Dynamic Epistemic Logic
Chapter 7, Completeness (7.3-7.8)

Maria Markella Diamantopoulou
«Applied Mathematical Sciences»
Spring Semester 2021

Overview

- Reminder
- Completeness for S5C
- Completeness for PA without common knowledge
- Completeness for PA with common knowledge
- Completeness in a generalised form for logic of action model without common knowledge
- Completeness in a generalised form for logic of action model with common knowledge
- Introduction to relativised common knowledge

Reminder

The proof system S5

all instantiations of propositional tautologies	
$K_a(\varphi \rightarrow \psi) \rightarrow (K_a\varphi \rightarrow K_a\psi)$	distribution of K_a over $\rightarrow$
$K_a\varphi \rightarrow \varphi$	truth
$K_a\varphi \rightarrow K_aK_a\varphi$	positive introspection
$\neg K_a\varphi \rightarrow K_a\neg K_a\varphi$	negative introspection
From φ and $\varphi \rightarrow \psi$, infer ψ	modus ponens
From φ, infer $K_a\varphi$	necessitation of K_a

- Definition 7.1 (Maximal consistent):

Let $\Gamma \subseteq \mathcal{L}_K$. Γ is maximal consistent iff

  - Γ is consistent: $\Gamma \nvdash \bot$
  - Γ is maximal: there is no $\Gamma' \subseteq \mathcal{L}_K$ such that $\Gamma \subset \Gamma'$ and Γ' is consistent.

- Definition 7.2(Canonical model):

The canonical model $M^c = \langle S^c, \sim^c, V^c \rangle$ is defined as follows:

  - S^c={Γ| Γ is maximal consistent}
  - $\Gamma \sim_a^c \Delta$ iff $\{K_a\varphi | K_a\varphi \in \Gamma\} = \{K_a\varphi | K_a\varphi \in \Delta\}$
  - $V_p^c = \{\Gamma \in S^c | p \in \Gamma\}$

- Lemma 7.3(Lindenbaum):

Every consistent set of formulas is a subset of a maximal consistent set of formulas.

Adolf Lindenbaum (12 June 1904 – August 1941)
Polish-Jewish logician and mathematician

- Definition (out of the book):

In mathematical logic, a set T of logical formulas is deductively closed if it contains every formula φ that can be logically deduced from T formally if $\mathrm{T} \vdash \varphi$ always implies $\varphi \in \mathrm{T}$.

- Lemma 7.4:

If Γ and Δ are maximal consistent sets, then:

1. Γ is deductively closed,
2. $\varphi \in \Gamma$ iff $\neg\varphi \notin \Gamma$,
3. $(\varphi \wedge \psi) \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$,
4. $\Gamma \sim_a^c \Delta$ iff $\{K_a\varphi | K_a\varphi \in \Gamma\} \subseteq \Delta$,
5. $\{K_a\varphi | K_a\varphi \in \Gamma\} \vdash \psi$ iff $\{K_a\varphi | K_a\varphi \in \Gamma\} \vdash K_a\psi$.

- Lemma 7.5(Truth):

For every $\varphi \in \mathcal{L}_K$ and every maximal consistent set Γ:

$$\varphi \in \Gamma \text{ iff } (M^c, \Gamma) \models \varphi$$

- Lemma 7.6 (Canonicity):

The canonical model is reflexive, transitive and Euclidean.

- Theorem 7.7 (Completeness) :

For every $\varphi \in \mathcal{L}_K$

$$\models \varphi \text{ implies } \vdash \varphi$$

The proof system S5C

all instantiations of propositional tautologies	
$K_a(\varphi \rightarrow \psi) \rightarrow (K_a\varphi \rightarrow K_a\psi)$	distribution of K_a over $\rightarrow$
$K_a\varphi \rightarrow \varphi$	truth
$K_a\varphi \rightarrow K_aK_a\varphi$	positive introspection
$\neg K_a\varphi \rightarrow K_a\neg K_a\varphi$	negative introspection
$C_B(\varphi \rightarrow \psi) \rightarrow (C_B\varphi \rightarrow C_B\psi)$	distribution of C_B over $\rightarrow$
$C_B\varphi \rightarrow (\varphi \wedge E_BC_B\varphi)$	mix
$C_B(\varphi \rightarrow E_B\varphi) \rightarrow (\varphi \rightarrow C_B\varphi)$	induction axiom
From φ and $\varphi \rightarrow \psi$, infer ψ	modus ponens
From φ, infer $K_a\varphi$	necessitation of K_a
From φ, infer $C_B\varphi$	necessitation of C_B

- Definition 7.8(Closure):

Let $cl: \mathcal{L}_{KC} \rightarrow \wp(\mathcal{L}_{KC})$, be the function such that for every $\varphi \in \mathcal{L}_{KC}$, $cl(\varphi)$ is the smallest set such that:

1. $\varphi \in cl(\varphi)$,
2. If $\psi \in cl(\varphi)$, then $Sub(\psi) \subseteq cl(\varphi)$ (where $Sub(\psi)$ is the set of subformulas of ψ),
3. If $\psi \in cl(\varphi)$ and ψ is not a negation, then $\neg\psi \in cl(\varphi)$,
4. If $C_B\psi \in cl(\varphi)$, then$\{K_a C_B \psi | a \in \mathrm{B}\} \subseteq cl(\varphi)$.

- Lemma 7.9:

$cl(\varphi)$ is finite for all formulas $\varphi \in \mathcal{L}_{KC}$.

Proof:

By induction to φ.

Base case: If φ is a propositional variable p, then the closure of φ is $\{p, \neg p\}$, which is finite.

Induction hypothesis: $cl(\varphi)$ and $cl(\psi)$ are finite.

Induction step:

- The case for $\neg\varphi$: the closure of our case is the set $\{\neg\varphi\} \cup cl(\varphi)$. By induction hypothesis we know that $cl(\varphi)$ is finite so the set is also finite.
- The case for $(\varphi \wedge \psi)$: the closure of our case is the set $\{(\varphi \wedge \psi), \neg(\varphi \wedge \psi)\} \cup cl(\varphi) \cup cl(\psi)$. By induction hypothesis $cl(\varphi)$ and $cl(\psi)$ are finite so our set is also finite.
- The case for $K_a\varphi$: the closure of our case is the set $\{K_a\varphi, \neg K_a\varphi\} \cup cl(\varphi)$. By induction hypothesis we know that $cl(\varphi)$ is finite so our set is also finite.
- The case for $C_B\varphi$: the closure of our case is the set $\{C_B\varphi, \neg C_B\varphi\} \cup \{K_a C_B\varphi, \neg K_a C_B\varphi | a \in \mathrm{B}\} \cup cl(\varphi)$. By induction hypothesis we know that $cl(\varphi)$ is finite so our set is also finite.

- Definition 7.10(Maximal consistent in Φ):

Let $\Phi \subseteq \mathcal{L}_{KC}$ be the closure of some formula. Γ is maximal consistent in Φ iff:

- $\Gamma \subseteq \Phi$
- Γ is consistent: $\Gamma \nvdash \bot$
- Γ is maximal: there is no $\Gamma' \subseteq \mathcal{L}_K$ such that $\Gamma \subset \Gamma'$ and Γ' is consistent.

- Lemma 7.12 (Lindenbaum):

Let Φ be the closure of some formula. Every consistent subset of Φ is a subset of a maximal consistent set in Φ.

Proof:

Let $\Delta \subseteq \Phi$ be a consistent set of formulas. Let $|\Phi| = n$. Let φ_k be the k-th formula in an enumeration of Φ. We consider the sequence of sets of formulas as follows:

$\Gamma_0 = \Delta$

$$\Gamma_{k+1} = \begin{cases} \Gamma_k \cup \{\varphi_{k+1}\}, \text{if } \Gamma_k \cup \{\varphi_{k+1}\} \text{ is consistent} \\ \Gamma_k \qquad\qquad\qquad\qquad\qquad\qquad\quad , \text{otherwise} \end{cases}$$

We can see that $\Delta \subseteq \Gamma_n$.What we need to show?

We need to show that Γ_n is maximal consistent. In order to see that Γ_n is consistent we will prove that Γ_k is consistent by induction on k, which means that Γ_n is also consistent.

Proof :

By assumption Γ_0 is consistent due to Δ is a consistent set of formulas. We can also see that if Γ_k is consistent then Γ_{k+1} is consistent. Thus we prove thatΓ_n is consistent

To see that Γ_n is maximal in Φ, take an arbitrary formula $\varphi_k \in \Phi$ such that $\varphi_k \notin \Gamma_n$. Then $\varphi_k \notin \Gamma_k$ too. Therefore $\Gamma_k \cup \{\varphi_k\}$ is inconsistent and so $\Gamma_n \cup \{\varphi_k\}$ is inconsistent too. Since φ_k was arbitrary there is no $\Gamma' \subseteq \Phi$ such that $\Gamma_n \subset \Gamma'$and Γ' is consistent.

- Definition 7.11(Canonical model for Φ):

Let Φ be the closure of some formula. The canonical model $M^c = \langle S^c, \sim^c, V^c \rangle$ is defined as follows:

- S^c={Γ| Γ is maximal consistent in Φ}
- $\Gamma \sim_a^c \Delta$ iff $\{K_a\varphi | K_a\varphi \in \Gamma\} = \{K_a\varphi | K_a\varphi \in \Delta\}$
- $V_p^c = \{\Gamma \in S^c | p \in \Gamma\}$

We construct a finite model for only a finite fragment of the language depending on the formula we are interested in.

