The Diffie-Hellman Key Exchange

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Algorithms and Complexity

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A cryptographic system is a single parameter family $\{S_K\}_{K\in\{K\}}$ of invertible transformations

$$S_K : \{P\} \to \{C\}$$

for a space $\{P\}$ of plaintext messages to a space $\{C\}$ of ciphertext messages. The parameter K is called the key and is selected from a finite set $\{K\}$ called the keyspace.

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The Caeser cipher with $\{P\} = \{C\} = \Sigma^*$ and $\{K\} = \Sigma$ uses the invertible transformations

$$S_K(a_1 a_2 \ldots a_n) = (a_1 + K \mod 26)(a_2 + K \mod 26) \ldots (a_n + K \mod 26).$$

Similarly, the One-Time-Pad with $\{P\} = \{C\} = \{K\} = \Sigma^*$ uses keys as long as the plaintext:

$$S_K(a_1 a_2 \ldots a_n) = (a_1 + k_1 \mod 26)(a_2 + k_2 \mod 26) \ldots (a_n + k_n \mod 26),$$

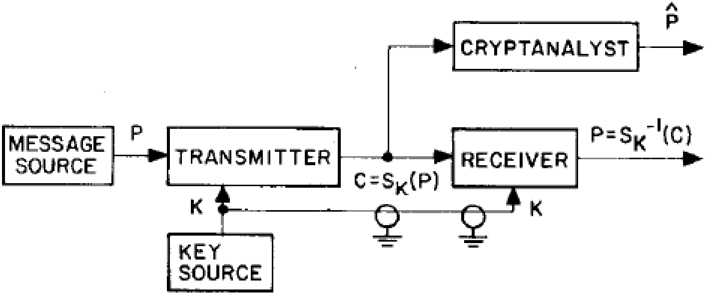
where $K = k_1 k_2 \ldots k_n$.

Classifying the threats:

- Ciphertext-only attack: totally insecure systems
- Known plaintext attack: not secure in case of later public disclosure
- Chosen plaintext attack: allows to opponents to plant messages

Private Key Exchange

A basic problem in conventional cryptography is to ensure a secure communication via a private channel



Private Key Exchange

Due to physical constraints, this is infeasible when developing large, secure, telecommunication systems.

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- immediate, fast communication
- communication of unknown parties
- n different users who wish to communicate privately from others

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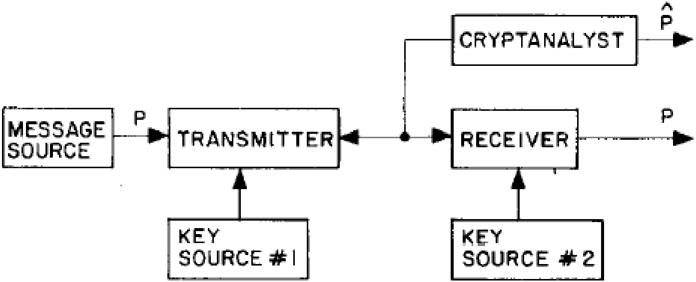
CRYPTANALYST
$\hat{P}$
MESSAGE SOURCE
P
TRANSMITTER
RECEIVER
P
KEY SOURCE #1
KEY SOURCE #2

A public key cryptosystem is a pair of families $\{E_K\}_{K\in\{K\}}, \{D_K\}_{K\in\{K\}}$ of algorithms representing invertible transformations:

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- for almost every $K \in \{K\}$, each easily computed algorithm equivalent to D_K is computationally infeasible to derive from E_K,
- for every $K \in \{K\}$ it is feasible to compute inverse pairs E_K and D_K from K.

Each user generates a pair of inverse transformations, E and D. The enciphering algorithm E can de made public, while the deciphering transformation D must be kept secret.

A naive example

Let the message to be enciphered be a binary n-vector $\mathbf{m}$, and E be an invertible $n \times n$ matrix, i.e., $\mathbf{c} = E\mathbf{m}$. Then $D = E^{-1}$ and $\mathbf{m} = D\mathbf{c}$.

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- Calculating D from E (matrix inversion) is a harder problem (but not hard enough!$\rightarrow$ n^3 operations)
- It is simpler to obtain a pair of inverse matrices than it is to invert a given matrix

Example: RSA

Let $K = (p, q)$, where p, q are prime numbers (secret), $N = pq$ (public) and $\{M\} = \mathbb{Z}_N^*$.
Then

$$E_K : \mathbb{Z}_N^* \to \mathbb{Z}_N^*, m \mapsto m^e \mod N,$$

where $e < N$ and

$$D_K : \mathbb{Z}_N^* \to \mathbb{Z}_N^*, c \mapsto c^d \mod N,$$

for some d such that $ed = 1 \mod \phi(N)$.

