

# Θεωρητική Πληροφορική Ι (ΣΗΜΜΥ) Αλγόριθμοι & Πολυπλοκότητα ΙΙ (ΜΠΛΑ)

Εργαστήριο Λογικής και Επιστήμης Υπολογισμών  
Εθνικό Μετσόβιο Πολυτεχνείο

2013-2014

# Πληροφορίες Μαθήματος

## Θεωρητική Πληροφορική Ι (ΣΗΜΜΥ)

## Αλγόριθμοι & Πολυπλοκότητα II (ΜΠΛΥ), Λ4-Υπ.

- Διδάσκοντες: Σ. Ζάχος, Ά. Παγουρτζής
- Βοηθός Διδασκαλίας, Επιμέλεια Διαφανειών: Α. Αντωνόπουλος
- Δευτέρα: 17:00 - 19:00 (1.1.31, Παλιά Κτίρια ΗΜΜΥ, ΕΜΠ)  
Πέμπτη: 14:00 - 17:00 (1.1.31, Παλιά Κτίρια ΗΜΜΥ, ΕΜΠ)
- Ώρες Γραφείου: Μετά από κάθε μάθημα, Παρασκευή 10:30-12:30
- Σελίδα: [www.corelab.ntua.gr/courses/grad-algo/](http://www.corelab.ntua.gr/courses/grad-algo/)
- Βαθμολόγηση:
  - Διαγώνισμα: 6 μονάδες
  - Ασκήσεις: 2 μονάδες
  - Ομιλία: 2 μονάδες
  - Quizes : 1 μονάδα

# Theoretical Computer Science I (ECE) Algorithms and Complexity II (MPLA)

Computation and Reasoning Laboratory  
National Technical University of Athens

2013-2014

## 1st Part

Introduction - Turing Machines - Undecidability - Complexity Classes - Oracles & The Polynomial Hierarchy

**Professors:**

S. Zachos, Professor

A. Pagourtzis, Ass. Professor

**TA-Slides:** Antonis Antonopoulos

# Bibliography

## Textbooks

- ① C. Papadimitriou, **Computational Complexity**, Addison Wesley, 1994
- ② S. Arora, B. Barak, **Computational Complexity: A Modern Approach**, Cambridge University Press, 2009
- ③ O. Goldreich, **Computational Complexity: A Conceptual Perspective**, Cambridge University Press, 2008

## Lecture Notes

- ① L. Trevisan, **Lecture Notes in Computational Complexity**, 2002, UC Berkeley
- ② E. Allender, M. Loui, and K. Regan, **Three chapters for the CRC Handbook on Algorithms and Theory of Computation** (M.J. Atallah, ed.), (Boca Raton: CRC Press, 1998).

# Contents

- **Introduction**
- Turing Machines
- Undecidability
- Complexity Classes
- Oracles & Optimization Problems
- Randomized Computation
- Non-Uniform Complexity
- Interactive Proofs
- Counting Complexity

- *Computational Complexity*: Quantifying the amount of computational resources required to solve a given task. Classify computational problems according to their inherent difficulty in complexity classes, and prove relations among them.
- *Structural Complexity*: “The study of the relations between various complexity classes and the global properties of individual classes. [...] The goal of structural complexity is a thorough understanding of the relations between the various complexity classes and the internal structure of these complexity classes.” [J. Hartmanis]

## Decision Problems

- Have answers of the form “yes” or “no”
- Encoding: each instance  $x$  of the problem is represented as a *string* of an alphabet  $\Sigma$  ( $|\Sigma| \geq 2$ ).
- Decision problems have the form “Is  $x$  in  $L$ ?”, where  $L$  is a *language*,  $L \subseteq \Sigma^*$ .

- So, for an encoding of the input, using the alphabet  $\Sigma$ , we associate the following language with the decision problem  $\Pi$ :

$L(\Pi) = \{x \in \Sigma^* \mid x \text{ is a representation of a “yes” instance of the problem } \Pi\}$

### Example

- Given a number  $x$ , is this number prime? ( $x \stackrel{?}{\in} \text{PRIMES}$ )
- Given graph  $G$  and a number  $k$ , is there a clique with  $k$  (or more) nodes in  $G$ ?

## Optimization Problems

- For each instance  $x$  there is a **set of Feasible Solutions**  $F(x)$ .
- To each  $s \in F(x)$  we map a positive integer  $c(x)$ , using **the objective function**  $c(s)$ .
- We search for the solution  $s \in F(x)$  which minimizes (or maximizes) the objective function  $c(s)$ .

