
$$\text{cignorant}=\wedge_{\text{q}\in\ \text{Q}} \wedge_{\text{n=a,b}} \neg\ K_c q_n$$

We have to remark here that these conditions must hold on the model and are therefore independent of the actual deal. They are part of the context of the agents. Therefore they are not just true but commonly known, Caknowsbs, Cbknowsas,, Ccignorant. We will see later, that to achieve a solution to the problem we must have a sequence of anouncements that preserve Ccignorant.

Let's explore some non-examples first

Example 6. *Anne says "I have {0,1,2} or Bill has {0,1,2}" and Bill says "I have {3,4,5} or Anne has {3,4,5}".*

An update in Russian,012.345.6 with the formula $012_a \vee 012_b$ *would result in an information state with eight card deals, where cignorant holds but not common knowledge of it. A subsequent update with* $345_a \vee 345_b$ *results in an epistemic state with only two deal 012.345.6 and 345.012.6 where that are the same for Cath but different for Anne and Bill. Thus aknowsbs, bknowsas, cignorant are all common knowledge. What's the catch? .*

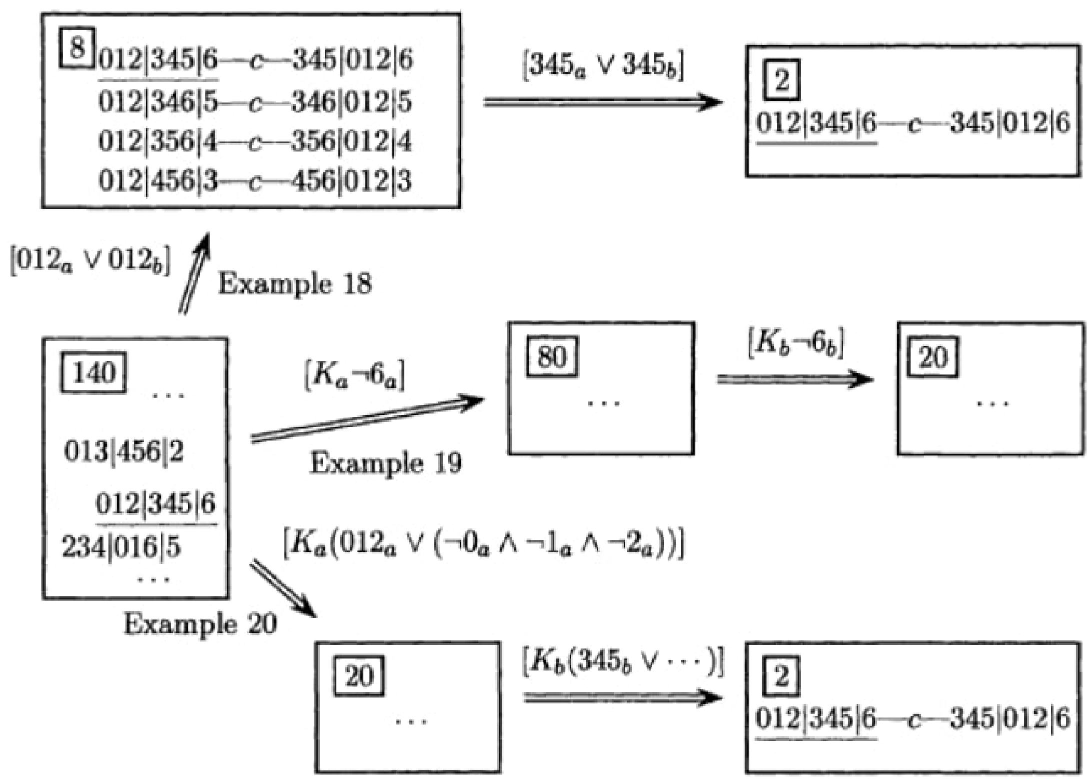
Our reasoning about the announcement is wrong. We treated it as an announcement by an insider, someone whose accesibility relation in the model is the identity, where knowledge and truth are equivalent. But this is not the case for Anne. Her announcements are based on her knowledge, and because she has less knowledge her announcements are more informative. Because a priori it is commonly known that Anne doesn't know Bill cards, in order to truthfully announce that she knows that either she or bill has 0,1,2 , she must have 0,1,2. To see this consider the following:

Suppose Anne had 3,4,6 then even if Bill had 012, making $012_a \vee 012$ *true, Anne couldn't have announced it because she didn't know it to be true.*