Example: RSA

- E_K is the inverse of D_K:

$$D_K(E_K(m)) = E_K(m)^d \mathrm{mod} N = (m^e)^d \mathrm{mod} N = m^{ed} \mathrm{mod} N = m$$

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- It is feasible to find e, d such that $ed = 1 \mod \phi(pq)$

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Require: $a, n, b_0, b_1, \ldots, b_{k-1}$ such that $m = (b_{k-1} \ldots b_1 b_0)_2$

$x := a$
$y := 1$
for $i = 0, \ldots k-1$ **do**
    if $b_i = 1$ **then**
        $y := y \cdot x \mod n$
        $x := x^2 \mod n$
    end if
end for
return y

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The Diffie-Hellman Protocol

A and B wish to exchange a private key via a public, insecure channel, in order to use a symmetric cryptosystem. Let q be a prime number and $a \in \{2, \ldots, q-1\}$. q and a are agreed upon by A and B, and they can be made public.

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- A computes $K = (Y')^X \mod q = a^{X'X} \mod q$
- B computes $Y^{X'} \mod q = a^{XX'} \mod q = K$

Both A and B have the same key K.

The Diffie-Hellman Protocol

Since any other user does not possess neither X nor X', they should be able to compute either $\log_a Y \mod q = X$ or $\log_a Y' \mod q = X'$, which is computationally infeasible:

The Discrete Logarithm Problem is in **NP**.

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Proof.

$$q \text{ is a prime and } a \text{ is a primitive root} \mod q$$

$$\text{iff}$$

$$\mathrm{ord}_q(a) = q - 1.$$

We can verify these conditions in non-deterministic polynomial time:

- guess all prime factors $p_1, \ldots, p_k$ of $q - 1$,
- use primality test for all of them,
- compute $(q-1)/p_i$ and verify that $a^{(q-1)/p_i} \not\equiv 1 \mod q$ for all p_i.

So, given q, a, n, m we can verify whether q is a prime and a a primitive root and if $a^m \equiv n \mod q$. This means that DLP is in **NP**.
On the other hand, if $a^m \not\equiv n \mod q$, then

- there are i, j with $0 < i, j < q$ such that $a^i \equiv a^j \mod q$ or
- there is an $\ell \leq m$ such that $a^\ell \equiv n \mod q$ or
- $n \geq q$

All of the above can be checked in polynomial time, so DLP is in **coNP**.
We have that DLP is in $\mathbf{NP} \cap \mathbf{coNP}$. □

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Conventional Cryptography

- Written signature $\rightarrow$ digital signature

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Digital signature must be recognizable without being known

The "login" problem

- User enters password PW
- Computer computes and stores a function $f(PW)$
- Each time a login with X is attempted, computer calculates $f(X)$ and compares with stored value $f(PW)$

Computation time of f must be small, computation of f^{-1} must be practically infeasible.

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Still not entirely secure

Public Key Cryptography

- A wants to send a message M to B
- A deciphers the message $D_K(M)$ using his (secret) algorithm D_K and sends it to B, along with M.
- B enciphers the message $E_K(D_K(M))$ using A's (public) algorithm E_K.
- Obviously, B obtains M and can therefore be assured of the authenticity of its sender.

Example: RSA digital signature

Let $E(x) = x^e \mod N$ be A's public encryption algorithm, and $D(y) = y^d \mod N$ their secret decryption algorithm.

- A sends (m, m') to B, where $m' = m^d \mod N$.
- B computes $E(m') = (m')^e \mod N = (m^d)^e \mod N = m$, and therefore recognizes the authenticity of the sender.

Conversely, suppose that a third user C claims to be A. Since C has no knowledge of d, he sends to user B (m, m'), where $m' = m^{d'} \mod N$ for some $d' \neq d$. B computes $E(m') = (m')^e \mod N = (m^{d'})^e \mod N \neq m$ and sees through C's forgery.

Bibliography

- W. Diffie, M. E. Hellman. New Directions in Cryptography. *IEEE Transactions of Information Theory*, Vol. IT-22, No.6 (1976): 644-654
- J. Katz, Y. Lindell. *Introduction to Modern Cryptography*. CRC Press, 2nd edition, 2015.
- G. Brassard, (1979). A note on the complexity of cryptography (Corresp.). *IEEE Transactions on Information Theory*, 25(2), 232–233. doi:10.1109/tit.1979.1056010
- B. Schneier. 1995. *Applied Cryptography: Protocols, Algorithms, and Source Code in C* (2nd. ed.). John Wiley & Sons, Inc., USA.

Thank You!