## Example

- The **Traveling Salesperson Problem** (TSP):  
Given a finite set  $C = \{c_1, \dots, c_n\}$  of cities and a distance  $d(c_i, c_j) \in \mathbb{Z}^+$ ,  $\forall (c_i, c_j) \in C^2$ , we ask for a permutation  $\pi$  of  $C$ , that minimizes this quantity:

$$\sum_{i=1}^{n-1} d(c_{\pi(i)}, c_{\pi(i+1)}) + d(c_{\pi(n)}, c_{\pi(1)})$$



# A Model Discussion

- There are many computational models (RAM, Turing Machines etc).
- The **Church-Turing Thesis** states that all computation models are equivalent. That is, every computation model can be simulated by a Turing Machine.
- In Complexity Theory, we consider **efficiently computable** the problems which are solved (aka the languages that are decided) in **polynomial number of steps** (*Edmonds-Cobham Thesis*).

**Efficiently Computable  $\equiv$  Polynomial-Time Computable**

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## Definition

A Turing Machine  $M$  is a quintuple  $M = (Q, \Sigma, \delta, q_0, F)$ :

- $Q = \{q_0, q_1, q_2, q_3, \dots, q_n, q_{\text{halt}}, q_{\text{yes}}, q_{\text{no}}\}$  is a finite set of states.
  - $\Sigma$  is the alphabet. The tape alphabet is  $\Gamma = \Sigma \cup \{\sqcup\}$ .
  - $q_0 \in Q$  is the initial state.
  - $F \subseteq Q$  is the set of final states.
  - $\delta : (Q \setminus F) \times \Gamma \rightarrow Q \times \Gamma \times \{S, L, R\}$  is the transition function.
- 
- A TM is a “programming language” with a single data structure (a tape), and a cursor, which moves left and right on the tape.
  - Function  $\delta$  is the *program* of the machine.

# Turing Machines and Languages

## Definition

Let  $L \subseteq \Sigma^*$  be a language and  $M$  a TM such that, for every string  $x \in \Sigma^*$ :

- If  $x \in L$ , then  $M(x) = \text{“yes”}$
- If  $x \notin L$ , then  $M(x) = \text{“no”}$

Then we say that  **$M$  decides  $L$** .

- We can alternatively say that  $M(x) = \chi_L(x)$ , where  $\chi_L(\cdot)$  is the *characteristic function* of  $L$  (if we consider 1 as “yes” and 0 as “no”).
- If  $L$  is decided by some TM  $M$ , then  $L$  is called a **recursive language**.

## Definition

If for a language  $L$  there is a TM  $M$ , which if  $x \in L$  then  $M(x) = \text{"yes"}$ , and if  $x \notin L$  then  $M(x) \uparrow$ , we call  $L$  **recursively enumerable**.

\*By  $M(x) \uparrow$  we mean that  $M$  does not halt on input  $x$  (it runs forever).

## Theorem

*If  $L$  is recursive, then it is recursively enumerable.*

**Proof:** *Exercise*

## Definition

If for a language  $L$  there is a TM  $M$ , which if  $x \in L$  then  $M(x) = \text{"yes"}$ , and if  $x \notin L$  then  $M(x) \uparrow$ , we call  $L$  **recursively enumerable**.

\*By  $M(x) \uparrow$  we mean that  $M$  does not halt on input  $x$  (it runs forever).

## Theorem

*If  $L$  is recursive, then it is recursively enumerable.*

**Proof:** *Exercise*

## Definition

If  $f$  is a function,  $f : \Sigma^* \rightarrow \Sigma^*$ , we say that a TM  $M$  computes  $f$  if, for any string  $x \in \Sigma^*$ ,  $M(x) = f(x)$ . If such  $M$  exists,  $f$  is called a **recursive function**.

- Turing Machines can be thought as algorithms for solving string related problems.

# Multitape Turing Machines

- We can extend the previous Turing Machine definition to obtain a Turing Machine with multiple tapes:

## Definition

A  $k$ -tape Turing Machine  $M$  is a quintuple  $M = (Q, \Sigma, \delta, q_0, F)$ :

- $Q = \{q_0, q_1, q_2, q_3, \dots, q_n, q_{\text{halt}}, q_{\text{yes}}, q_{\text{no}}\}$  is a finite set of states.
- $\Sigma$  is the alphabet. The tape alphabet is  $\Gamma = \Sigma \cup \{\sqcup\}$ .
- $q_0 \in Q$  is the initial state.
- $F \subseteq Q$  is the set of final states.
- $\delta : (Q \setminus F) \times \Gamma^k \rightarrow Q \times (\Gamma \times \{S, L, R\})^k$  is the transition function.