In other words, an update with $K_a(012_a \vee 012_b$ *already restricts the model to the four epistemic states where Anne holds 0,1,2 and Cath knows the card deal. With the subsequent update leading to the sigleton where the card deal is commonly known! Formally*:

$$\textit{Russian,012.345.6} \models [012_a \vee 012_b]\,\textit{cignorant}$$

$$\textit{Russian,012.345.6} \not\models [K_a(012_a \vee 012_b)]\,\textit{cignorant}$$



Example 7. *Anne says " I don't have 6" and Bill says "Neither do I"*

After the Anne's announcement the remaining deals are 80 and after Bill's 20. At the final state aknowsbs and bknowsas are both commonly known and even Ccignorant holds. But Anne cannot distinguise between 012.345.6 and 012.456.3 and at this state after her announcement cignorant doesn't hold, so she can't make that announcement. Formally: $Russian, 012.345.6 \models [K_a \neg 6]\, cignorant$

$Russian, 012.345.6 \nvDash [K_a \neg 6] K_a\, cignorant$

This example is unsafe in the sense that a different execution of the underlying protocol would result in Cath knowing the deal.

Example 8. *Anne says " I have 012 or I don't have any of these cards" and Bill says "I have 345 or I don't have any of these cards".*

$\boxed{140}$

. . .

013.456.2

<u>012.345.6</u>

234.016.5

. . .

$$\Downarrow \; K_a(012_a \vee (\neg 0_a \wedge \neg 1_a \wedge \neg 2_a))$$

$\boxed{20}$

$$\begin{tikzcd}[arrows=dash]
\underline{012.345.6} \arrow[r, "a"] \arrow[d, "c"] & 012.346.5 \arrow[r, "a"] \arrow[d, "c"] & 012.356.4 \arrow[r, "a"] \arrow[d, "c"] & 012.456.3 \arrow[d, "c"] \\
345.012.6 \arrow[r, "b"] \arrow[d, "a"] & 346.012.5 \arrow[r, "b"] \arrow[d, "a"] & 356.012.4 \arrow[r, "b"] \arrow[d, "a"] & 456.012.3 \arrow[d, "a"] \\
345.016.2 \arrow[r, "c"] \arrow[d, "a"] & 346.015.2 \arrow[r, "c"] \arrow[d, "a"] & 356.014.2 \arrow[r, "c"] \arrow[d, "a"] & 456.013.2 \arrow[d, "a"] \\
345.026.1 \arrow[r, "c"] \arrow[d, "a"] & 346.025.1 \arrow[r, "c"] \arrow[d, "a"] & 356.024.1 \arrow[r, "c"] \arrow[d, "a"] & 456.023.1 \arrow[d, "a"] \\
345.126.0 \arrow[r, "c"] & 346.125.0 \arrow[r, "c"] & 356.124.0 \arrow[r, "c"] & 456.123.0
\end{tikzcd}$$

We can easily see that the following hold:

Let first be the formula $012_a \vee (\neg 0_a \wedge \neg 1_a \wedge \neg 2_a)$. *Then,*

$$Russian,012.345.6 \models [K_a first]\, cignorant$$
$$Russian,012.345.6 \models [K_a first] K_a\, cignorant$$

So what is the problem?

Even tho Cath is ignorant and Anne knows that, Cath doesn't know that and she can draw information from her ignorance.

When Anne anounces K_a*first she isn't merely announcing that, but also that after first she knows that Cath is ignorant. That is, she is saying first with intension to solve the Russian problem.*

So her true announcement is $[K_a first \wedge [K_a first] K_a\, cignorant]$.

Now Cath after the update with K_a*first only confuses 012.345.6 and 345.012.6. If the latter was the case then Anne could have imagined it to be for example 345.016.2 and there* K_a*first would have been informative for Cath: She would have known that Anne doesn't have 0,1 and 2! So cignorant fails in 345.016.2 after the update with* K_a*first and thus* K_a *cignorant doesn't hold in 345.012.6 after update with* K_a*first. Hence* $K_a first \wedge [K_a first] K_a\, cignorant$ *doesn't hold in epistemic state 345.012.6 and Cath deletes it. But since it was her only other alternative she now knows the deal! Formally*:

$$Russian,012.345.6 \not\models [K_a first] K_c K_a\, cignorant$$
$$Russian,012.345.6 \models [K_a first \wedge [K_a first] K_a\, cignorant] \neg cignorant$$

And to make the unsuccesful update stand out:

$$Russian|K_a first,012.345.6 \models \langle K_a\, cignorant \rangle \neg K_a\, cignorant$$

In other words, Cath does not learn Anne's cards from the mere fact that her announcement is based on her information. Instead, she learns them from Anne's intentntions to prevent Cath from learning her cards. Without that intention Cath could not have known her cards.

