An introduction to Quantum Complexity

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Advanced Topics on Algorithms and Complexity
$\mu\Pi\lambda\forall$

July 3, 2014

Outline

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ECT Thesis

Goal of computational complexity:

classify problems according to *amount* of resources needed for solving them

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Any "reasonable" model of computation can be efficiently simulated on a probabilistic Turing Machine or random access machine.

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Any "reasonable" model of computation can be efficiently simulated on a probabilistic Turing Machine or random access machine.

However, there is evidence that ECT doesn't hold for the quantum world.
Why?
Turing Machine is based on a classical physics model of the Universe, whereas current physical theory asserts that the Universe is quantum physical.

Evidence and Meaning

Some evidence:

- **Feynman '82:** it's not clear how to simulate a quantum system on a computer without exponential penalty
- **Bernstein & Vazirani '97:** relative to an oracle, quantum poly-time properly contains probabilistic poly-time
- **Simon '97:** relative to an oracle, quantum poly-time is not contained in subexponential probabilistic time
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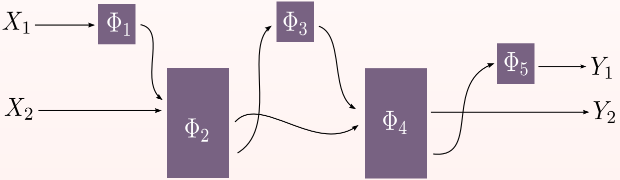
So, one of the following must hold:

- ECT thesis is false
- Quantum Physics is false
- Our picture of computational complexity theory is false

Outline

Quantum Circuit Model

A **quantum circuit** is an acyclic network of quantum gates connected by qubit wires. For example:



- convenient model when study the complexity of quantum computation
- acyclic to preserve time ordering of things
- introduced by Deutsch in '85

Qubit: intuition

Qubit is the basic unit of quantum information. Some math intuition:

An event with n possible outcomes is a vector in $\mathbb{R}^n$: $v = (p_1, \ldots, p_n)$

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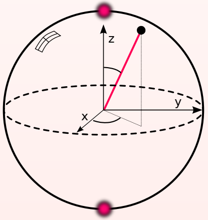
why not use 2-norm?

- vector $v' = (a, b)$ where $a, b \in \mathbb{C}$
- $\|v'\|_2 = 1 \Rightarrow a^2 + b^2 = 1$
- operation: unitary matrix $(U^H U = I)$



Qubit

Qubit is a 2D quantum system in Hilbert Space C^2

- **basis** of C^2: $(0,1)$ and $(1,0)$
- **state** of qubit: vector in C^2
- **Dirac notation:** $\psi = (a,b) \Longrightarrow |\psi\rangle = a|0\rangle + b|1\rangle$

Properties of qubits:

- Normalization: $|a|^2 + |b|^2 = 1 = \langle\psi|\psi\rangle$
- Superposition: linear combination
- Measurement: state collapses irreversibly to one of the basis states
- Non-Clonability: cannot copy unknown quantum state
- Entanglement: see in a while

Physical implementation:

- electron spin
- photon polarization etc.

2 Qubits

- space now is $C^2 \otimes C^2$
- 4 basis states: $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$
- 2-qubit state: $|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$
- $\sum_{x \in \{0,1\}^2} |a_x|^2 = 1$

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- Entanglement: perfect (anti) correlation
- n-qubit state is a linear superposition of 2^n basis states
- Huge computational power of quantum computers!

Quantum Gates

- quantum operations: unitary matrices
- search for a pattern in superposition
- rotate Hilbert space
- same number of input and output qubits
- reversible: no info is lost
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Solovay-Kitaev theorem

Informally: any universal gate set can be simulated by another universal gate set with only a polynomial increase of gates.