# Bounds on Turing Machines

- We will characterize the “performance” of a Turing Machine by the amount of *time* and *space* required on instances of size  $n$ , when these amounts are expressed as a function of  $n$ .

## Definition

Let  $T : \mathbb{N} \rightarrow \mathbb{N}$ . We say that machine  $M$  operates within time  $T(n)$  if, for any input string  $x$ , the time required by  $M$  to reach a final state is at most  $T(|x|)$ . Function  $T$  is a **time bound** for  $M$ .

## Definition

Let  $S : \mathbb{N} \rightarrow \mathbb{N}$ . We say that machine  $M$  operates within space  $S(n)$  if, for any input string  $x$ ,  $M$  visits at most  $S(|x|)$  locations on its work tapes (excluding the input tape) during its computation. Function  $S$  is a **space bound** for  $M$ .



# Multitape Turing Machines

## Theorem

*Given any  $k$ -tape Turing Machine  $M$  operating within time  $T(n)$ , we can construct a TM  $M'$  operating within time  $\mathcal{O}(T^2(n))$  such that, for any input  $x \in \Sigma^*$ ,  $M(x) = M'(x)$ .*

**Proof:** See Th.2.1 (p.30) in [1].

- This is a strong evidence of the robustness of our model: *Adding a bounded number of strings does not increase their computational capabilities, and affects their efficiency only polynomially.*

# Linear Speedup

## Theorem

*Let  $M$  be a TM that decides  $L \subseteq \Sigma^*$ , that operates within time  $T(n)$ . Then, for every  $\varepsilon > 0$ , there is a TM  $M'$  which decides the same language and operates within time  $T'(n) = \varepsilon T(n) + n + 2$ .*

**Proof:** See Th.2.2 (p.32) in [1].

- If, for example,  $T$  is linear, i.e. something like  $cn$ , then this theorem states that the constant  $c$  can be made arbitrarily close to 1. So, it is fair to start using the  $\mathcal{O}(\cdot)$  notation in our time bounds.
- A similar theorem holds for space:

## Theorem

*Let  $M$  be a TM that decides  $L \subseteq \Sigma^*$ , that operates within space  $S(n)$ . Then, for every  $\varepsilon > 0$ , there is a TM  $M'$  which decides the same language and operates within space  $S'(n) = \varepsilon S(n) + 2$ .*

# Nondeterministic Turing Machines

- We will now introduce an **unrealistic** model of computation:

## Definition

A Turing Machine  $M$  is a quintuple  $M = (Q, \Sigma, \delta, q_0, F)$ :

- $Q = \{q_0, q_1, q_2, q_3, \dots, q_n, q_{\text{halt}}, q_{\text{yes}}, q_{\text{no}}\}$  is a finite set of states.
- $\Sigma$  is the alphabet. The tape alphabet is  $\Gamma = \Sigma \cup \{\sqcup\}$ .
- $q_0 \in Q$  is the initial state.
- $F \subseteq Q$  is the set of final states.
- $\delta : (Q \setminus F) \times \Gamma \rightarrow \text{Pow}(Q \times \Gamma \times \{S, L, R\})$  is the transition **relation**.

# Nondeterministic Turing Machines

- In this model, an input is accepted if there is *some sequence* of nondeterministic choices that results in “yes”.
- An input is rejected if there is *no sequence* of choices that lead to acceptance.
- Observe the similarity with recursively enumerable languages.

## Definition

We say that  $M$  operates within bound  $T(n)$ , if for every input  $x \in \Sigma^*$  and every sequence of nondeterministic choices,  $M$  reaches a final state within  $T(|x|)$  steps.

- The above definition requires that  $M$  does not have computation paths longer than  $T(n)$ , where  $n = |x|$  the length of the input.
- The amount of time charged is the *depth* of the **computation tree**.

# Contents

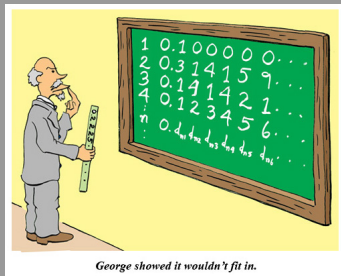
- Introduction
- Turing Machines
- **Undecidability**
- Complexity Classes
- Oracles & Optimization Problems
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- Non-Uniform Complexity
- Interactive Proofs
- Counting Complexity

# Diagonalization



*Suppose there is a town with just one barber, who is male. In this town, the barber shaves all those, and only those, men in town who do not shave themselves. Who shaves the barber?*

Diagonalization is a technique that was used in many different cases:



# Diagonalization

## Theorem

*The functions from  $\mathbb{N}$  to  $\mathbb{N}$  are uncountable.*

**Proof:** Let, for the sake of contradiction that are countable:

$\phi_1, \phi_2, \dots$ . Consider the following function:  $f(x) = \phi_x(x) + 1$ .