- Definition 7.13(Paths):
  - A B-path from Γ is a sequence $\Gamma_0, \ldots, \Gamma_n$ of maximal consistent sets in Φ such that for all k $(0 \leq k < n)$ there is an agent $a \in B$ such that $\Gamma_k \sim_a^c \Gamma_{k+1}$ and $\Gamma_0 = \Gamma$.
  - A φ-path is a sequence $\Gamma_0, \ldots, \Gamma_n$ of maximal consistent sets in Φ such that for all k $(0 \leq k < n)$ $\varphi \in \Gamma_k$.

Note: We take the length of a path $\Gamma_0, \ldots, \Gamma_n$ to be n.

- Lemma 7.14:

Let Φ be the closure of some formula. Let $M^c = (S^c, \sim^c, V^c)$ be the canonical model for Φ. If Γ and Δ are maximal consistent sets, then:

1. Γ is deductively closed in Φ (for all formulas $\varphi \in \Phi$, if $\vdash \underline{\Gamma} \to \varphi$, then $\varphi \in \Gamma$. Note that $\underline{\Gamma} = \bigwedge \Gamma$)
2. If $\neg\varphi \in \Phi$, then $\varphi \in \Gamma$ iff $\neg\varphi \notin \Gamma$
3. If $(\varphi \wedge \psi) \in \Phi$, then $(\varphi \wedge \psi) \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$
4. If $\underline{\Gamma} \wedge \widehat{K_a}\underline{\Delta}$ is consistent, then $\Gamma \sim_a^c \Delta$
5. If $K_a\psi \in \Phi$, then $\{K_a\varphi | K_a\varphi \in \Gamma\} \vdash \psi$ iff $\{K_a\varphi | K_a\varphi \in \Gamma\} \vdash K_a\psi$
6. If $C_B\varphi \in \Phi$, then $C_B\varphi \in \Gamma$ iff every B-path from Γ is a φ-path.

In other case, it doesn't hold that Γ is maximal consistent set in Φ by the definition of maximal consistent set

Exercise 7.15

Proof :

1. Suppose that $\varphi \in \Phi$. Suppose that $\Gamma \vdash \varphi$. We know by assumption that Γ is maximal consistent which means that Γ is consistent and maximal in Φ. Because of consistency of Γ in Φ, $\Gamma \cup \{\varphi\}$ is also consistent. Therefore, by maximality of Γ in Φ, it must be the case that $\varphi \in \Gamma$.
2. Suppose $\neg\varphi \in \Phi$. Therefore $\varphi \in \Phi$.

" $\Rightarrow$ " Suppose that $\varphi \in \Gamma$ then by consistency $\neg\varphi \in \Gamma$

" $\Leftarrow$ " Suppose that $\neg\varphi \notin \Gamma$. By maximality, $\Gamma \cup \{\neg\varphi\}$ is inconsistent. Therefore $\Gamma \vdash \varphi$ and by the item 1 of this Lemma, $\varphi \in \Gamma$.

3. Suppose that $(\varphi \wedge \psi) \in \Phi$.

" $\Rightarrow$ " Suppose that $(\varphi \wedge \psi) \in \Gamma$. Then $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$. Since Φ is closed, also $\varphi \in \Phi$ and $\psi \in \Phi$. Therefore $\varphi \in \Gamma$ and $\psi \in \Gamma$ by the item 1 of this Lemma.

" $\Leftarrow$ " Suppose that $\varphi \in \Gamma$ and $\psi \in \Gamma$. Therefore $\Gamma \vdash (\varphi \wedge \psi)$ and by the item 1 of this Lemma $(\varphi \wedge \psi) \in \Gamma$.

Lemma 7.17 (Truth):

Let Φ be the closure of some formula. Let $M^c = (S^c, \sim^c, V^c)$ be the canonical model for Φ. For all $\Gamma \in S^c$ and all $\varphi \in \Phi$:

$$\varphi \in \Gamma \text{ iff } (M^c, \Gamma) \vDash \varphi$$

Proof: Suppose that $\varphi \in \Phi$

Base case: Suppose that φ is a propositional variable p. Then by the definition of V^c, $p \in \Gamma$ iff $\Gamma \in S^c$ which by semantics is equivalent to $(M^c, \Gamma) \vDash p$.

Induction hypothesis: For every maximal consistent set Γ, $\varphi \in \Gamma$ iff $(M^c, \Gamma) \vDash \varphi$.

Induction step:

- The case for $\neg\varphi$: $\neg\varphi \in \Gamma$ is equivalent to $\varphi \notin \Gamma$ by the item 2 in Lemma 7.14. By induction hypothesis and the semantics this is equivalent to $(M^c, \Gamma) \vDash \neg\varphi$.
- The case for $(\varphi \wedge \psi)$: $(\varphi \wedge \psi) \in \Gamma$ is equivalent to $\varphi \in \Gamma$ and $\psi \in \Gamma$ by the item 3 of Lemma 7.14 . By induction hypothesis this is equivalent to $(M^c, \Gamma) \vDash \varphi$ and $(M^c, \Gamma) \vDash \psi$ which by semantics is equivalence to $(M^c, \Gamma) \vDash (\varphi \wedge \psi)$.
- The case for $K_a\varphi$:

Suppose that $K_a\varphi \in \Gamma$.Take an arbitrary maximal consistent set Δ in Φ. Suppose that $\Gamma \sim_a^c \Delta$, so $K_a\varphi \in \Delta$ by the definition of the relation $\sim_a^c$. Since $\vdash K_a\varphi \rightarrow \varphi$ by the truth, and Δ is deductively closed (due to Δ is maximal consistent and by the item 1 in Lemma 7.14) then $\varphi \in \Delta$. By the induction hypothesis, this is equivalent to $(M^c, \Delta) \vDash \varphi$. Since we chose an arbitrary Δ , then $(M^c, \Delta) \vDash \varphi$ holds for all Δ such that $\Gamma \sim_a^c \Delta$. Therefore by semantics this is equivalence to $(M^c, \Gamma) \vDash K_a\varphi$.

2. If $\neg\varphi \in \Phi$, then $\varphi \in \Gamma$ iff $\neg\varphi \notin \Gamma$
3. If $(\varphi \wedge \psi) \in \Phi$, then $(\varphi \wedge \psi) \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$

Truth: $K_a\varphi \rightarrow \varphi$

– The case for $C_B\varphi$: Suppose that $C_B\varphi \in \Gamma$. From item 6 in Lemma 7.14 this is the case iff every B-path from Γ is a φ-path. By induction hypothesis, this is the case that iff every B-path along φ is true. Therefore by semantics this is equivalence to $(M^c, \Gamma) \vDash C_B\varphi$.

- Lemma 7.18(Canonicity):

Let Φ be the closure of some formula. The canonical model for Φ is reflexive, transitive and Euclidean.

Proof:

The same as the proof of Lemma 7.6 which follows straightforwardly from the definition of the relation $\sim_a^c$.

- Theorem 7.19 (Completeness):

For every $\varphi \in \mathcal{L}_{KC}$

$$\vDash \varphi \text{ implies } \vdash \varphi$$

Proof:

We will prove this theorem by contraposition. Thus we suppose that $\nvdash \varphi$. Therefore $\{\neg\varphi\}$ is a consistent set. By the Lindenbaum Lemma $\{\neg\varphi\}$ is a subset of some Γ which is maximal consistent in $cl(\neg\varphi)$. Let M^c be the canonical model for $cl(\neg\varphi)$. By the Truth Lemma $(M^c, \Gamma) \vDash \neg\varphi$. Therefore $\nvDash\varphi$.

The proof system PA:

all instantiations of propositional tautologies	
$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$	distribution of K_a over $\to$
$K_a\varphi \to \varphi$	truth
$K_a\varphi \to K_aK_a\varphi$	positive introspection
$\neg K_a\varphi \to K_a\neg K_a\varphi$	negative introspection
$[\varphi]p \leftrightarrow (\varphi \to p)$	atomic permanence
$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$	announcement and negation
$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$	announcement and conjunction
$[\varphi]K_a\psi \leftrightarrow (\varphi \to K_a[\varphi]\psi)$	announcement and knowledge
$[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$	announcement composition
From φ and $\varphi \to \psi$, infer ψ	modus ponens
From φ, infer $K_a\varphi$	necessitation of K_a

- <u>Definition 7.20 (Translation):</u>

The translation $t\colon \mathcal{L}_{K[]} \rightarrow \mathcal{L}_K$ is defined as follows:

$t(p) = p$
$t(\neg\varphi) = \neg t(\varphi)$
$t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi)$
$t(K_a\varphi\ ) = K_a t(\varphi)$
$t([\varphi]p) = t(\varphi \rightarrow p)$
$t([\varphi]\neg\psi) = t(\varphi \rightarrow \neg[\varphi]\psi$
$t\big([\varphi](\psi \wedge \chi)\big) = t\big([\varphi](\psi) \wedge [\varphi]\chi)\big)$
$t([\varphi]K_a\psi) = t(\varphi \rightarrow K_a[\varphi]\psi)$
$t([\varphi][\psi]\chi) = t([\varphi \wedge [\varphi]\psi]\chi)$

Where p is a propositional variable, φ and ψ are formulas

- <u>Definition 7.21 (Complexity):</u>

The complexity $c : \mathcal{L}_{K[]} \rightarrow \mathbb{N}$ is defined as follows:

$c(p) = 1$
$c(\neg\varphi) = 1 + c(\varphi)$
$c(\varphi \wedge \psi) = 1 + \max\big(c(\varphi), c(\psi)\big)$
$c(K_a\varphi) = 1 + c(\varphi)$
$c([\varphi]\psi) = \big(4 + c(\varphi)\big) \cdot c(\psi)$

- Lemma 7.22:

For all φ, ψ and χ:

1. $c(\psi) \geq c(\varphi)$ if $\varphi \in Sub(\psi)$
2. $c([\varphi]p) > c(\varphi \to p)$
3. $c([\varphi]\neg\psi) > c(\varphi \to \neg[\varphi]\psi)$
4. $c\big([\varphi](\psi \wedge \chi)\big) > c([\varphi]\psi \wedge [\varphi]\chi)$
5. $c([\varphi]K_a\psi) > c(\varphi \to K_a[\varphi]\psi)$
6. $c([\varphi][\psi]\chi) > c([\varphi \wedge [\varphi]\psi]\chi)$

$Sub(\psi)$ is the set of subformulas of ψ

- Exercise 7.23:

Prove Lemma 7.22

2. $c([\varphi]p) > c(\varphi \rightarrow p)$

Proof: ??