Universal gate set

Hadamard Gate

$$|a\rangle \;—\boxed{H}—\; H\,|a\rangle$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Longrightarrow H|a\rangle = \frac{|0\rangle + (-1)^a |1\rangle}{\sqrt{2}}$$

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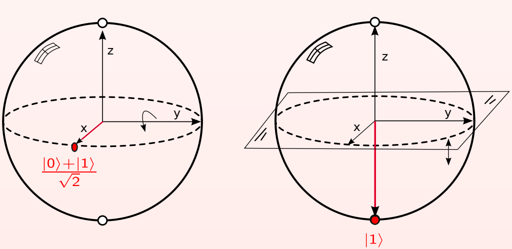
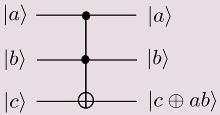


Universal gate set

Toffoli Gate



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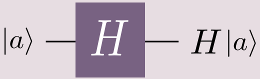

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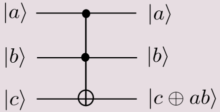

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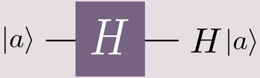


$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Longrightarrow H|a\rangle = \frac{|0\rangle + (-1)^a |1\rangle}{\sqrt{2}}$$

- proved to be quantum universal by Shi, '02
- real entries -- how approximate complex unitary matrices?
- no strict but computational universality
- can be used for fault tolerant purposes

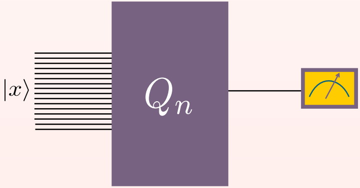
Poly-time Quantum Algorithms

Definition

In the quantum circuit model, a **quantum algorithm** Q is described by a family of quantum circuits

$$Q = \{Q_n : n \in \mathbb{N}\}$$

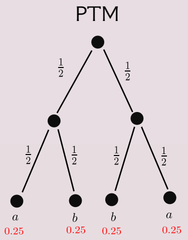
- We require that such a family is poly-time uniform
- To run this algorithm on input $x \in \{0,1\}^n$ we apply Q_n to $|x\rangle$ and measure the output in the standard basis:



- $Q(x)$ denotes the outcome, which is a random variable in general.

QTM: informal

- Reminder: internal state of PTM changes in a probabilistic way
- **description** of configurations: a probability vector $\vec{p}$
- step of computation: $M \cdot \vec{p} = \vec{q}$ where M is a stochastic matrix.
- QTM is the same
- just change M to be unitary and $\vec{p}$ to be 2-norm unit vector



$\Pr(a) = 0.5$

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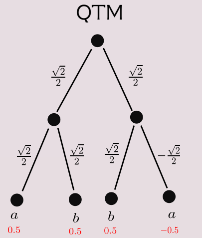

$$\Pr(a) = |\sum_{i:a} c_i|^2 = |\tfrac{1}{2} - \tfrac{1}{2}|^2 = 0$$

$$\Pr(b) = |\sum_{i:b} c_i|^2 = |\tfrac{1}{2} + \tfrac{1}{2}|^2 = 1$$

QTM: formal

Quantum Turing Machine [Deutsch, '85]

A QTM is defined by a triplet (Σ, Q, δ), where Σ is the alphabet, Q is a finite set of states and δ is the quantum transition function

$$\delta : Q \times \Sigma \longrightarrow \tilde{C}^{\Sigma \times Q \times D}$$

with $D = \{L, R\}$ and $\tilde{C}$ the set of "efficiently computable" complex numbers.

- each state of QTM is a linear combination $\sum_c a_c|c\rangle$ of all classical configurations $c = |a, q, m\rangle$ (tape content, state, head position)
- $\delta(p, \sigma)$ gives a superposition of all possible (finite) configs which the machine will take when in state p reading a σ.
- so δ is like a unitary matrix

Query Complexity Model

- Almost all quantum algorithms operate in the **query complexity model**.
- In this model, input is not a bit-string but a "black box" computing some function $f : \{0,1\}^n \to \{0,1\}$ which returns $f(x)$ when x is passed in.
    - put it in quantum words:

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- Algorithm can also apply arbitrary unitary transformations as long as values of f are not involved in their definitions.
- Pros: if there is a circuit simulating U_f just plug it in and return to computational complexity model.
- Cons: quantum-classical separations are relative to an oracle.