This function must appear somewhere in this enumeration, so let

$\phi_y = f(x)$ . Then  $\phi_y(x) = \phi_x(x) + 1$ , and if we choose  $y$  as an argument, then  $\phi_y(y) = \phi_y(y) + 1$ .  $\square$

# Machines as strings

- It is obvious that we can represent a Turing Machine as a string: *just write down the description and encode it using an alphabet, e.g.  $\{0, 1\}$ .*
- We denote by  $\lfloor M \rfloor$  the TM  $M$ 's representation as a string.
- Also, if  $x \in \Sigma^*$ , we denote by  $M_x$  the TM that  $x$  represents.

## Keep in mind that:

- **Every string represents some Turing Machine.**
- **Every TM is represented by infinitely many strings.**



# The Universal Turing Machine

- So far, our computational models are specified to solve a single problem.
- Turing observed that there is a TM that can simulate any other TM  $M$ , given  $M$ 's description as input.

## Theorem

*There exists a TM  $\mathcal{U}$  such that for every  $x, w \in \Sigma^*$ ,  $\mathcal{U}(x, w) = M_w(x)$ .*

*Also, if  $M_w$  halts within  $T$  steps on input  $x$ , then  $\mathcal{U}(x, w)$  halts within  $CT \log T$  steps, where  $C$  is a constant independent of  $x$ , and depending only on  $M_w$ 's alphabet size number of tapes and number of states.*

**Proof:** See section 3.1 in [1], and Th. 1.9 and section 1.7 in [2].

# The Halting Problem

- Consider the following problem: “Given the description of a TM  $M$ , and a string  $x$ , will  $M$  halt on input  $x$ ? ” This is called the HALTING PROBLEM.
- **We want to compute this problem ! ! !** (Given a computer program and an input, will this program enter an infinite loop?)
- In language form:  $H = \{ \langle M \rangle ; x \mid M(x) \downarrow \}$ , where “ $\downarrow$ ” means that the machine halts, and “ $\uparrow$ ” that it runs forever.

## Theorem

$H$  is recursively enumerable.

**Proof:** See Th.3.1 (p.59) in [1]

- In fact,  $H$  is not just a recursively enumerable language:  
If we had an algorithm for deciding  $H$ , then we would be able to derive an algorithm for deciding any r.e. language (**RE-complete**).

# The Halting Problem

- But....

## Theorem

$H$  is not recursive.

## Proof:

See Th.3.1 (p.60) in [1]

- Suppose, for the sake of contradiction, that there is a TM  $M_H$  that decides  $H$ .
- Consider the TM  $D$ :
 

$$D(\langle M \rangle) : \text{if } M_H(\langle M \rangle; \langle M \rangle) = \text{"yes"} \text{ then } \uparrow \text{ else "yes"}$$
- What is  $D(\langle D \rangle)$ ?

# The Halting Problem

- But....

## Theorem

$H$  is not recursive.

## Proof:

See Th.3.1 (p.60) in [1]

- Suppose, for the sake of contradiction, that there is a TM  $M_H$  that decides  $H$ .
- Consider the TM  $D$ :
 

$$D(\ulcorner M \urcorner) : \text{if } M_H(\ulcorner M \urcorner; \ulcorner M \urcorner) = \text{“yes” then } \uparrow \text{ else “yes”}$$
- What is  $D(\ulcorner D \urcorner)$ ?
- If  $D(\ulcorner D \urcorner) \uparrow$ , then  $M_H$  accepts the input, so  $\ulcorner D \urcorner; \ulcorner D \urcorner \in H$ , so  $D(D) \downarrow$ .
- If  $D(\ulcorner D \urcorner) \downarrow$ , then  $M_H$  rejects  $\ulcorner D \urcorner; \ulcorner D \urcorner$ , so  $\ulcorner D \urcorner; \ulcorner D \urcorner \notin H$ , so  $D(D) \uparrow$ .  $\square$

- Recursive languages are a *proper* subset of recursive enumerable ones.
- Recall that the complement of a language  $L$  is defined as:

$$\bar{L} = \{x \in \Sigma^* \mid x \notin L\} = \Sigma^* \setminus L$$

### Theorem

- ① *If  $L$  is recursive, so is  $\bar{L}$ .*
- ②  *$L$  is recursive if and only if  $L$  and  $\bar{L}$  are recursively enumerable.*

**Proof:** Exercise

# More Undecidability

- The HALTING PROBLEM, our first undecidable problem, was the first, but not the only undecidable problem. It spawns a wide range of such problems, via *reductions*.
- To show that a problem  $A$  is undecidable we establish that, if there is an algorithm for  $A$ , then there would be an algorithm for  $H$ , which is absurd.