$c([\varphi]p) = \big(4 + c(\varphi)\big) = 4 + c(\varphi)$

And

$c(\varphi \rightarrow p) = c(\neg\varphi \vee p\ )$

$= c\big(\neg(\varphi \wedge \neg p)\big)$

$= 1 + c(\varphi \wedge \neg p)$

$= 2 + \max(c(\varphi), 2\ )$

So $c([\varphi]p) > c(\varphi \rightarrow p)$

3. $c([\varphi]\neg\psi) > c(\varphi \rightarrow \neg[\varphi]\psi)$

Proof:??

$c([\varphi]\neg\psi) = \big(4 + c(\varphi)\big) \cdot c(\neg\psi) = \big(4 + c(\varphi)\big) \cdot \big(1 + c(\psi)\big)$

$= 4 + c(\varphi) + 4c(\psi) + c(\varphi) \cdot c(\psi)$

And

$c(\varphi \rightarrow \neg[\varphi]\psi) = c\big(\neg\varphi \vee (\neg[\varphi]\psi)\big) = c\Big(\neg\big(\varphi \wedge \neg(\neg[\varphi]\psi)\big)\Big)$

$= 1 + c\big(\varphi \wedge \neg(\neg[\varphi]\psi)\big) =$

Reminder:

$c(p) = 1$

$c(\neg\varphi) = 1 + c(\varphi)$

$c(\varphi \wedge \psi) =$

$1 + \max\big(c(\varphi), c(\psi)\big)$

$c(K_a\varphi) = 1 + c(\varphi)$

$c([\varphi]\psi) = \big(4 + c(\varphi)\big) \cdot c(\psi)$

The abbriviation of

$(\varphi \rightarrow p\ )$ is $(\neg\varphi \vee p\ )$

$= 1 + c(\varphi \wedge \neg(\neg[\varphi]\psi))$

$= 1 + 1 + \max\left(c(\varphi), c\left(\neg(\neg[\varphi]\psi)\right)\right)$

$= 2 + \max(c(\varphi), +c(\neg[\varphi]\psi))$

$= 2 + \max(c(\varphi), 2 + c([\varphi]\psi))$

$= 2 + \max\left(c(\varphi), 2 + \left(4 + c(\varphi)\right) \cdot c(\psi)\right)$

$= 2 + \max\left(c(\varphi), 2 + \left(4c(\psi) + c(\varphi)c(\psi)\right)\right)$

$2 + \max(c(\varphi), 2 + 4c(\psi) + c(\varphi)c(\psi))$

Thus $c([\varphi]\neg\psi) > c(\varphi \rightarrow \neg[\varphi]\psi)$.

4. $c\left([\varphi](\psi \wedge \chi)\right) > c([\varphi]\psi \wedge [\varphi]\chi)$

Proof:??

Assume without loss of generality, that $c(\psi) \geq c(\chi)$. Then :

$c\left([\varphi](\psi \wedge \chi)\right) = \left(4 + c(\varphi)\right) \cdot c(\psi \wedge \chi) = \left(4 + c(\varphi)\right)\left(1 + \max(c(\psi), c(\chi))\right)$

$= \left(4 + c(\varphi)\right)\left(1 + c(\psi)\right) = 4 + 4c(\psi) + c(\varphi) + c(\varphi)c(\psi)$

Reminder:

$c(p) = 1$

$c(\neg\varphi) = 1 + c(\varphi)$

$c(\varphi \wedge \psi) =$
$1 + \max(c(\varphi), c(\psi))$

$c(K_a\varphi) = 1 + c(\varphi)$

$c([\varphi]\psi) = \left(4 + c(\varphi)\right) \cdot c(\psi)$

And

$c([\varphi]\psi \wedge [\varphi]\chi)$

$= 1 + \max\Big(\big(4 + c(\varphi)\big)c(\psi), \big(4 + c(\varphi)\big)c(\chi)\Big)$

$= 1 + \Big(\big(4 + c(\varphi)\big)c(\psi)\Big)$

$= 1 + 4c(\psi) + c(\varphi)c(\psi)$

So $c\big([\varphi](\psi \wedge \chi)\big) > c([\varphi]\psi \wedge [\varphi]\chi)$.

6. $c([\varphi][\psi]\chi) > c([\varphi \wedge [\varphi]\psi]\chi)$

Proof:??

$c([\varphi][\psi]\chi) = \big(4 + c(\varphi)\big)\big(4 + c(\psi)\big)c(\chi)$

$= \big(16 + 4c(\varphi) + 4c(\psi) + c(\varphi)c(\psi)c(\chi)\big)$

And

$c([\varphi \wedge [\varphi]\psi]\chi) = \Bigg(4 + \bigg(1 + \max\Big(c(\varphi), \big(4 + c(\varphi)\big)c(\psi)\Big)\bigg)\Bigg)c(\chi)$

$= \bigg(5 + \Big(\big(4 + c(\varphi)\big)c(\psi)\Big)\bigg)c(\chi)$

$= \big(5 + 4c(\psi) + c(\varphi)c(\psi)c(\chi)\big)$

Reminder:

$c(p) = 1$

$c(\neg\varphi) = 1 + c(\varphi)$

$c(\varphi \wedge \psi) =$

$1 + \max\big(c(\varphi), c(\psi)\big)$

$c(K_a\varphi) = 1 + c(\varphi)$

$c([\varphi]\psi) = \big(4 + c(\varphi)\big) \cdot c(\psi)$

We will show that every formula is provably equivalent to its translation.

- Lemma 7.24:

For all formulas $\varphi \in \mathcal{L}_{K[]}$ it is the case that

$\vdash \varphi \leftrightarrow t(\varphi)$

Proof:

We will prove this Lemma by induction on $c(\varphi)$.

Base case: If φ is a propositional variable p, it's trivial that $\vdash p \leftrightarrow t(p) = p$.

Induction hypothesis: For all φ such that $c(\varphi) \leq n$: $\vdash \varphi \leftrightarrow t(\varphi)$.

Induction step: The case for $\neg$, $\wedge$, K_a follows straightforwardly from the induction hypothesis and item 1 of Lemma 7.22.

- The case for $[\varphi]p$: This case follows straightforwardly from the atomic permanence axiom, item 2 of Lemma 7.22 and the induction hypothesis.

atomic permanence axiom:
$[\varphi]p \leftrightarrow (\varphi \rightarrow p)$
Item 2 of Lemma 7.22
$c([\varphi]p) > c(\varphi \rightarrow p)$

– The case for $[\varphi]\neg\psi$: This case follows straightforwardly from the announcement and negation axiom, item 3 of Lemma 7.22 and the induction hypothesis

– The case for $[\varphi](\varphi \wedge \psi)$: This case follows straightforwardly from the announcement and conjunction axiom, item 4 of Lemma 7.22 and the induction hypothesis.

– The case for $[\varphi]\, K_a\psi$: This case follows straightforwardly from the announcement and knowledge axiom, item 5 of Lemma 7.22 and the induction hypothesis.

– The case for $[\varphi][\psi]\chi$: This case follows straightforwardly from the announcement composition axiom, item 6 of Lemma 7.22 and the induction hypothesis.

$[\varphi]\neg\psi \leftrightarrow \varphi \rightarrow \neg[\varphi]\psi$
$[\varphi](\psi \wedge \chi) \leftrightarrow [\varphi]\psi \wedge [\varphi]\chi$
$[\varphi]K_a\psi \leftrightarrow \varphi \rightarrow K_a[\varphi]\psi$
$[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$

3. $c([\varphi]\neg\psi) > c(\varphi \rightarrow \neg[\varphi]\psi)$
4. $c\big([\varphi](\psi \wedge \chi)\big) > c([\varphi]\psi \wedge [\varphi]\chi)$
5. $c([\varphi]K_a\psi) > c(\varphi \rightarrow K_a[\varphi]\psi)$
6. $c([\varphi][\psi]\chi) > c([\varphi \wedge [\varphi]\psi]\chi)$

- Theorem 7.26 (Completeness):

For every $\varphi \in \mathcal{L}_{K[]}$ $\models \varphi$ implies $\vdash \varphi$

Proof:

Suppose that $\models \varphi$. Therefore $\models t(\varphi)$ (by soundness) and by Lemma 7.24 holds that $\vdash \varphi \leftrightarrow t(\varphi)$. Because of the fact that $t(\varphi)$ doesn't contain any announcement operators, $S5 \vdash t(\varphi)$ (Theorem 7.7) We also have that $\mathrm{PA} \vdash t(\varphi)$ as $S5$ is subsystem of PA. Since $\mathrm{PA} \vdash \varphi \leftrightarrow t(\varphi)$, it follows that $\mathrm{PA} \vdash \varphi$.