Deutsch's Algorithm

- initially proposed by David Deutsch in '85 - improved by Cleve, Ekert, Macchiavello, and Mosca in '92
- combines quantum parallelism with *interference*

Deutsch's Problem

given $f : \{0,1\} \to \{0,1\}$ we wish to compute $f(0) \oplus f(1)$

- classical query complexity is 2
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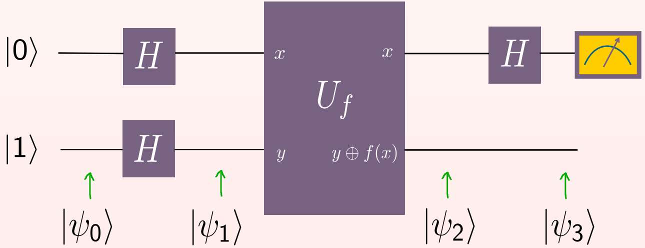

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  - Observe that: $U_f|x\rangle H|1\rangle = (-1)^{f(x)}|x\rangle H|1\rangle$ (remember phase oracle)
  - So: $U_f|\psi_1\rangle = \frac{(-1)^{f(0)}|0\rangle+(-1)^{f(1)}|1\rangle}{\sqrt{2}} H|1\rangle$
  - Therefore: $|\psi_2\rangle = \begin{cases} \pm\left[\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right] H|1\rangle & \text{if } f(0) = f(1) \\ \pm\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right] H|1\rangle & \text{if } f(0) \neq f(1) \end{cases}$

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- interference: just apply Hadamard gate to first qubit
  - $|\psi_3\rangle = \begin{cases} \pm|0\rangle H|1\rangle & \text{if } f(0)=f(1) \\ \pm|1\rangle H|1\rangle & \text{if } f(0)\neq f(1) \end{cases}$
  - notice that if $f(0)=f(1)$ then $f(0)\oplus f(1) = 0$
  - finally: $|\psi_3\rangle = |f(0)\oplus f(1)\rangle\, H|1\rangle \Rightarrow$ just measure first qubit!

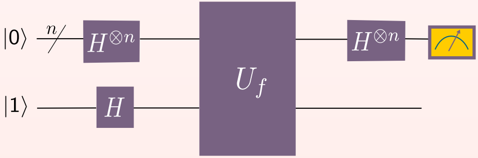
Some query complexity separations (1)

- We've seen only a 2-speedup factor in computing the XOR of n qubits
- Is there a bigger quantum-classical gap?

Deutsch-Jozsa Problem

We have a function $f : \{0,1\}^n \to \{0,1\}$ which is either constant or balanced (0 for half the inputs, 1 for the other half). The goal is to find out what it is.

- in classical world, we need $2^{n-1}+1$ queries (error prob. is not allowed)
- in quantum world, a generalization of prev. algorithm uses only 1 query



Some query complexity separations (2)

Simon's Problem

We have a function $f:\{0,1\}^n \to \{0,1\}^n$ and we are promised that there exists a "secret XOR mask" $s \in \{0,1\}^n$ s.t. $f(x) = f(y) \Leftrightarrow y = x \oplus s$ for all distinct (x,y) pairs. The goal is to find out the identity of s.

- Deutsch's Problem is a special case for $n = 1$.
- Classically, we know that any algorithm in the query model (even with error probability at most ϵ) will make $\Omega(\sqrt{2^n \log \frac{1}{\epsilon}})$ queries.
- Quantumly, it can be solved with $O(n \log \frac{1}{\epsilon})$ queries.

So, in the query complexity model, there are quantum algorithms which *do* achieve an exponential separation between quantum and classical.