## Theorem

*The following languages are not recursive:*

- ①  $\{M \mid M \text{ halts on all inputs}\}$
- ②  $\{M; x \mid \text{There is a } y \text{ such that } M(x) = y\}$
- ③  $\{M; x \mid \text{The computation of } M \text{ uses all states of } M\}$
- ④  $\{M; x; y \mid M(x) = y\}$

# Rice's Theorem

- The previous problems lead us to a more general conclusion:

**Any non-trivial property of  
Turing Machines is undecidable**

- If a TM  $M$  accepts a language  $L$ , we write  $L = L(M)$ :

Theorem (Rice's Theorem)

*Suppose that  $\mathcal{C}$  is a proper, non-empty subset of the set of all recursively enumerable languages. Then, the following problem is undecidable:*

*Given a Turing Machine  $M$ , is  $L(M) \in \mathcal{C}$ ?*

# Rice's Theorem

## Proof:

See Th.3.2 (p.62) in [1]

- We can assume that  $\emptyset \notin \mathcal{C}$  (*why?*).
- Since  $\mathcal{C}$  is nonempty,  $\exists L \in \mathcal{C}$ , accepted by the TM  $M_L$ .
- Let  $M_H$  the TM deciding the HALTING PROBLEM for an arbitrary input  $x$ . For each  $x \in \Sigma^*$ , we construct a TM  $M$  as follows:

$M(y) : \text{if } M_H(x) = \text{"yes"} \text{ then } M_L(y) \text{ else } \uparrow$

- We claim that:  $L(M) \in \mathcal{C}$  if and only if  $x \in H$ .



# Rice's Theorem

## Proof:

See Th.3.2 (p.62) in [1]

- We can assume that  $\emptyset \notin \mathcal{C}$  (*why?*).
- Since  $\mathcal{C}$  is nonempty,  $\exists L \in \mathcal{C}$ , accepted by the TM  $M_L$ .
- Let  $M_H$  the TM deciding the HALTING PROBLEM for an arbitrary input  $x$ . For each  $x \in \Sigma^*$ , we construct a TM  $M$  as follows:

$M(y) : \text{if } M_H(x) = \text{"yes"} \text{ then } M_L(y) \text{ else } \uparrow$

- We claim that:  $L(M) \in \mathcal{C}$  if and only if  $x \in H$ .

### Proof of the claim:

- If  $x \in H$ , then  $M_H(x) = \text{"yes"}$ , and so  $M$  will accept  $y$  or never halt, depending on whether  $y \in L$ . Then the language accepted by  $M$  is exactly  $L$ , which is in  $\mathcal{C}$ .
- If  $M_H(x) \uparrow$ ,  $M$  never halts, and thus  $M$  accepts the language  $\emptyset$ , which is not in  $\mathcal{C}$ .  $\square$

# Contents

- Introduction
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**Proof** (*cont'd*):

## Definition (Configuration Graph of a TM)

The configuration graph of  $M$  on input  $x$ , denoted  $G(M, x)$ , has as **vertices** all the possible configurations, and there is an **edge** between two vertices  $C$  and  $C'$  if and only if  $C'$  can be reached from  $C$  in one step, according to  $M$ 's transition function.

- So, we have reduced this simulation to REACHABILITY\* problem (also known as S-T CONN), for which we know there is a poly-time ( $\mathcal{O}(n^2)$ ) algorithm.
- So, the simulation takes  $c_2 c_1^{2(\log n + S(n))} \sim k^{\log n + S(n)}$  steps.  $\square$

\*REACHABILITY: Given a graph  $G$  and two nodes  $v_1, v_n \in V$ , is there a path from  $v_1$  to  $v_n$ ?