The proof system PAC

all instantiations of propositional tautologies	
$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$	distribution of K_a over $\to$
$K_a\varphi \to \varphi$	truth
$K_a\varphi \to K_aK_a\varphi$	positive introspection
$\neg K_a\varphi \to K_a\neg K_a\varphi$	negative introspection
$[\varphi]p \leftrightarrow (\varphi \to p)$	atomic permanence
$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$	announcement and negation
$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$	announcement and conjunction
$[\varphi]K_a\psi \leftrightarrow (\varphi \to K_a[\varphi]\psi)$	announcement and knowledge
$[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$	announcement composition
$C_B(\varphi \to \psi) \to (C_B\varphi \to C_B\psi)$	distribution of C_B over $\to$
$C_B\varphi \to (\varphi \wedge E_BC_B\varphi)$	mix of common knowledge
$C_B(\varphi \to E_B\varphi) \to (\varphi \to C_B\varphi)$	induction of common knowledge
From φ and $\varphi \to \psi$, infer ψ	modus ponens
From φ, infer $K_a\varphi$	necessitation of K_a
From φ, infer $C_B\varphi$	necessitation of C_B
From φ, infer $[\psi]\varphi$	necessitation of $[\psi]$
From $\chi \to [\varphi]\psi$ and $\chi \wedge \varphi \to E_B\chi$, infer $\chi \to [\varphi]C_B\psi$	announcement and common knowledge

- Definition 7.27 (Closure):

Let $cl: \mathcal{L}_{KC[]} \rightarrow \wp(\mathcal{L}_{KC[]})$, be the function such that for every $\varphi \in \mathcal{L}_{KC[]}$, $cl(\varphi)$ is the smallest set such that:

1. $\varphi \in cl(\varphi)$,
2. If $\psi \in cl(\varphi)$, then $Sub(\psi) \subseteq cl(\varphi)$ (where $Sub(\psi)$ is the set of subformulas of ψ),
3. If $\psi \in cl(\varphi)$ and ψ is not a negation, then $\neg\psi \in cl(\varphi)$,
4. If $C_B\psi \in cl(\varphi)$, then$\{K_a C_B \psi | a \in \mathrm{B}\} \subseteq cl(\varphi)$,
5. If $[\psi]p \in cl(\varphi)$, then $(\psi \rightarrow p) \in cl(\varphi)$,
6. If $[\psi]\neg\chi \in cl(\varphi)$, then $(\psi \rightarrow \neg[\psi]\chi) \in cl(\varphi)$,
7. If $[\psi](\chi \wedge \xi) \in cl(\varphi)$, then $([\psi]\chi \wedge [\psi]\xi) \in cl(\varphi)$,
8. If $[\psi]K_a\chi \in cl(\varphi)$, then $(\psi \rightarrow K_a[\psi]\chi) \in cl(\varphi)$,
9. If $[\psi]C_B\chi \in cl(\varphi)$, then $[\psi]\chi \in cl(\varphi)$ and $\{K_a[\psi]C_B\chi | a \in \mathrm{B}\} \subseteq cl(\varphi)$,
10. If $[\psi][\chi]\xi \in cl(\varphi)$, then $[\psi \wedge [\psi]\chi]\xi \in cl(\varphi)$.

- Lemma 7.28:

$cl(\varphi)$ is finite for all formulas $\varphi \in \mathcal{L}_{KC[]}$.

- Lemma 7.29 (Lindenbaum):

Let Φ be the closure of some formula. Every consistent subset of Φ is a subset of a maximal consistent set in Φ.

Proof:

The same as Lemma 7.12

- Definition 7.30 (B-φ-path):

A B-φ-path from Γ is a B-path that is also φ-path.

- Lemma 7.31:

Let Φ be the closure of some formula. Let $M^c = (S^c, \sim^c, V^c)$ be the canonical model for Φ. If Γ and Δ are maximal consistent sets, then:

1. Γ is deductively closed in Φ (for all formulas $\varphi \in \Phi$, if $\vdash \underline{\Gamma} \rightarrow \varphi$, then $\varphi \in \Gamma$. Note that $\underline{\Gamma} = \bigwedge \Gamma$)
2. If $\neg\varphi \in \Phi$, then $\varphi \in \Gamma$ iff $\neg\varphi \notin \Gamma$
3. If $(\varphi \wedge \psi) \in \Phi$, then $(\varphi \wedge \psi) \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$
4. If $\underline{\Gamma} \wedge \widehat{K_a}\underline{\Delta}$ is consistent, then $\Gamma \sim_a^c \Delta$
5. If $K_a\psi \in \Phi$, then $\{K_a\varphi | K_a\varphi \in \Gamma\} \vdash \psi$ iff $\{K_a\varphi | K_a\varphi \in \Gamma\} \vdash K_a\psi$
6. If $C_B\varphi \in \Phi$, then $C_B\varphi \in \Gamma$ iff every B-path from Γ is a φ-path.
7. If $[\varphi]C_B\psi \in \Phi$, then $[\varphi]C_B\psi \in \Gamma$ iff every B−path from Γ is a $[\varphi]\psi$−path.

- Definition 7.32 (Complexity):

The complexity $c : \mathcal{L}_{KC[]} \to \mathbb{N}$ is defined as follows:

$c(p) = 1$

$c(\neg\varphi) = 1 + c(\varphi)$

$c(\varphi \wedge \psi) = 1 + \max\big(c(\varphi), c(\psi)\big)$

$c(K_a\varphi) = 1 + c(\varphi)$

$c(C_B\varphi) = 1 + c(\varphi)$

$c([\varphi]\psi) = \big(4 + c(\varphi)\big) \cdot c(\psi)$

- Lemma 7.33:

For all φ, ψ and χ:

1. $c(\psi) \geq c(\varphi)$ for all $\varphi \in Sub(\psi)$
2. $c([\varphi]p) > c(\varphi \to p)$
3. $c([\varphi]\neg\psi) > c(\varphi \to \neg[\varphi]\psi)$
4. $c\big([\varphi](\psi \wedge \chi)\big) > c([\varphi]\psi \wedge [\varphi]\chi)$
5. $c([\varphi]K_a\psi) > c(\varphi \to K_a[\varphi]\psi)$
6. $c([\varphi]C_B\psi) > c([\varphi]\psi)$
7. $c([\varphi][\psi]\chi) > c([\varphi \wedge [\varphi]\psi]\chi)$

- Lemma 7.34 (Truth):

Let Φ be the closure of some formula. Let $M^c = (S^c, \sim^c, V^c)$ be the canonical model for Φ. For all $\Gamma \in S^c$ and all $\varphi \in \Phi$:

$$\varphi \in \Gamma \text{ iff } (M^c, \Gamma) \vDash \varphi$$

Proof: (by induction on $c(\varphi)$)

Suppose that $\varphi \in \Phi$.

Base case: Suppose that φ is a propositional variable p. Then by the definition of V^c, $p \in \Gamma$ iff $\Gamma \in S^c$ which by semantics is equivalent to $(M^c, \Gamma) \vDash p$.

Induction hypothesis: For all φ such that $c(\varphi) \leq n$:

$\varphi \in \Gamma$ iff $(M^c, \Gamma) \vDash \varphi$.

Induction step: Suppose that $c(\varphi) = n + 1$. The cases for $\neg$, $\wedge$, K_a, C_B are just like Lemma 7.17.

- The case for $[\psi]p$: Suppose that $[\psi]p \in \Gamma$. Given that $[\psi]p \in \Phi$, $[\psi]p \in \Gamma$ is equivalent to $(\psi \to p)$ by the atomic permanence axiom. By item 2 of Lemma 7.33, we can apply the induction hypothesis. Therefore this is equivalent to
  $(M^c, \Gamma) \vDash (\psi \to p)$ which is equivalent by semantics to $(M^c, \Gamma) \vDash [\psi]p$.
- The case for $[\psi]\neg\chi$: Suppose that $[\psi]\neg\chi \in \Gamma$. Given that$[\psi]\neg\chi \in \Phi$, $[\psi]p \in \Gamma$ is equivalent to $(\psi \to \neg[\psi]\chi) \in \Gamma$ by the announcement and negation axiom. By item 3 of Lemma 7.33, we can apply the induction hypothesis. Therefore this is equivalent to
  $(M^c, \Gamma) \vDash \psi \to \neg[\psi]\chi$which is equivalent by semantics to $(M^c, \Gamma) \vDash [\psi]\neg\chi$.