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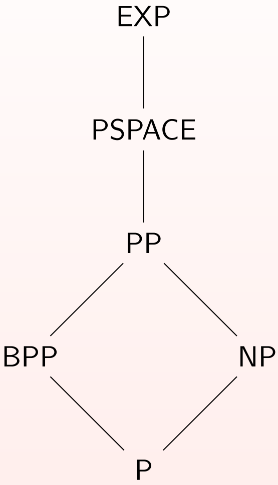
Recap



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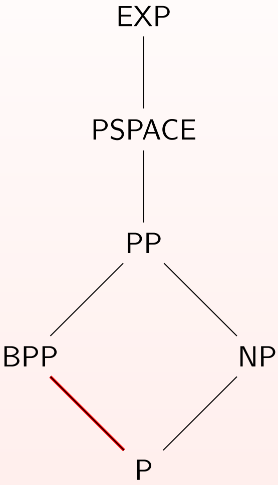

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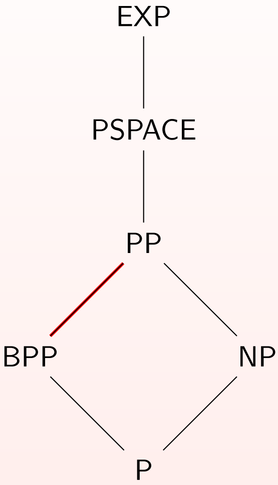

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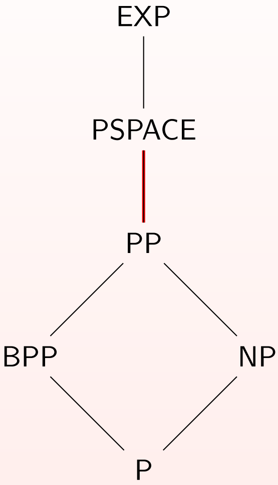

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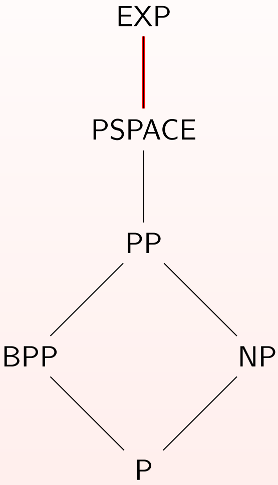

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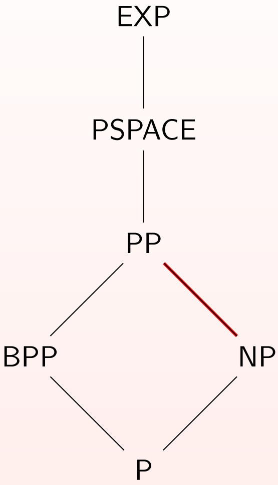


BQP

- One of the fundamental classes in quantum complexity.
- It describes what we can efficiently solve with a quantum computer.

Definition

BQP: is the class containing all languages $L \subset \{0,1\}^*$ for which there exists a poly-time uniform family $Q = \{Q_n : n \in \mathbb{N}\}$ of quantum circuits s.t. for all inputs x it holds that:

$$x \in L \Rightarrow \Pr[Q(x) = 1] \geq 2/3$$

$$x \notin L \Rightarrow \Pr[Q(x) = 0] \geq 2/3$$

- Error reduction: just like BPP, repeat computation and take majority vote
- Assumption: circuits use gates form a universal gate set
- Auxiliary qubits are bounded by some polynomial q:

$$Q(x) = Q(|x\rangle|0\rangle^{\otimes q(n)})$$

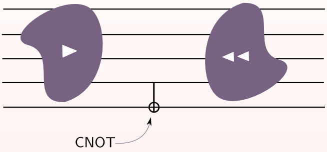
Some Structural Properties of BQP

1. BQP is closed under complement
2. BQP is closed under intersection (and union)
3. BQP is low for itself, meaning $\mathrm{BQP}^{\mathrm{BQP}} = \mathrm{BQP}$

- If you can't prove 1. and 2. by now, then I completely failed to attract your interest ☹
- The proof about 3. is like that of BPP with one exception:
  - when a quantum algo terminates, we measure only the output qubit
  - all other qubits are considered as garbage
  - so when we replace BQP oracle with a BQP subroutine, we have some subroutine garbage left
  - in case of pure states, we just throw them away
  - but in case of mixed states, they may annoy interference
  - what can we do to avoid this?

