Descriptive Complexity

Fagin’s Theorem

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«ΑΛΓΟΡΙΘΜΟΙ, ΛΟΓΙΚΗ ΚΑΙ ΔΙΑΚΡΙΤΑ ΜΑΘΗΜΑΤΙΚΑ»

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Introduction

Computational Complexity

The ”hardness” of a problem is related to the *computational* resources a *mechanical* procedure requires to solve the problem.

Descriptive Complexity

The ”hardness” of a problem is related to the *logical* resources that are required, inorder to formally express the problem.

Introduction

- An algorithm can be seen as a precise description of a mapping from inputs to outputs
- The most usual way to describe such a mapping, is via Turing machines.
- However we may choose to describe such a mapping with another, precise way.
- For example, using formal logic.

Preliminaries

Vocabulary

A vocabulary $\tau = \langle R_1^{a_1}, \ldots, R_k^{a_k}, c_1, \ldots, c_s, f_1^{r_1}, \ldots, f_t^{r_t} \rangle$ is a tuple of relation, constant and function symbols. R_i is a relation of arity a_i and f_j is a function of arity r_j.

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- We do not care about function symbols.
- (Our vocabularies will therefore be called relational.)

Preliminaries

Let $\tau = \langle R_1^{a_1}, \ldots, R_k^{a_k}, c_1, \ldots, c_s \rangle$ be a vocabulary.

τ-structure

A structure of vocabulary τ is a tuple $\mathcal{A} = \langle |A|, R_1^{\mathcal{A}}, \ldots, R_k^{\mathcal{A}}, c_1^{\mathcal{A}}, \ldots, c_s^{\mathcal{A}} \rangle$, whose universe is the nonempty set $|A|$, for each relation R_i of arity a_i in τ, $\mathcal{A}$ has a relation $R_i^{\mathcal{A}}$ of arity a_i defined on $|A|$ and for each constant $c_j \in \tau$, $\mathcal{A}$ has a specified element of its universe.

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- In descriptive complexity we study finite structures.
- To be more accurate, descriptive complexity is the connection of finite model theory and complexity.

Preliminaries

e.x: $\tau_s = \langle \leq^2, S^1 \rangle$ is the vocabulary of binary strings, where S is a unary relation that tells us which bit of the binary string is equal to 1.

Let $\mathcal{U}$ be a τ_s-structure such that:

$$\mathcal{U} = \langle |\mathcal{U}| = \{1, 2, 3, 4\}, \leq = \{\text{"}the\ usual\ order\text{"}\}, S = \{2, 3\} \rangle$$

Then $\mathcal{U}$ "represents" the binary string 0110.
($i \in S$ iff the ith bit of the string is equal to 1)

Second-order logic

Second-order logic = first-order logic + quantification $(\exists, \forall)$ over relations on the universe.

[1] R is a relation (or predicate) variable, while x is an individual variable (the usual variables we have in first-order logic $\ldots$)

Second-order logic

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e.x: Let $\mathcal{U}$ be a structure with universe $|\mathcal{U}| = \{0, 1, 2, 3\}$ and ϕ the second-order sentence $\exists R \forall x R(x, x)$.[1]

Then $\mathcal{U} \models \exists R \forall x R(x, x)$ iff there exist a relation $r \subseteq |\mathcal{U}| \times |\mathcal{U}|$ such that for all $x \in |\mathcal{U}|$, $(x, x) \in r$.

$$r = \{(0,0), (1,1), (2,2), (3,3)\}$$

[1] R is a relation (or predicate) variable, while x is an individual variable (the usual variables we have in first-order logic ...)

($\exists$)Second-order logic

Existential second-order logic ($SO(\exists)$) is a fragment of second-order logic. In particular $\phi \in SO(\exists)$ iff ϕ is of the following form:

$$\phi \equiv \exists R_1, \ldots, \exists R_n \psi$$

Where $R_i, i \in [n]$ relation symbols and ψ is a first-order formula.

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e.x:

$$\exists R \exists G \exists B \forall x \forall y \Big(\big(R(x) \oplus G(x) \oplus B(x)\big) \wedge \big(E(x, y) \rightarrow \big(\neg(R(x) \wedge R(y)) \wedge \neg(G(x) \wedge G(y)) \wedge \neg(B(x) \wedge B(y))\big)\big)\Big)$$

Queries

Queries - boolean queries

Let C_{τ_1}, C_{τ_2} be classes of *finite* structures of vocabulary τ_1, τ_2 accordingly, that are closed under isomorphisms.

- A query is a mapping $Q : C_{\tau_1} \to C_{\tau_2}$.
- A boolean query is a mapping $Q_b : C_{\tau_1} \to \{0, 1\}$.

Where both Q and Q_b preserve isomorphisms.

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e.x: Let $\sigma = \langle E \rangle$ be the vocabulary of graphs and C_σ the class of all structures of vocabulary σ ($\mathcal{A} \in C_\sigma \Rightarrow \mathcal{A} = \langle |\mathcal{A}|, E^{\mathcal{A}} \rangle$).

Then for all $G \in C_\sigma$ the disconnectivity query is:

$$DC(G) = \begin{cases} 1 & \textit{if } G \textit{ is a disconnected graph} \\ 0 & \textit{otherwise} \end{cases}$$

Queries - Definability

Let $\mathcal{L}$ be a logic and C a class of τ-structures.

$\mathcal{L}$-definability

A boolean query Q_b on C is $\mathcal{L}$-definable if there is a $\mathcal{L}$-sentence ϕ such that for all $\mathcal{U} \in C$:

$$Q_b(\mathcal{U}) = 1 \Leftrightarrow \mathcal{U} \models \phi$$

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e.x: The boolean query *disconnectivity* is definable by the second-order sentence:

$$\exists S \Big(\exists x S(x) \land \exists y \neg S(y) \land \forall z \forall w \big(S(z) \land \neg S(w) \rightarrow \neg E(z,w) \big) \Big)$$

Which intuitively says "there are two disjoint, nonempty, sets of nodes with no edge between them".