- The case for$[\psi](\chi \wedge \xi)$: Suppose that$[\psi](\chi \wedge \xi) \in \Gamma$. Given that$[\psi](\chi \wedge \xi) \in \Phi$,$[\psi](\chi \wedge \xi) \in \Gamma$ is equivalent to $([\psi]\chi \wedge [\psi]\xi) \in \Gamma$ by the announcement and conjunction axiom. By item 4 of Lemma 7.33, we can apply the induction hypothesis. Therefore this is equivalent to $(M^c, \Gamma) \vDash [\psi]\chi \wedge [\psi]\xi$ which is equivalent by semantics to $(M^c, \Gamma) \vDash [\psi](\chi \wedge \xi)$.
- The case $[\psi]K_a\chi$: Suppose that $[\psi]K_a\chi \in \Gamma$. Given that$[\psi]K_a\chi \in \Phi$,$[\psi]K_a\chi \in \Gamma$ is equivalent to $(\psi \rightarrow K_a[\psi]\chi) \in \Gamma$ by the announcement and knowledge axiom. By item 5 of Lemma 7.33, we can apply the induction hypothesis. Therefore this is equivalent to $(M^c, \Gamma) \vDash \psi \rightarrow K_a[\psi]\chi$ which is equivalent by semantics to $(M^c, \Gamma) \vDash [\psi]K_a\chi$.
- The case for $[\psi]C_B\chi$: Suppose that$[\psi]C_B\chi \in \Gamma$. Given that$[\psi]C_B\chi \in \Phi$,$[\psi]C_B\chi \in \Gamma$ iff every B−path from Γ is a $[\psi]\chi$−path by item 6 of Lemma 7.31. By item 6 of Lemma 7.33 we can apply the induction hypothesis. Therefore this is equivalent to every B−path from Γ is along which $[\psi]\chi$ is true which is equivalent by semantics to $(M^c, \Gamma) \vDash [\psi]C_B\chi$.
- The case for $[\psi][\chi]\xi$: Suppose that$[\psi][\chi]\xi \in \Gamma$. Given that$[\psi][\chi]\xi \in \Phi$,$[\psi][\chi]\xi \in \Gamma$ is equivalent to $[\psi \wedge [\psi]\chi]\xi \in \Gamma$ by the announcement and composition axiom. By item 7 of Lemma 7.33, we can apply the induction hypothesis. Therefore this is equivalent to $(M^c, \Gamma) \vDash [\psi \wedge [\psi]\chi]\xi$which is equivalent by semantics to

  $(M^c, \Gamma) \vDash [\psi][\chi]\xi$.

- Lemma 7.35 (Canonicity):

The canonical model is reflexive, transitive and Euclidean.

- Theorem 7.36 (Completeness) :

For every $\varphi \in \mathcal{L}_{KC[]}$

$$\vDash \varphi \text{ implies } \vdash \varphi$$

Proof:

We will prove this theorem by contraposition. Thus we suppose that $\nvdash \varphi$. Therefore $\{\neg\varphi\}$ is a consistent set. By the Lindenbaum Lemma $\{\neg\varphi\}$ is a subset of some Γ which is maximal consistent in $cl(\neg\varphi)$. Let M^c be the canonical model for $cl(\neg\varphi)$. By the Truth Lemma $(M^c, \Gamma) \vDash \neg\varphi$.

Therefore $\nvDash \varphi$.

The proof system AM

all instantiations of propositional tautologies	
$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$	distribution of K_a over $\to$
$K_a\varphi \to \varphi$	truth
$K_a\varphi \to K_aK_a\varphi$	positive introspection
$\neg K_a\varphi \to K_a\neg K_a\varphi$	negative introspection
$[\mathsf{M},\mathsf{s}]p \leftrightarrow (\mathsf{pre}(\mathsf{s}) \to p)$	atomic permanence
$[\mathsf{M},\mathsf{s}]\neg\varphi \leftrightarrow (\mathsf{pre}(\mathsf{s}) \to \neg[\mathsf{M},\mathsf{s}]\varphi)$	action and negation
$[\mathsf{M},\mathsf{s}](\varphi \wedge \psi) \leftrightarrow ([\mathsf{M},\mathsf{s}]\varphi \wedge [\mathsf{M},\mathsf{s}]\psi)$	action and conjunction
$[\mathsf{M},\mathsf{s}]K_a\varphi \leftrightarrow (\mathsf{pre}(\mathsf{s}) \to \bigwedge_{\mathsf{s}\sim_a\mathsf{t}} K_a[\mathsf{M},\mathsf{t}]\varphi)$	action and knowledge
$[\mathsf{M},\mathsf{s}][\mathsf{M}',\mathsf{s}']\varphi \leftrightarrow [(\mathsf{M},\mathsf{s});(\mathsf{M}',\mathsf{s}')]\varphi$	action composition
$[\alpha \cup \alpha']\varphi \leftrightarrow ([\alpha]\varphi \wedge [\alpha']\varphi)$	non-deterministic choice
From φ and $\varphi \to \psi$, infer ψ	modus ponens
From φ, infer $K_a\varphi$	necessitation of K_a
From φ, infer $[\mathsf{M},\mathsf{s}]\varphi$	necessitation of (M,s)

- <u>Definition 7.37 (Translation):</u>

The translation $t: \mathcal{L}_{K\otimes} \rightarrow \mathcal{L}_K$ is defined as follows:

$t(p) = p$

$t(\neg\varphi) = \neg t(\varphi)$

$t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi)$

$t(K_a\varphi\ ) = K_a t(\varphi)$

$t([\mathrm{M}, \mathrm{s}]p) = t(pre(s) \rightarrow p)$

$t([\mathrm{M}, s]\neg\varphi) = t(pre(s) \rightarrow \neg[\mathrm{M}, s]\varphi)$

$t\big([\mathrm{M}, s](\varphi \wedge \psi)\big) = t([\mathrm{M}, s]\varphi \wedge [\mathrm{M}, s]\psi)$

$t([\mathrm{M}, s]K_a\varphi) = t(pre(s) \rightarrow K_a[\mathrm{M}, s]\varphi)$

$t([\mathrm{M}, s][\mathrm{M}', s']\varphi) = t([\mathrm{M}, s; \mathrm{M}', s']\varphi)$

$t([\alpha \cup \alpha']\varphi) = t([\alpha]\varphi) \wedge t([\alpha']\varphi)$

- Definition 7.38 (Complexity):

The complexity $c : \mathcal{L}_{K\otimes} \to \mathbb{N}$ is defined as follows:

$$c(p) = 1$$
$$c(\neg\varphi) = 1 + c(\varphi)$$
$$c(\varphi \wedge \psi) = 1 + \max\big(c(\varphi), c(\psi)\big)$$
$$c(K_a\varphi) = 1 + c(\varphi)$$
$$c([\alpha]\varphi) = \big(4 + c(\alpha)\big) \cdot c(\varphi)$$
$$c([\mathrm{M}, \mathrm{s}]) = \max\big\{c\big(pre(t)\big) | t \in \mathrm{M}\big\}$$
$$c([\alpha \cup \alpha']) = 1 + \max\big(c(\alpha), c(\alpha')\big)$$

- Lemma 7.39:

For all φ, ψ and χ:

1. $c(\psi) \geq c(\varphi)$ if $\varphi \in Sub(\psi)$
2. $c([\mathrm{M}, \mathrm{s}]p) > c(pre(s) \rightarrow p)$
3. $c([\mathrm{M}, s]\neg\varphi) > c(pre(s) \rightarrow \neg[\mathrm{M}, s]\varphi)$
4. $\mathrm{c}\big([\mathrm{M}, s](\varphi \wedge \psi)\big) > c([\mathrm{M}, s]\varphi \wedge [\mathrm{M}, s]\psi)$
5. $\mathrm{c}([\mathrm{M}, s]K_a\varphi) > c(pre(s) \rightarrow K_a[\mathrm{M}, s]\varphi)$
6. $\mathrm{c}([\mathrm{M}, s][\mathrm{M}', s']\varphi) > c([\mathrm{M}, s; \mathrm{M}', s']\varphi)$
7. $\mathrm{c}([\alpha \cup \alpha']\varphi) > c([\alpha]\varphi \wedge [\alpha']\varphi)$

- Lemma 7.40:

For all formulas $\varphi \in \mathcal{L}_{K\otimes}$ it is the case that

$\vdash \varphi \leftrightarrow t(\varphi)$

(i.e. every formula is provably equivalent to its translation)

Proof:

Is similar to the proof of Lemma 7.24

- Theorem 7.41 (Completeness):

For every $\varphi \in \mathcal{L}_{K\otimes}$ $\vDash \varphi$ implies $\vdash \varphi$

Proof:

Suppose that $\vDash \varphi$. Therefore $\vDash t(\varphi)$ (by the soundness) and by Lemma 7.40 holds that $\vdash \varphi \leftrightarrow t(\varphi)$. Because of the fact that $t(\varphi)$ doesn't contain any action models, $S5 \vdash t(\varphi)$ (Theorem 7.7) We also have that $\mathrm{AM} \vdash t(\varphi)$ as $S5$ is subsystem of AM. Since $\mathrm{AM} \vdash \varphi \leftrightarrow t(\varphi)$, it follows that $\mathrm{AM} \vdash \varphi$.