Uncomputing

1. play subroutine
2. copy answer qubit to separate location
3. rewind subroutine



- solution proposed by Bennett in the '80s
- quantum mechanics cleans its mess
- if subroutine has some error probability, it won't erase everything
  - solution: apply probability amplification in the subroutine part

BPP $\subseteq$ BQP

How can we simulate randomness?

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- whenever a BQP machine wants to flip a coin, just apply a Hadamard gate on input $|0\rangle$ and you'll have a random source for 0 and 1.

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- make classical logic gates reversible: e.g. a Toffoli gate can simulate a NAND gate, which is universal in the classical set
- By **Solovay-Kitaev** theorem, with a universal quantum gate set we can approximate efficiently any other unitary transformation: simulating arbitrary gates up to exponentially small error, costs only a polynomial overhead

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So, a quantum computer is at least as powerful as a classical one.

BQP $\subseteq$ EXP

- We've seen that a quantum state is $|\psi\rangle = \sum_i a_i |i\rangle$ where $i \in \{0,1\}^n$
- so, this state vector moves inside an exponential space
- to simulate with a classical computer the evolution of this vector, exponential time should suffice
- conclusion: quantum computers can offer no more than an exponential advantage over classical ones.
- can we find better lower bound?

BQP $\subseteq$ PSPACE [Bernstein & Vazirani '93]

Basic Idea: integrating over computational paths

- We have a language $L \in$ BQP.
- So, there exists a BQP machine $\mathcal{M}$ that decides L within time $p(n)$, for some polynomial p and input $x \in \{0,1\}^n$.
- The tree of the computation has depth $p(n)$.
- For now, let the transition amplitudes be computed in polynomial time (and therefore in polynomial space).
- For each path on the tree:
  - If path ends up accepting, add its amplitude to a running total.
  - Reuse space an repeat process for all paths ($2^{p(n)}$).
- We conclude that the total amplitude needs poly-space to be stored.
- If we square it, we get the probability that $\mathcal{M}$ accepts.
- So $L \in$ PSPACE.

BQP $\subseteq$ PSPACE [Bernstein & Vazirani '93]

How to remove the assumption?

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How to remove the assumption?

- Given an arbitrary $\mathcal{M}' \in \mathrm{BQP}$, Bernstein & Vazirani showed it suffices to use a similar machine $\mathcal{M}''$ that its transition amplitudes can be exactly calculated.
- If the amplitude of $\mathcal{M}''$ is at least $\frac{7}{12}$ we accept, otherwise we reject.
- They proved that this simulation requires polynomial space.

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Some backstage notes:

- In a universal gate set, each gate operates in a bounded number of qubits.
- a complex number is represented by two integers (one for the real, and one for the imaginary part) with some accuracy they fix.

BQP $\subseteq$ PP [Adleman, Demarrais & Huang '97]

- Like before, proof is based on Feynman path integral.
- Let S be the set of basis states where the output qubit will be $|1\rangle$ (accepting states)
- for each $|x\rangle \in S$ loop over all paths that contribute amplitude to it:
  - the total amplitude of $|x\rangle$ is $a_x = \sum_i a_{x,i}$
  - each $a_{x,i}$ is the amplitude of a path that has $|x\rangle$ as its leaf.
- So $\mathsf{P}_{\text{accept}} = \sum_{x \in S} |\sum_i a_{x,i}|^2 = \sum_{x \in S} \sum_{i,j} a_{x,i} \cdot a^*_{x,j}$
- This is a sum of exponentially many terms, where each term can be computed in poly-time.
- Recall the definition of PP: in order to decide a language, such a machine take the sum of exponentially many terms and decides if it's above or below some threshold.