# The essential Complexity Hierarchy

## Definition

$$\mathbf{L} = \mathbf{DSPACE}[\log n]$$

$$\mathbf{NL} = \mathbf{NSPACE}[\log n]$$

$$\mathbf{P} = \bigcup_{c \in \mathbb{N}} \mathbf{DTIME}[n^c]$$

$$\mathbf{NP} = \bigcup_{c \in \mathbb{N}} \mathbf{NTIME}[n^c]$$

$$\mathbf{PSPACE} = \bigcup_{c \in \mathbb{N}} \mathbf{DSPACE}[n^c]$$

$$\mathbf{NPSPACE} = \bigcup_{c \in \mathbb{N}} \mathbf{NSPACE}[n^c]$$

# The essential Complexity Hierarchy

## Definition

$$\mathbf{EXP} = \bigcup_{c \in \mathbb{N}} \mathbf{DTIME}[2^{n^c}]$$

$$\mathbf{NEXP} = \bigcup_{c \in \mathbb{N}} \mathbf{NTIME}[2^{n^c}]$$

$$\mathbf{EXPSPACE} = \bigcup_{c \in \mathbb{N}} \mathbf{DSPACE}[2^{n^c}]$$

$$\mathbf{NEXPSPACE} = \bigcup_{c \in \mathbb{N}} \mathbf{NSPACE}[2^{n^c}]$$

# The essential Complexity Hierarchy

## Definition

$$\mathbf{EXP} = \bigcup_{c \in \mathbb{N}} \mathbf{DTIME}[2^{n^c}]$$

$$\mathbf{NEXP} = \bigcup_{c \in \mathbb{N}} \mathbf{NTIME}[2^{n^c}]$$

$$\mathbf{EXPSPACE} = \bigcup_{c \in \mathbb{N}} \mathbf{DSPACE}[2^{n^c}]$$

$$\mathbf{NEXPSPACE} = \bigcup_{c \in \mathbb{N}} \mathbf{NSPACE}[2^{n^c}]$$

$$\mathbf{L} \subseteq \mathbf{NL} \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE} \subseteq \mathbf{NPSPACE} \subseteq \mathbf{EXP} \subseteq \mathbf{NEXP}$$

# Certificate Characterization of NP

## Definition

Let  $R \subseteq \Sigma^* \times \Sigma^*$  a binary relation on strings.

- $R$  is called **polynomially decidable** if there is a DTM deciding the language  $\{x; y \mid (x, y) \in R\}$  in polynomial time.
- $R$  is called **polynomially balanced** if  $(x, y) \in R$  implies  $|y| \leq |x|^k$ , for some  $k \geq 1$ .

## Theorem

*Let  $L \subseteq \Sigma^*$  be a language.  $L \in \mathbf{NP}$  if and only if there is a polynomially decidable and polynomially balanced relation  $R$ , such that:*

$$L = \{x \mid \exists y R(x, y)\}$$

- This  $y$  is called **succinct certificate**, or **witness**.

**Proof:**

See Pr.9.1 (p.181) in [1]

( $\Leftarrow$ ) If such an  $R$  exists, we can construct the following NTM deciding  $L$ :

“On input  $x$ , *guess* a  $y$ , such that  $|y| \leq |x|^k$ , and then test (in poly-time) if  $(x, y) \in R$ . If so, accept, else reject.” Observe that an accepting computation exists if and only if  $x \in L$ .

( $\Rightarrow$ ) If  $L \in \mathbf{NP}$ , then  $\exists$  an NTM  $N$  that decides  $L$  in time  $|x|^k$ , for some  $k$ . Define the following  $R$ :

“ $(x, y) \in R$  if and only if  $y$  is an **encoding** of an accepting computation of  $N(x)$ .”

$R$  is polynomially balanced and decidable (*why?*), so, given by assumption that  $N$  decides  $L$ , we have our conclusion.  $\square$

# Can creativity be automated?

As we saw:

- Class **P**: Efficient *Computation*
- Class **NP**: Efficient *Verification*
- So, if we can efficiently verify a mathematical proof, can we create it efficiently?

If  $P = NP$ ...

- For every mathematical statement, and given a page limit, we would (quickly) generate a proof, if one exists.
- Given detailed constraints on an engineering task, we would (quickly) generate a design which meets the given criteria, if one exists.
- Given data on some phenomenon and modeling restrictions, we would (quickly) generate a theory to explain the data, if one exists.

# Complements of complexity classes

- Deterministic complexity classes are in general closed under complement ( $coL = L$ ,  $coP = P$ ,  $coPSPACE = PSPACE$ ).
- Complements of non-deterministic complexity classes are very interesting:
- The class  $coNP$  contains all the languages that have **succinct disqualifications** (the analogue of *succinct certificate* for the class  $NP$ ). The “no” instance of a problem in  $coNP$  has a short proof of its being a “no” instance.
- So:

$$P \subseteq NP \cap coNP$$

- Note the *similarity* and the *difference* with  $R = RE \cap coRE$ .





# Savitch's Theorem

- REACHABILITY  $\in$  NL.

See Ex.2.10 (p.48) in [1]

Theorem (Savitch's Theorem)

REACHABILITY  $\in$  DSPACE[ $\log^2 n$ ]

**Proof:**

See Th.7.4 (p.149) in [1]

$PATH(x, y, i)$  : "There is a path from  $x$  to  $y$ , of length  $\leq 2^i$ ".