The proof system AMC

all instantiations of propositional tautologies	
$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$	distribution of K_a over $\to$
$K_a\varphi \to \varphi$	truth
$K_a\varphi \to K_aK_a\varphi$	positive introspection
$\neg K_a\varphi \to K_a\neg K_a\varphi$	negative introspection
$[\mathsf{M},\mathsf{s}]p \leftrightarrow (\mathsf{pre}(\mathsf{s}) \to p)$	atomic permanence
$[\mathsf{M},\mathsf{s}]\neg\varphi \leftrightarrow (\mathsf{pre}(\mathsf{s}) \to \neg[\mathsf{M},\mathsf{s}]\varphi)$	action and negation
$[\mathsf{M},\mathsf{s}](\varphi \land \psi) \leftrightarrow ([\mathsf{M},\mathsf{s}]\varphi \land [\mathsf{M},\mathsf{s}]\psi)$	action and conjunction
$[\mathsf{M},\mathsf{s}]K_a\varphi \leftrightarrow (\mathsf{pre}(\mathsf{s}) \to \bigwedge_{\mathsf{s}\sim_a\mathsf{t}} K_a[\mathsf{M},\mathsf{t}]\varphi)$	action and knowledge
$[\mathsf{M},\mathsf{s}][\mathsf{M}',\mathsf{s}']\varphi \leftrightarrow [(\mathsf{M},\mathsf{s});(\mathsf{M}',\mathsf{s}')]\varphi$	action composition
$[\alpha \cup \alpha']\varphi \leftrightarrow ([\alpha]\varphi \land [\alpha']\varphi)$	non-deterministic choice
$C_B(\varphi \to \psi) \to (C_B\varphi \to C_B\psi)$	distribution of C_B over $\to$
$C_B\varphi \to (\varphi \land E_BC_B\varphi)$	mix
$C_B(\varphi \to E_B\varphi) \to (\varphi \to C_B\varphi)$	induction axiom
From φ and $\varphi \to \psi$, infer ψ	modus ponens
From φ, infer $K_a\varphi$	necessitation of K_a
From φ, infer $C_B\varphi$	necessitation of C_B
From φ, infer $[\mathsf{M},\mathsf{s}]\varphi$	necessitation of (M,s)
Let (M,s) be an action model and let a set of formulas χ_t for every t such that $\mathsf{s} \sim_B \mathsf{t}$ be given. From $\chi_\mathsf{t} \to [\mathsf{M},\mathsf{t}]\varphi$ and $(\chi_\mathsf{t} \land \mathsf{pre}(\mathsf{t})) \to K_a\chi_\mathsf{u}$ for every $\mathsf{t} \in \mathsf{S}$, $a \in B$ and $\mathsf{t} \sim_a \mathsf{u}$, infer $\chi_\mathsf{s} \to [\mathsf{M},\mathsf{s}]C_B\varphi$.	action and common knowledge

- Definition 7.42 (Closure):

Let $cl: \mathcal{L}_{KC\otimes} \to \wp(\mathcal{L}_{KC\otimes})$, be the function such that for every $\varphi \in \mathcal{L}_{KC\otimes}$, $cl(\varphi)$ is the smallest set such that:

1. $\varphi \in cl(\varphi)$,
2. If $\psi \in cl(\varphi)$, then $Sub(\psi) \subseteq cl(\varphi)$ (where $Sub(\psi)$ is the set of subformulas of ψ),
3. If $\psi \in cl(\varphi)$ and ψ is not a negation, then $\neg\psi \in cl(\varphi)$,
4. If $C_B\psi \in cl(\varphi)$, then$\{K_a C_B \psi | a \in \mathrm{B}\} \subseteq cl(\varphi)$,
5. If $[\mathrm{M}, \mathrm{s}]p \in cl(\varphi)$, then $(pre(s) \to p) \in cl(\varphi)$,
6. If $([\mathrm{M}, s]\neg\psi) \in cl(\varphi)$, then$(pre(s) \to \neg[\mathrm{M}, s]\psi) \in cl(\varphi)$,
7. If $[\mathrm{M}, s](\psi \wedge \chi) \in cl(\varphi)$, then $([\mathrm{M}, s]\psi \wedge [\mathrm{M}, s]\chi) \in cl(\varphi)$,
8. If $[\mathrm{M}, s]K_a\varphi \in cl(\varphi)$ and $\mathrm{s}\sim_\alpha t$, then $(pre(s) \to K_a[\mathrm{M}, s]\varphi) \in cl(\varphi)$,
9. If $[\mathrm{M}, s]C_B\psi \in cl(\varphi)$, then $\{[\mathrm{M}, t]\psi | \ \mathrm{s}\sim_B t\} \subseteq cl(\varphi)$ and $\{K_a[\mathrm{M}, t]C_B\psi | a \in \mathrm{B}\ and s\sim_B t\} \subseteq cl(\varphi)$,
10. If $[\mathrm{M}, s][\mathrm{M}', s']\psi \in cl(\varphi)$, then $[\mathrm{M}, s; \mathrm{M}', s']\psi \in cl(\varphi)$,
11. if $[\alpha \cup \alpha']\psi \in cl(\varphi)$, then $([\alpha]\psi \wedge [\alpha']\psi) \in cl(\varphi)$,

- Lemma 7.43:

$cl(\varphi)$ is finite for all formulas $\varphi \in \mathcal{L}_{KC\otimes}$.

- Lemma 7.44 (Lindenbaum):

Let Φ be the closure of some formula. Every consistent subset of Φ is a subset of a maximal consistent set in Φ.

Proof:

The same as Lemma 7.12

- Definition 7.45 (BMst-path):

A BMst-path from Γ is a B-path $\Gamma_0, \dots, \Gamma_n$ from Γ such that there is a B-path $s_0, \dots, s_n$ from s to t in M and for all $k < n$ there is an agent $a \in B$ such that $\Gamma_k \sim_a^c \Gamma_{k+1}$ and $s_k \sim_a s_{k+1}$ and for all $k \leq n$ it is the case that $pre(s_k) \in \Gamma_k$.

- <u>Lemma 7.46:</u>

Let Φ be the closure of some formula. Let $M^c = (S^c, \sim^c, V^c)$ be the canonical model for Φ. If Γ and Δ are maximal consistent sets, then:

1. Γ is deductively closed in Φ
2. If $\neg\varphi \in \Phi$, then $\varphi \in \Gamma$ iff $\neg\varphi \notin \Gamma$
3. If $(\varphi \wedge \psi) \in \Phi$, then $(\varphi \wedge \psi) \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$
4. If $\underline{\Gamma} \wedge \widehat{K_a}\underline{\Delta}$ is consistent, then $\Gamma \sim_a^c \Delta$
5. If $K_a\psi \in \Phi$, then $\{K_a\varphi | K_a\varphi \in \Gamma\} \vdash \psi$ iff $\{K_a\varphi | K_a\varphi \in \Gamma\} \vdash K_a\psi$
6. If $C_B\varphi \in \Phi$, then $C_B\varphi \in \Gamma$ iff every B-path from Γ is a φ-path.
7. If $[\mathrm{M}, s]C_B\varphi \in \Phi$, then $[\mathrm{M}, s]C_B\varphi \in \Gamma$ iff for all $t \in S$ every BMst–path from Γ ends in a $[\mathrm{M}, \mathrm{t}]\varphi$–state.

- Definition 7.47 (Complexity):

The complexity $c : \mathcal{L}_{KC\otimes} \to \mathbb{N}$ is defined as follows:

$c(p) = 1$

$c(\neg\varphi) = 1 + c(\varphi)$

$c(\varphi \wedge \psi) = 1 + \max\big(c(\varphi), c(\psi)\big)$

$c(K_a\varphi) = 1 + c(\varphi)$

$c(C_B\varphi) = 1 + c(\varphi)$

$c([\alpha]\varphi) = \big(4 + c(\alpha)\big) \cdot c(\varphi)$

$c([\mathrm{M}, \mathrm{s}]) = \max\big\{c\big(pre(t)\big) | t \in \mathrm{M}\big\}$

$c([\alpha \cup \alpha']) = 1 + \max\big(c(\alpha), c(\alpha')\big)$

- Lemma 7.48:

For all φ, ψ and χ:

1. $c(\psi) \geq c(\varphi)$ for all $\varphi \in Sub(\psi)$
2. $c([\mathrm{M}, \mathrm{s}]p) > c(pre(s) \to p)$
3. $c([\mathrm{M}, s]\neg\varphi) > c(pre(s) \to \neg[\mathrm{M}, s]\varphi)$
4. $\mathrm{c}\big([\mathrm{M}, s](\varphi \wedge \psi)\big) > c([\mathrm{M}, s]\varphi \wedge [\mathrm{M}, s]\psi)$
5. $\mathrm{c}([\mathrm{M}, s]K_a\varphi) > c(pre(s) \to K_a[\mathrm{M}, t]\varphi)$ for all $t \in \mathrm{M}$
6. $\mathrm{c}([\mathrm{M}, s]C_B\varphi) > c([\mathrm{M}, t]\varphi)$ for all $t \in \mathrm{M}$
7. $\mathrm{c}([\mathrm{M}, s][\mathrm{M}', s']\varphi) > c([(\mathrm{M}, s); (\mathrm{M}', s')]\varphi)$
8. $\mathrm{c}([\alpha \cup \alpha']\varphi) > c([\alpha]\varphi \wedge [\alpha']\varphi)$

- Lemma 7.50 (Truth):

Let Φ be the closure of some formula. Let $M^c = (S^c, \sim^c, V^c)$ be the canonical model for Φ. For all $\Gamma \in S^c$ and all $\varphi \in \Phi$:

$$\varphi \in \Gamma \text{ iff } (M^c, \Gamma) \vDash \varphi$$

Proof: (by induction on $c(\varphi)$)

Suppose that $\varphi \in \Phi$.