BQP $\subseteq$ PP [Adleman, Demarrais & Huang '97]

- Let $L \in$ BQP.
- Non deterministically guess x, i, j.
  - If $a_{x,i} \cdot a^*_{x,j} > 0$ then make $|$accepting paths$| \sim |a_{x,i} \cdot a^*_{x,j}|$.
  - If $a_{x,i} \cdot a^*_{x,j} < 0$ then make $|$rejecting paths$| \sim |a_{x,i} \cdot a^*_{x,j}|$.
  - If $a_{x,i} \cdot a^*_{x,j} = 0$ then $|$accepting paths$| \sim |$rejecting paths$|$.
- Notice $x \in L \Rightarrow \mathsf{P}_{\text{accept}} \geq \frac{2}{3} > \frac{1}{2}$ and $x \notin L \Rightarrow \mathsf{P}_{\text{accept}} \leq \frac{1}{3} < \frac{1}{2}$
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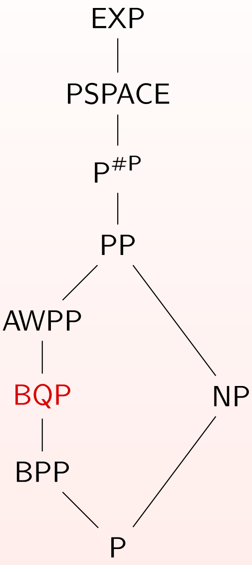
- So, $L \in PP$.

Further Notes:

- Best classical upper bound: BQP ⊆ AWPP [Fortnow & Rogers '99].
- They also showed that BQP is low for PP.
- Scott Aaronson, via "post-selection", proved that PostBQP=PP.

What we've seen so far



$\mathrm{BQP} \stackrel{?}{\neq} \mathrm{BPP}$

- If $\mathrm{BQP} \neq \mathrm{BPP}$ then a quantum computer would be more powerful than a classical one.
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- If $\mathrm{BQP} \neq \mathrm{BPP}$ then a quantum computer would be more powerful than a classical one.
- Furthermore, that would imply that $\mathrm{P} \neq \mathrm{PSPACE}$.

- Recall Simon's algorithm (find hidden XOR mask s): does it prove that $\mathrm{BQP} \neq \mathrm{BPP}$?
- No! Due to its black box formulation, it only proves that there is an oracle A for which it holds $\mathrm{BQP}^A \neq \mathrm{BPP}^A$.
- still lack of formal evidence..

what about NP?

- Let's say we have a space of 2^n possible solutions and we are looking for the right one.
- Assume we are in the query model, where we feed a black box oracle with a solution and it replies if it's correct.
- Classically, we need $\sim 2^{n-1}$ queries on average.
- Quantumly, Grover's algorithm makes $2^{n/2}$ queries.
- Actually, Bennett et al. proved that this result is optimal.
- So, for "unstructured" search problems, quantum computers give only quadratic speedup!

- We don't know if $\mathsf{NP} \not\subseteq \mathsf{BQP}$ (unrelativised)
- We don't even know $\mathsf{P} \neq \mathsf{NP} \Rightarrow \mathsf{NP} \not\subseteq \mathsf{BQP}$.
- Abrams & Lloyd in '98 proved that if we remove linearity from quantum mechanics then quantum computers can solve NP-complete problems.

Outline

Recap

NP: A promise problem A is in NP iff there exists:

1. a polynomial p
2. a poly-time deterministic V s.t.

Completeness: if $x \in A_{\text{yes}}$, then $\exists y \; |y| = p(|x|)$ s.t. $V(x,y) = 1$

Soundness: if $x \in A_{\text{no}}$, then $\forall y \; |y| = p(|x|)$ it holds that $V(x,y) = 0$

MA: A promise problem A is in MA iff there exists:

1. a polynomial p
2. a poly-time probabilistic V s.t.

Completeness: if $x \in A_{\text{yes}}$, then $\exists y \; |y| = p(|x|)$ s.t. $\Pr[V(x,y) = 1] \geq \frac{2}{3}$

Soundness: if $x \in A_{\text{no}}$, then $\forall y \; |y| = p(|x|)$ it holds that $\Pr[V(x,y) = 0] \geq \frac{2}{3}$

QMA

- The natural quantum analogue of NP is actually the quantum analogue of MA
- name QMA was coined by Watrous
- briefly: make V quantum and allow proof to be a quantum state

QMA$_p$: A promise problem A is in QMA$_p$ iff there exists:

1. a polynomial p
2. a family $Q = \{Q_n : n \in \mathbb{N}\}$ of quantum circuits s.t.

