Advanced Algorithms: Solution of Homework 2

1. Consider the eigendecomposition: $A = U\Lambda U^{-1}$.
   Then, $\lambda I + A = \lambda UU^{-1} + U\Lambda U^{-1} = U(\lambda I + \Lambda)U^{-1}$.

2. Let $x \in \mathbb{R}^n$. Then, $x^\top ABA^\top x = (A^\top x)^\top B(A^\top x) \geq 0$.

3. Suppose $A \succcurlyeq 0$. Then $A = U\Lambda U^\top$, where U is orthonormal, and Λ is diagonal with nonnegative diagonal entries. Let $V := U\Lambda^{1/2}U^\top$ (V is symmetric and is called square root of A and denoted by $A^{1/2}$). We have $VV^\top = V^2 = A$. Suppose now $A = VV^\top$ for some matrix V. Let $x \in \mathbb{R}^n$. We have $x^\top Ax = x^\top VV^\top x = \|V^\top x\|^2 \geq 0$.

4. Consider the eigendecomposition: $A = U\Lambda U^\top$. Since the map $x \mapsto U^\top x$ is an invertible map from the unit sphere onto itself, we have $\max_{\|x\|=1} x^\top U\Lambda U^\top x = \max_{\|x\|=1}(U^\top x)^\top \Lambda U^\top x = \max_{\|y\|=1} y^\top \Lambda y$. The proof finishes by observing that for any $y \in \mathbb{R}^n$ such that $\|y\| = 1$, we have $y^\top \Lambda y = \sum_{i=1}^n \lambda_i y_i^2 \leq \lambda_{\max} \sum_{i=1}^n y_i^2 = \lambda_{max}$. The proof for $\lambda_{\min}$ is identical.

5. Let $x \in \mathbb{R}^n$ nonzero. Then, $\|Ax\|^2 = x^\top A^\top Ax = \|x\|^2 \left(x/\|x\|\right)^\top A^\top A \left(x/\|x\|\right) \leq \|x\|^2 \lambda_{\max}(A^\top A)$.

6. Consider the eigendecomposition: $A = U\Lambda U^\top$. Then, $A^\top A = A^2 = U\Lambda^2 U^\top$.