Base case: Suppose that φ is a propositional variable p. Then by the definition of V^c, $p \in \Gamma$ iff $\Gamma \in S^c$ which by semantics is equivalent to $(M^c, \Gamma) \vDash p$.

Induction hypothesis: For all φ such that $c(\varphi) \leq n$:

$\varphi \in \Gamma$ iff $(M^c, \Gamma) \vDash \varphi$.

Induction step: Suppose that $c(\varphi) = n + 1$. The cases for $\neg$, $\wedge$, K_a, C_B are just like Lemma 7.17.

- The case for $[\mathrm{M}, s]p$: Suppose that $[\mathrm{M}, s]p \in \Gamma$. Given that $[\mathrm{M}, s]p \in \Phi$, $[\mathrm{M}, s]p \in \Gamma$ is equivalent to $(pre(s) \to p)$ by the atomic permanence axiom. By item 2 of Lemma 7.48, we can apply the induction hypothesis. Therefore this is equivalent to $(M^c, \Gamma) \vDash (pre(s) \to p)$ which is equivalent by semantics to $(M^c, \Gamma) \vDash [\mathrm{M}, s]p$.
- The case for $[\mathrm{M}, s]\neg\chi$: Suppose that $[\mathrm{M}, s]\neg\chi \in \Gamma$. Given that $[\mathrm{M}, s]\neg\chi \in \Phi$, $[\mathrm{M}, s]\neg\chi \in \Gamma$ is equivalent to $(pre(s) \to \neg[\mathrm{M}, s]\chi) \in \Gamma$ by the action and negation axiom. By item 3 of Lemma 7.48, we can apply the induction hypothesis. Therefore this is equivalent to $(M^c, \Gamma) \vDash pre(s) \to \neg[\mathrm{M}, s]\chi$ which is equivalent by semantics to $(M^c, \Gamma) \vDash [\mathrm{M}, s]\neg\chi$.

- The case for $[\mathrm{M}, s](\chi \wedge \xi)$: Suppose that $[\mathrm{M}, s](\chi \wedge \xi) \in \Gamma$. Given that $[\mathrm{M}, s](\chi \wedge \xi) \in \Phi$, $[\mathrm{M}, s](\chi \wedge \xi) \in \Gamma$ is equivalent to $([\mathrm{M}, s]\chi \wedge [\mathrm{M}, s]\xi) \in \Gamma$ by the action and conjunction axiom. By item 4 of Lemma 7.48, we can apply the induction hypothesis. Therefore this is equivalent to $(M^c, \Gamma) \vDash [\mathrm{M}, s]\chi \wedge [\mathrm{M}, s]\xi$ which is equivalent by semantics to $(M^c, \Gamma) \vDash [\mathrm{M}, s](\chi \wedge \xi)$.
- The case $[\mathrm{M}, \mathrm{s}]K_a\chi$: Suppose that $[\mathrm{M}, \mathrm{s}]K_a\chi \in \Gamma$. Given that $[\mathrm{M}, \mathrm{s}]K_a\chi \in \Phi$, $[\mathrm{M}, s]K_a\chi \in \Gamma$ is equivalent to $(pre(s) \rightarrow K_a[\mathrm{M}, t]\chi) \in \Gamma$ for all $t \in \mathrm{M}$ by the action and knowledge axiom. By item 5 of Lemma 7.48, we can apply the induction hypothesis. Therefore this is equivalent to $(M^c, \Gamma) \vDash pre(s) \rightarrow K_a[\mathrm{M}, t]\chi$ which is equivalent by semantics to $(M^c, \Gamma) \vDash [\mathrm{M}, \mathrm{s}]K_a\chi$.
- The case for $[\mathrm{M}, s][\mathrm{M}', s']\xi$: Suppose that $[\mathrm{M}, s][\mathrm{M}', s']\xi \in \Gamma$. Given that $[\mathrm{M}, s][\mathrm{M}', s']\xi \in \Phi$, $[\mathrm{M}, s][\mathrm{M}', s']\xi \in \Gamma$ is equivalent to $[\mathrm{M}, s; \mathrm{M}', s']\xi \in \Gamma$ by the action and composition axiom. By item 7 of Lemma 7.48, we can apply the induction hypothesis. Therefore this is equivalent to $(M^c, \Gamma) \vDash [\mathrm{M}, s; \mathrm{M}', s']\xi$ which is equivalent by semantics to
  $(M^c, \Gamma) \vDash [\mathrm{M}, s][\mathrm{M}', s']\xi$.
- The case for $[\alpha \cup \alpha']\xi$: Suppose that $[\alpha \cup \alpha']\xi \in \Gamma$. Given that $[\alpha \cup \alpha']\xi \in \Phi$, $[\alpha \cup \alpha']\xi \in \Gamma$ is equivalent to $[\alpha]\xi \wedge [\alpha']\xi \in \Gamma$ by the non-deterministic choice axiom. By item 8 of Lemma 7.48, we can apply the induction hypothesis. Therefore this is equivalent to $(M^c, \Gamma) \vDash [\alpha]\xi \wedge [\alpha']\xi$ which is equivalent by semantics to
  $(M^c, \Gamma) \vDash [\alpha \cup \alpha']\xi$.

👍

- Lemma 7.51 (Canonicity):

The canonical model is reflexive, transitive and Euclidean.

- Theorem 7.52 (Completeness) :

For every $\varphi \in \mathcal{L}_{KC\otimes}$

$$\vDash \varphi \text{ implies} \vdash \varphi$$

Proof:

We will prove this theorem by contraposition. Thus we suppose that $\nvdash \varphi$. Therefore $\{\neg\varphi\}$ is a consistent set. By the Lindenbaum Lemma $\{\neg\varphi\}$ is a subset of some Γ which is maximal consistent in $cl(\neg\varphi)$. Let M^c be the canonical model for $cl(\neg\varphi)$. By the Truth Lemma

$(M^c, \Gamma) \vDash \neg\varphi$. Therefore $\nvDash \varphi$.

The proof system S5RC

all instantiations of propositional tautologies	
$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$	distribution of K_a over $\to$
$K_a\varphi \to \varphi$	truth
$K_a\varphi \to K_aK_a\varphi$	positive introspection
$\neg K_a\varphi \to K_a\neg K_a\varphi$	negative introspection
$C_B(\varphi, \psi \to \chi) \to (C_B(\varphi, \psi) \to C_B(\varphi, \chi))$	distribution of $C_B(\cdot, \cdot)$ over $\to$
$C_B(\varphi, \psi) \leftrightarrow E_B(\varphi \to (\psi \wedge C_B(\varphi, \psi)))$	mix of relativised common knowledge
$C_B(\varphi, \psi \to E_B(\varphi \to \psi)) \to (E_B(\varphi \to \psi) \to C_B(\varphi, \psi))$	induction of relativised common knowledge
From φ and $\varphi \to \psi$, infer ψ	modus ponens
From φ, infer $K_a\varphi$	necessitation of K_a
From φ, infer $C_B(\psi, \varphi)$	necessitation of $C_B(\cdot, \cdot)$

- Definition 7.53:

Given are a set of agents A and a set of atoms P. The language $\mathcal{L}_{KRC}$ consists of all formulas given by the following BNF:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_\alpha\varphi \mid C_B(\varphi, \varphi)$$

Where $p \in P, a \in A \; and \; B \subseteq A$.

- Definition 7.54(Semantics):

Given is an epistemic model $M = \langle S, \sim, V \rangle$. The semantics for atoms, negations, conjunctions and individuals operators are as usual.

$(\mathrm{M}, s) \vDash C_B(\varphi, \psi)$ iff $(\mathrm{M}, s) \vDash \psi$ for all t such that $(s, t) \in (\bigcup_{a \in B} \sim_a \cap \, (S \times [\![\varphi]\!]_M))^+$

$$\left(\bigcup_{a \in B} \sim_a \cap \, (S \times [\![\varphi]\!]_M)\right)^+$$

is the transitive closure

- Theorem 7.56:

The proof system S5RC is sound , i.e., if $\vdash \varphi$, then $\vDash \varphi$

- Definition 7.58 (Closure):

Let $cl: \mathcal{L}_{KRC} \rightarrow \wp(\mathcal{L}_{KRC})$, be the function such that for every $\varphi \in \mathcal{L}_{KRC}$, $cl(\varphi)$ is the smallest set such that:

1. $\varphi \in cl(\varphi)$,
2. If $\psi \in cl(\varphi)$, then $Sub(\psi) \subseteq cl(\varphi)$ (where $Sub(\psi)$ is the set of subformulas of ψ),
3. If $\psi \in cl(\varphi)$ and ψ is not a negation, then $\neg\psi \in cl(\varphi)$,
4. If $C_B(\psi, \chi) \in cl(\varphi)$, then $\left\{K_a\left(\psi \rightarrow \left(\chi \wedge C_B(\psi, \chi)\right)\right) | a \in \mathrm{B}\right\} \subseteq cl(\varphi)$

- Lemma 7.59:

$cl(\varphi)$ is finite for all formulas $\varphi \in \mathcal{L}_{KRC}$.