- We can solve REACHABILITY if we can compute  $PATH(x, y, \lceil \log n \rceil)$ , for any nodes  $x, y \in V$ , since any path in  $G$  can be at most  $n \log n$  long.
- If  $i = 0$ , we can check whether  $PATH(x, y, i)$ .
- If  $i \geq 1$ :

**forall** nodes  $z$  test whether  $PATH(x, z, i - 1)$  and  $PATH(z, y, i - 1)$

# Savitch's Theorem

## Proof (*cont'd*):

- We generate all nodes  $z$  one after the other, *reusing* space.
- Once a  $z$  is generated, we add  $(x, z, i - 1)$  to the tape, and start working on this recursively.
- If a negative answer is obtained to  $PATH(x, z, i - 1)$ , we erase this triple and move to the next  $z$ .
- If a positive answer is obtained to  $PATH(x, z, i - 1)$ , we erase the triple and move to  $PATH(z, y, i - 1)$ .
- If this is negative, we erase it and move to the next  $z$ .
- If it is positive, we compare it to  $(x, y, i)$  to check that this is the second recursive call, and then return a positive answer to  $PATH(x, y, i)$ .
- The work tape contains at any moment at most  $\lceil \log n \rceil$ , each of length at most  $3 \log n$ .  $\square$

# Savitch's Theorem

## Corollary

**NSPACE** $[S(n)] \subseteq$  **DSPACE** $[S^2(n)]$ , for any space-constructible function  $S(n) \geq \log n$ .

## Proof:

- Let  $M$  be the nondeterministic TM to be simulated.
- We run the algorithm of Savitch's Theorem proof on the configuration graph of  $M$  on input  $x$ .
- Since the configuration graph has  $c^{S(n)}$  nodes,  $\mathcal{O}(S^2(n))$  space suffices.  $\square$

## Corollary

**PSPACE = NSPACE**

# NL-Completeness

- In Complexity Theory, we “connect” problems in a complexity class with partial ordering relations, called **reductions**, which formalize the notion of “*a problem that is at least as hard as another*”.
- A reduction must be computationally weaker than the class in which we use it.

## Definition

A language  $L_1$  is **logspace reducible** to a language  $L_2$ , denoted  $L_1 \leq_l L_2$ , if there is a function  $f : \Sigma^* \rightarrow \Sigma^*$ , computable by a DTM in  $\mathcal{O}(\log n)$  space, such that for all  $x \in \Sigma^*$ :

$$x \in L_1 \Leftrightarrow f(x) \in L_2$$

We say that a language  $L$  is **NL-complete** if it is in **NL** and for every  $A \in \mathbf{NL}$ ,  $A \leq_l L$ .

# NL-Completeness

## Theorem

*REACHABILITY* is **NL**-complete.

# NL-Completeness

## Theorem

*REACHABILITY* is **NL**-complete.

## Proof:

See Th.4.18 (p.89) in [2]

- We've argued why  $REACHABILITY \in \mathbf{NL}$ .
- Let  $L \in \mathbf{NL}$ , that is, it is decided by a  $\mathcal{O}(\log n)$  NTM  $N$ .
- Given input  $x$ , we can construct the *configuration graph* of  $N(x)$ .
- We can assume that this graph has a *single* accepting node.
- We can construct this in logspace: Given configurations  $C, C'$  we can in space  $\mathcal{O}(|C| + |C'|) = \mathcal{O}(\log |x|)$  check the graph's adjacency matrix if they are connected by an edge.
- It is clear that  $x \in L$  if and only if the produced instance of  $REACHABILITY$  has a "yes" answer.  $\square$

# Certificate Definition of NL

- We want to give a characterization of **NL**, similar to the one we gave for **NP**.
- A certificate may be polynomially long, so a logspace machine may not have the space to store it.
- So, we will assume that the certificate is provided to the machine on a separate tape that is **read once**.

# Certificate Definition of NL

## Definition

A language  $L$  is in **NL** if there exists a deterministic TM  $M$  with an additional special read-once input tape, such that for every  $x \in \Sigma^*$ :

$$x \in L \Leftrightarrow \exists y, |y| \in \text{poly}(|x|), M(x, y) = 1$$

where by  $M(x, y)$  we denote the output of  $M$  where  $x$  is placed on its input tape, and  $y$  is placed on its special read-once tape, and  $M$  uses at most  $\mathcal{O}(\log |x|)$  space on its read-write tapes for every input  $x$ .

- What if remove the read-once restriction and allow the TM's head to move back and forth on the certificate, and read each bit multiple times?