Completeness: if $x \in A_{\text{yes}}$, then $\exists$ state ρ on $p(|x|)$ qubits s.t. $\Pr[Q(x,y) = 1] \geq \frac{2}{3}$

Soundness: if $x \in A_{\text{no}}$, then $\forall$ state ρ on $p(|x|)$ qubits $\Pr[Q(x,y) = 0] \geq \frac{2}{3}$

- QMA $= \bigcup_p$ QMA$_p$
- QMA is unrealistic because ρ may be difficult to prepare
- but the point of QMA is quantum verification
- QCMA is like QMA but Merlin is classical.

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- $\mathsf{QCMA} \subseteq \mathsf{QMA}$: a classical Merlin can be simulated by a quantum one
  - We don't know if $\mathsf{QCMA} \neq \mathsf{QMA}$ (not even relativized)

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- QMA $\subseteq$ NEXP: Arthur simulates all witness states that Merlin could send
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- QCMA $\subseteq$ QMA: a classical Merlin can be simulated by a quantum one
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Quantum Oracle Separation [Aaronson & Kuperberg, '07]

There exists a quantum oracle A s.t. $\mathrm{QCMA}^A \neq \mathrm{QMA}^A$

Play with QMA's conditions

Perfect Completeness

- $\mathsf{MA} = \mathsf{MA}_1$ (Zachos & Fürer, '87)
- $\mathsf{QMA} \stackrel{?}{=} \mathsf{QMA}_1$
  - $\exists$ quantum oracle A s.t. $\mathsf{QMA}_1^A \subset \mathsf{QMA}^A$ [Aaronson, '09]
  - we need a quantumly nonrelativizing proof
- $\mathsf{QCMA} = \mathsf{QCMA}_1$ [Jordan, Kobayashi, Nagaj & Nishimura, '12]

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Perfect Soundness

- if perfect soundness, then we have NQP [Kobayashi, Matsumoto & Yamakami, '08]
- NQP is the quantum analogue of probabilistic characterization of NP
- QMA is the quantum analogue of quantifier characterization of NP
- $\mathsf{NQP} = co\mathsf{C}_{=}\mathsf{P}$ [Yamakami & Yao, '99]

Error reduction of QMA

- if we copy the quantum proof it will be damaged
- no need for a fresh copy each time - find another Verifier

Strong error reduction of QMA [Marriott & Watrous, '04]

For any choice of p and completeness and soundness probabilities a and b with $a(n) - b(n) \geq \frac{1}{q(n)}$ for some polynomial q, it holds that $\forall$ polynomial r

$$\mathrm{QMA}_p(a,b) = \mathrm{QMA}_p(1-2^{-r}, 2^{-r})$$

- we can make r bigger than p
- error will be smaller than the reciprocal of Hilbert space dimension

QMA $\subseteq$ PP

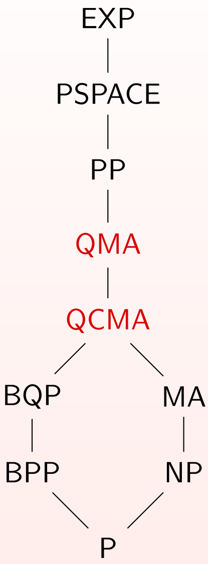
- Let $L \in \mathsf{QMA}$
- So for some p we have $L \in \mathsf{QMA}_p(\frac{2}{3}, \frac{1}{3})$
- by strong reduction $L \in \mathsf{QMA}_p(1 - \frac{1}{2^{p+2}}, \frac{1}{2^{p+2}})$

We consider the following algorithm:

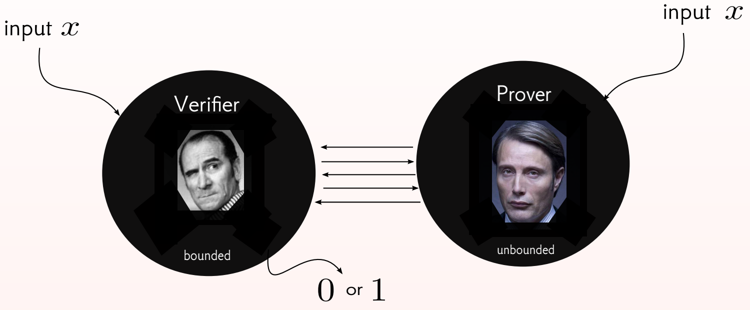
1. randomly guess a quantum proof on p qubits
2. feed this proof to a Verifier $V \in \mathsf{BQP}(1 - \frac{1}{2^{p+2}}, \frac{1}{2^{p+2}})$

- $\forall x \in L$ V accepts w.p. $\geq \frac{1}{2^{p(|x|)+1}}$
- $\forall x \notin L$ V accepts w.p. $\leq \frac{1}{2^{p(|x|)+2}}$
- V is not good but gives tiny amount of info about the correct answer
- $V \in \mathsf{PQP}$ (quantum analogue of PP)
- $\mathsf{PQP} = \mathsf{PP}$ [Watrous, '09]
- $\mathsf{QMA} = \mathsf{PP} \Rightarrow \mathsf{PH} \subseteq \mathsf{PP}$ [Vyali, '03]

What we've seen so far



Last Recap



- extend the notion of verification to interactive setting
- replace proof with an entity that answers questions

A language $L \subset \{0,1\}^*$ has an **interactive proof system** if:

Completeness: $\forall x \in L$, $\exists$ prover-strategy s.t. Verifier accepts with high prob.

Soundness: $\forall x \notin L$, for every prover-strategy, Verifier rejects with high prob.

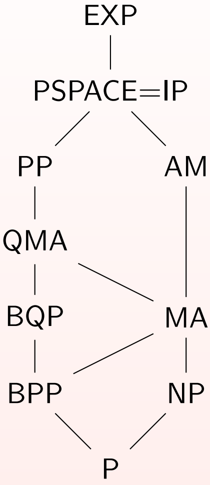
AM: class of languages that have classical interactive proof systems with constant number of rounds

- AM(m) =AM(2)

IP: class of languages that have classical interactive proof systems with polynomial number of rounds

IP=PSPACE [Shamir, '90]

quantum interactive proof systems: the same, just allow Prover and Verifier to be quantum



QIP

- QIP is the same as IP but with quantum interactive proof systems
- QIP(m): at most m rounds, where $m \in \mathbb{Z}^+$

[Jain, Ji, Upadhyay & Watrous '09]

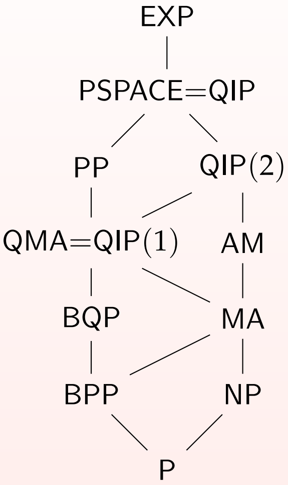
QIP(3) =QIP=PSPACE

- So quantum int. proof systems no more powerful than classical ones.
- with only 3 rounds, you get full power of QIP, even for polynomial number of rounds
- it's not believed that AM=PSPACE
- quantumly, there is a significant reduction in the number of rounds
- problems in PSPACE probably need polynomial number of rounds

[Kitaev & Watrous, '03]

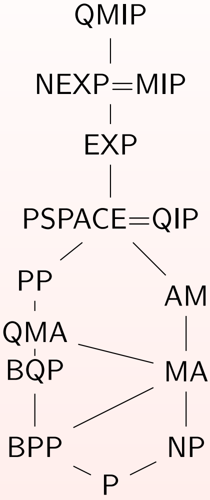
QIP(1) =QMA
QAM$\subseteq$QIP(2)

What we've finally seen so far



(More) Open Problems

- BQP ? PH
- what if we limit quantum models?
  - linear optical quantum computers
  - one-clean-qubit model
  - matchgate circuits
- Upper bounds on entangled provers?
  - we know MIP=NEXP
  - NEXP$\subseteq$ QMIP [Ito & Vidick, '12]



Outline

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Thank you!