- Lemma 7.60:

Let Φ be the closure of some formula. Let $M^c = (S^c, \sim^c, V^c)$ be the canonical model for Φ. Let Γ be maximal consistent set in Φ. If $C_B(\varphi, \psi) \in \Phi$, then $C_B(\varphi, \psi) \in \Gamma$ iff every B-φ-path from a $\Delta \in S^c$ such that there is an agent $a \in B$ and $\Gamma \sim_a^c \Delta$ is a ψ-path.

- Lemma 7.61 (Truth):

Let Φ be the closure of some formula. Let $M^c = (S^c, \sim^c, V^c)$ be the canonical model for Φ. For all $\Gamma \in S^c$ and all $\varphi \in \Phi$:

$$\varphi \in \Gamma \text{ iff } (M^c, \Gamma) \vDash \varphi$$

Proof : Suppose $\varphi \in \Phi$. We will prove this Lemma by induction on φ. We focus on the case for relativised common knowledge due to the fact that the other cases are the same as in proof of Lemma 7.17.

– The case for $C_B(\varphi, \psi)$: Suppose that $C_B(\varphi, \psi) \in \Gamma$.

From the Lemma 7.60 this is the case iff every B-φ-path from a $\Delta \in S^c$ such that there is an agent $a \in B$ and $\Gamma \sim_a^c \Delta$ is a ψ-path. By induction hypothesis this is the case iff every B-path where φ is true along the path, is a path along which ψ is true. By semantics this is equivalent to $(M^c, \Gamma) \vDash C_B(\varphi, \psi)$.

- Theorem 7.62 (Completeness) :

For every $\varphi \in \mathcal{L}_{KRC}$

$$\vDash \varphi \text{ implies } \vdash \varphi$$

Proof:

We will prove this theorem by contraposition. Thus we suppose that $\nvdash \varphi$. Therefore $\{\neg\varphi\}$ is a consistent set. By the Lindenbaum Lemma $\{\neg\varphi\}$ is a subset of some Γ which is maximal consistent in $cl(\neg\varphi)$. Let M^c be the canonical model for $cl(\neg\varphi)$. By the Truth Lemma

$(M^c, \Gamma) \vDash \neg\varphi$. Therefore $\nvDash \varphi$.

The proof system PARC

all instantiations of propositional tautologies	
$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$	distribution of K_a over $\to$
$K_a\varphi \to \varphi$	truth
$K_a\varphi \to K_aK_a\varphi$	positive introspection
$\neg K_a\varphi \to K_a\neg K_a\varphi$	negative introspection
$C_B(\varphi, \psi \to \chi) \to (C_B(\varphi,\psi) \to C_B(\varphi,\chi))$	distribution of $C_B(\cdot,\cdot)$ over $\to$
$C_B(\varphi,\psi) \leftrightarrow E_B(\varphi \to (\psi \land C_B(\varphi,\psi)))$	mix of relativised common knowledge
$C_B(\varphi, \psi \to E_B(\varphi \to \psi)) \to (E_B(\varphi \to \psi) \to C_B(\varphi,\psi))$	induction of relativised common knowledge
From φ and $\varphi \to \psi$, infer ψ	modus ponens
From φ, infer $K_a\varphi$	necessitation of K_a
From φ, infer $C_B(\psi,\varphi)$	necessitation of $C_B(\cdot,\cdot)$

$[\varphi]p \leftrightarrow (\varphi \to p)$	atomic permanence
$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$	announcement and negation
$[\varphi](\psi \land \chi) \leftrightarrow ([\varphi]\psi \land [\varphi]\chi)$	announcement and conjunction
$[\varphi]K_a\psi \leftrightarrow (\varphi \to K_a[\varphi]\psi)$	announcement and knowledge
$[\varphi]C_B(\psi,\chi) \leftrightarrow (\varphi \to C(\varphi \land [\varphi]\psi, [\varphi]\chi))$	announcement and relativised common knowledge
$[\varphi][\psi]\chi \leftrightarrow [\varphi \land [\varphi]\psi]\chi$	announcement composition
From ψ, infer $[\varphi]\psi$	necessitation of $[\varphi]$

- Definition 7.63:

Given are a set of agents A and a set of atoms P. The language $\mathcal{L}_{KRC[]}$ consists of all formulas given by the following BNF:

$$\varphi ::= p| \neg\varphi \mid (\varphi \wedge \varphi) \mid K_{\alpha}\varphi \mid C_B(\varphi, \varphi) \mid [\varphi]\varphi$$

Where $p \in P, a \in A \; and \; B \subseteq A$.

- Lemma 7.65:

$\models [\varphi]C_a(\psi, \chi) \leftrightarrow \varphi \rightarrow C_a(\varphi \wedge [\varphi]\psi, [\varphi]\chi)$

- Definition 7.66 (Translation):

The translation $t: \mathcal{L}_{KRC[]} \to \mathcal{L}_{KRC}$ is defined as follows:

$t(p) = p$
$t(\neg\varphi) = \neg t(\varphi)$
$t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi)$
$t(K_a\varphi\ ) = K_a t(\varphi)$
$t([\varphi]p) = t(\varphi \to p)$
$t([\varphi]\neg\psi) = t(\varphi \to \neg[\varphi]\psi)$
$t\big([\varphi](\psi \wedge \chi)\big) = t\big([\varphi](\psi) \wedge [\varphi]\chi)\big)$
$t([\varphi]K_a\psi) = t(\varphi \to K_a[\varphi]\psi)$
$t\big([\varphi]C_a(\psi, \chi)\big) = t\big(\varphi \to C_a(\varphi \wedge [\varphi]\psi, [\varphi]\chi)\big)$
$t([\varphi][\psi]\chi) = t([\varphi \wedge [\varphi]\psi]\chi)$

- Definition 7.67 (Complexity):

The complexity $c : \mathcal{L}_{KRC[]} \to \mathbb{N}$ is defined as follows:

$c(p) = 1$
$c(\neg\varphi) = 1 + c(\varphi)$
$c(\varphi \wedge \psi) = 1 + \max\big(c(\varphi), c(\psi)\big)$
$c(K_a\varphi) = 1 + c(\varphi)$
$c(C_B(\varphi, \psi)) = 1 + \max\big(c(\varphi), c(\psi)\big)$
$c([\varphi]\psi) = \big(5 + c(\varphi)\big) \cdot c(\psi)$

- Lemma 7.68:

For all φ, ψ and χ:

1. $c(\psi) \geq c(\varphi)$ if $\varphi \in Sub(\psi)$
2. $c([\varphi]p) > c(\varphi \to p)$
3. $c([\varphi]\neg\psi) > c(\varphi \to \neg[\varphi]\psi)$
4. $c\big([\varphi](\psi \wedge \chi)\big) > c([\varphi]\psi \wedge [\varphi]\chi)$
5. $c([\varphi]K_a\psi) > c(\varphi \to K_a[\varphi]\psi)$
6. $c\big([\varphi]C_B(\psi, \chi)\big) > c\big(\varphi \to C_B(\varphi \wedge [\varphi]\psi, [\varphi]\chi)\big)$
7. $c([\varphi][\psi]\chi) > c([\varphi \wedge [\varphi]\psi]\chi)$

Proof:

$$c([\varphi]C_B(\psi,\chi)) > c(\varphi \to C_B(\varphi \land [\varphi]\psi, [\varphi]\chi))$$

Assume without loss of generality, that $c(\psi) \geq c(\chi)$. Then :

$$c([\varphi]C_B(\psi,\chi)) = (5 + c(\varphi))\left(1 + \max(c(\psi), c(\chi))\right)$$

$$= 5 + c(\varphi) + 5\max(c(\psi), c(\chi)) + c(\varphi)\max(c(\psi), c(\chi))$$

$$= 5 + c(\varphi) + 5c(\psi) + c(\varphi)c(\psi)$$

And

$$c(\varphi \to C_B(\varphi \land [\varphi]\psi, [\varphi]\chi)) = c\left(\neg(\varphi \land \neg C_B(\varphi \land [\varphi]\psi, [\varphi]\chi))\right)$$

$$= 2 +$$

$$\max\left(c(\varphi), 1 + 1 + \max\left(1 + \max\left(c(\varphi), (5 + c(\varphi))c(\psi)\right), (5 + c(\varphi))c(\chi)\right)\right)$$

$$= 5 + \left((5 + c(\varphi))c(\psi)\right)$$

$$= 5 + 5\,c(\psi) + c(\varphi)\,c(\psi)$$

- <u>Lemma 7.69:</u>

For all formulas $\varphi \in \mathcal{L}_{KRC[]}$ it is the case that

$\vdash \varphi \leftrightarrow t(\varphi)$

<u>Proof:</u>

Is similar to the proof of Lemma 7.24

- <u>Theorem 7.70 (Completeness):</u>

For every $\varphi \in \mathcal{L}_{KRC[]}$ $\vDash \varphi$ implies $\vdash \varphi$

<u>Proof:</u>

Suppose that $\vDash \varphi$. Therefore $\vDash t(\varphi)$ (by the soundness) and by Lemma 7.69 holds that $\vdash \varphi \leftrightarrow t(\varphi)$. Because of the fact that $t(\varphi)$ doesn't contain any announcement operators, $S5RC \vdash t(\varphi)$ (Theorem 7.62) We also have that $\mathrm{PARC} \vdash t(\varphi)$ as $S5RC$ is subsystem of PARC. Since $\mathrm{PARC} \vdash \varphi \leftrightarrow t(\varphi)$, it follows that $\mathrm{PARC} \vdash \varphi$.