Capturing complexity classes

$\mathcal{L}$ captures $\mathcal{C}$

We will say that a logic $\mathcal{L}$ captures a complexity class $\mathcal{C}$ on a domain of structures $\mathcal{D}$ if:

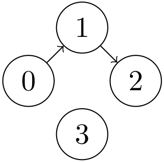
- For every fixed sentence $\phi \in \mathcal{L}$, the complexity of evaluating ϕ ($\mathcal{U} \models \phi$) on structures of $\mathcal{D}$ is a problem in class $\mathcal{C}$.
- Every boolean query from structures of $\mathcal{D}$ that can be decided in $\mathcal{C}$ is $\mathcal{L}$-definable

Binary strings

- Everything a TM does may be thought of as a query from binary strings to binary strings.
- In descriptive complexity we saw the vocabulary of binary strings.
- However we have a lot more structures that use other vocabularies.
- To relate them with Turing machines, we have to somehow encode them into binary strings.

Encoding structures

Define an encoding query $bin_\tau : C_\tau \to C_{\tau_s}$, where C_τ is the class of all (*ordered!*) structures of vocabulary τ and C_{τ_s} the class of all structures of the vocabulary of binary strings $\tau_s = \langle \leq^2, S^1 \rangle$.
Let $G = (V, E, s, t)$ where $V = \{0, 1, 2, 3\}$, $E^G = \{(0, 1), (1, 2)\}$ and $s^G = 0, t^G = 3$.



$E^G \subseteq V \times V =$
$\{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), \ldots\}$
E is a 2-ary relation, V has 4 elements $\to$ we need 4^2 bits.

Binary encoding:

$$bin(G) = \overbrace{0100001000000000}^{E}\overbrace{00}^{s}\overbrace{10}^{t}, \text{ with } |bin(G)| = O(n^2).$$

Encoding structures

The particular choice of an encoding is not important. However there are some conditions that must be satisfied by the encoding.

An encoding query must be:

- Order-independent (identifies isomorphic structures).
- Polynomially bounded. (space efficient)
- First-order definable. (easy to encode and (…) to decode)

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Note that inorder to encode a structure into a binary string, an ordering must exist!

Fagin's Theorem

Fagin's Theorem

Existential second-order logic $SO(\exists)$ captures NP, on the domain of all(!) finite structures.

1. For a fixed $\phi \in SO(\exists)$, for all structures $\mathcal{U}$, checking if $\mathcal{U} \models \phi$ is in NP.
2. Every boolean query that can be decided in NP can be defined in $SO(\exists)$.

Fagin's Theorem

In other words …

1. For every $\phi \in SO(\exists)$ of some vocabulary τ and forall $\mathcal{A} \in C_\tau$ there exists a polynomial-time NTM M such that:

$$\mathcal{A} \models \phi \Leftrightarrow M(bin(\mathcal{A}) \downarrow_{yes})$$

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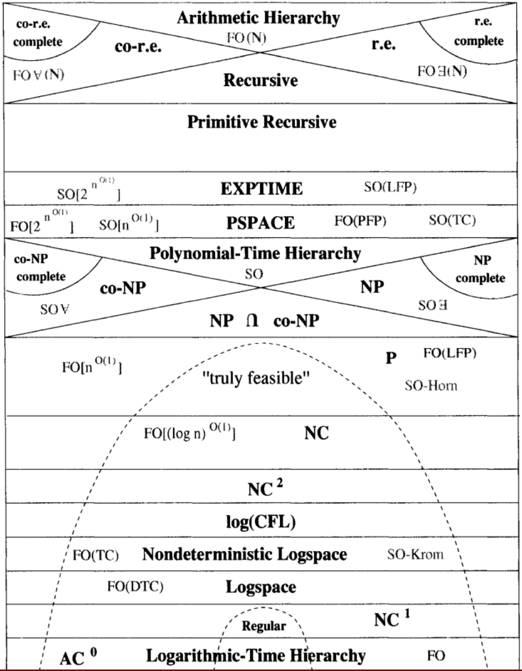
2. For every polynomial-time NTM M that decides a language $L \in NP$, there exists a $SO(\exists)$ sentence ϕ such that:

$$w \in L \text{ iff } w = bin(\mathcal{A}) \text{ where } \mathcal{A} \text{ is a model of } \phi$$

The End

- By Fagin's theorem $SO(\exists) = NP$
- Corollary: $SO(\forall) = coNP$
- We know that $NP \neq coNP \to P \neq NP$.
- Corollary: $SO(\exists) \neq SO(\forall) \to P \neq NP$.


Arithmetic Hierarchy
FO(N)
co-r.e. complete
co-r.e.
r.e.
r.e. complete
FO∀(N)
FO∃(N)
Recursive
Primitive Recursive
SO[$2^{n^{O(1)}}$]
EXPTIME
SO(LFP)
FO[$2^{n^{O(1)}}$]
SO[$n^{O(1)}$]
PSPACE
FO(PFP)
SO(TC)
Polynomial-Time Hierarchy
SO
co-NP complete
co-NP
NP
NP complete
SO∀
SO∃
NP ∩ co-NP
FO[$n^{O(1)}$]
"truly feasible"
P
FO(LFP)
SO-Horn
FO[$(\log n)^{O(1)}$]
NC
NC^2
log(CFL)
FO(TC)
Nondeterministic Logspace
SO-Krom
FO(DTC)
Logspace
Regular
NC^1
AC^0
Logarithmic-Time Hierarchy
FO

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