Fortunately we see that these kind of unsuccessful updates can be avoided if the intentions of Anne is to guarantee that Ccignorant, because updates with publicly known information are always successful.

$$\text{M,s} \models [K_a \phi \wedge [K_a \phi][\text{Ccignorant}]]\text{Ccignorant}$$
$$\Leftrightarrow$$
$$\text{M,s} \models [K_a \phi][\text{Ccignorant}]\text{Ccignorant}$$
$$\Leftarrow$$
$$\text{M}|K_a \phi\text{,s} \models [\text{Ccignorant}]\text{Ccignorant}$$
$$\Leftrightarrow$$
$$\text{truth}$$

Example 9. *We will now show that the following is a solution to the puzzle*:
Anne says " I have one of 012,034,054,135,246" and Bill says "Cath has 6"
Ms Let π=($012_a \vee 034_a \vee 056_a \vee 135_a \vee 246_a$) . *We have to show that the following hold*:

$$i)\ Russia, 012.345.6 \models K_a\pi$$
$$ii)\ Russia|K_a\pi, 012.345.6 \models Ccignorant$$
$$iii)\ Russia|K_a\pi, 012.345.6 \models K_b 6_c$$
$$iv)\ Russia|K_a\pi|6_c, 012.345.6 \models Ccignorant \wedge aknowsbs \wedge bknowsas$$

To prove that Ccignorant holds in a given model we procced in the following systematic way:
For an arbitary card c, we remove all of the states that include this card, because the actual $a-$*hand cannot have the actual* $c-$*card. Then we show that all other cards occur at least once and are absent at least once in the remaining hands. This implies whatever the hand of a, for each of* a'*s cards in that hand, there is at least one other remaining hand where that card is not included. This means that c stays ignorant about the ownership of the cards. We will now prove conditions i to iv*:

i) Hand 012 is included in π *so i holds*
ii) If c holds 0, the remaing hands are {135,246}. Now each of 1,2,3,4,5,6 is included and absent in at least one of the hands. Same can be seen for the other cards for Cath. Thus ii holds.
iii) From π *Bill can remove all hands that have 3 or 4 or 5 and is left with 012. Therefore Bill knows the deal and also* 6_c. *iii holds.*
iv) After both communications, the following are still possible:
{012.345.6,034.125.6,135.024.6}
Which are all different for Anne and Bill and the same for Cath. Each of 0,1,2,3,4,5 occurs in at least one of {012,034,135} and is absent in at least one. So cignorant remains and iv holds.

The same way it can be shown that the following communications consisting of Anne anouncing 6 and 7 hands respectively are solutions to the puzzle:
Anne says " I have one of {012, 034, 056, 135, 146, 236}" and Bill says " Cath has 6"
Anne says "I have one of {012, 034, 056, 135, 146, 236, 245}" and Bill says "Catg has 6"

We are now going to show general results about how many hands Anne must reveal.

Proposition 2. *Every card must occur at least twice in Anne*$'$*s revealed hands.*

Proof. Suppose i occurs only once in Anne$'$s announcement of π. Cath can then reason as follows:
Suppose a didn$'$t have card i then see can imagine c not to have it too. Let ijk be the hand that includes i. Suppose a didn$'$t have j. Then she could imagine c to have it, in which case she could eliminate this hand and then know that b has i, so Ccignorant fails. Therefore a must have j and Ccignorant fails again. Same for k. But because c$'$s assumption of a not having i leads to ¬Ccignorant , a must have i. But then again Ccignorant doesn$'$t hold.

□

Proposition 3. *Anne*$'$*s announcement must consist of at least 5 hands and no more than 7 hands*

Proof. Every card must be included in at least 2 hands. So in Anne$'$s announcement a total of 2x7=14 cards must occur. But an announcement of 4 hands consists of a total of 4x3=12 cards.

□

3 References

For an in depth analysis of the Muddy Children puzzle including a syntactic proof, I refer to J.D. Gerbrandy. Bisimulations on Planet Kripke. Ph.D. thesis, University of Amsterdam, 1998. ILLC Dissertation Series DS-1999-01.

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