# Immerman-Szelepcsényi

Theorem (The Immerman-Szelepcsényi Theorem)

REACHABILITY  $\in$  **NL**

# Immerman-Szelepcényi

## Theorem (The Immerman-Szelepcényi Theorem)

REACHABILITY  $\in$  **NL**

### Proof:

See Th.4.20 (p.91) in [2]

- It suffices to show a  $\mathcal{O}(\log n)$  verification algorithm  $A$  such that:  $\forall (G, s, t), \exists$  a polynomial certificate  $u$  such that:  $A((G, s, t), u) = \text{"yes"}$  iff  $t$  is not reachable from  $s$ .
- $A$  has read-once access to  $u$ .
- $G$ 's vertices are identified by numbers in  $\{1, \dots, n\} = [n]$
- $C_i$ : "*The set of vertices reachable from  $s$  in  $\leq i$  steps.*"
- Membership in  $C_i$  is easily certified:
- $\forall i \in [n]: v_0, \dots, v_k$  along the path from  $s$  to  $v$ ,  $k \leq i$ .
- The certificate is at most polynomial in  $n$ .

# The Immerman-Szelepcényi Theorem

## Proof (*cont'd*):

- We can check the certificate using read-once access:
  - ①  $v_0 = s$
  - ② for  $j > 0$ ,  $(v_{j-1}, v_j) \in E(G)$
  - ③  $v_k = v$
  - ④ Path ends within at most  $i$  steps
- We now construct two types of certificates:
  - ① A certificate that a vertex  $v \notin C_i$ , given  $|C_i|$ .
  - ② A certificate that  $|C_i| = c$ , for some  $c$ , given  $|C_{i-1}|$ .
- Since  $C_0 = \{s\}$ , we can provide the 2nd certificate to convince the verifier for the sizes of  $C_1, \dots, C_n$
- $C_n$  is the set of vertices *reachable* from  $s$ .

# The Immerman-Szelepcényi Theorem

## Proof (cont'd):

- Since the verifier has been convinced of  $|C_n|$ , we can use the 1st type of certificate to convince the verifier that  $t \notin C_n$ .
- **Certifying that  $v \notin C_i$ , given  $|C_i|$**

The certificate is the list of certificates that  $u \in C_i$ , for every  $u \in C_i$ .

The verifier will check:

- ① Each certificate is valid
- ② Vertex  $u$ , given a certificate for  $u$ , is larger than the previous.
- ③ No certificate is provided for  $v$ .
- ④ The total number of certificates is exactly  $|C_i|$ .

# The Immerman-Szelepcényi Theorem

**Proof** (*cont'd*):

**Certifying that  $v \notin C_i$ , given  $|C_{i-1}|$**

The certificate is the list of certificates that  $u \in C_{i-1}$ , for every  $u \in C_{i-1}$

The verifier will check:

- ① Each certificate is valid
- ② Vertex  $u$ , given a certificate for  $u$ , is larger than the previous.
- ③ No certificate is provided for  $v$  or for a neighbour of  $v$ .
- ④ The total number of certificates is exactly  $|C_{i-1}|$ .

**Certifying that  $|C_i| = c$ , given  $|C_{i-1}|$**

The certificate will consist of  $n$  certificates for each vertex.

The verifier will check all certificates, and count the vertices that have been certified to be in  $C_i$ . If  $|C_i| = c$ , it accepts.  $\square$

# The Immerman-Szelepcényi Theorem

## Corollary

For every space constructible  $S(n) > \log n$ :

$$\mathbf{NSPACE}[S(n)] = \mathbf{coNSPACE}[S(n)]$$

## Proof:

- Let  $L \in \mathbf{NSPACE}[S(n)]$ . We will show that  $\exists S(n)$  space-bounded NTM  $\overline{M}$  deciding  $\overline{L}$ :
- $\overline{M}$  on input  $x$  uses the above certification procedure on the *configuration graph* of  $M$ .  $\square$

## Corollary

$$\mathbf{NL} = \mathbf{coNL}$$

# What about Undirected Reachability?

- **UNDIRECTED REACHABILITY** captures the phenomenon of configuration graphs with both directions.
- H. Lewis and C. Papadimitriou defined the class **SL** (**S**ymmetric **L**ogspace) as the class of languages decided by a **Symmetric Turing Machine** using logarithmic space.
- Obviously,

$$\boxed{L \subseteq SL \subseteq NL}$$

- As in the case of **NL**, **UNDIRECTED REACHABILITY** is **SL**-complete.
- But in 2004, Omer Reingold showed, using expander graphs, a deterministic logspace algorithm for **UNDIRECTED REACHABILITY**, so:

Theorem (Reigold, 2004)

$$L = SL$$

# Our Complexity Hierarchy Landscape

