Υπολογιστική Πολυπλοκότητα (ΣΗΜΜΥ)

Δομική Πολυπλοκότητα (ΑΛΜΑ)

Εργαστήριο Λογικής και Επιστήμης Υπολογισμών
Εθνικό Μετσόβιο Πολυτεχνείο

Πληροφορίες Μαθήματος

Θεωρητική Πληροφορική Ι (ΣΗΜΜΥ)
Υπολογιστική Πολυπλοκότητα (ΑΛΜΑ)

- Διδάσκοντες: Σ. Ζάχος, Ά. Παγουρτζής
- Βοηθοί Διδασκαλίας: Α. Αντωνόπουλος, Σ. Πετσαλάκης
- Επιμέλεια Διαφανειών: Α. Αντωνόπουλος
- Δευτέρα: 17:00 - 20:00 (1.1.31, Παλιά Κτίρια ΗΜΜΥ, ΕΜΠ)
  Πέμπτη: 15:00 - 17:00 (1.1.31, Παλιά Κτίρια ΗΜΜΥ, ΕΜΠ)
- Ώρες Γραφείου: Μετά από κάθε μάθημα
- Σελίδα: http://courses.corelab.ntua.gr/complexity
- Βαθμολόγηση:

Διαγώνισμα:	6 μονάδες
Ασκήσεις:	2 μονάδες
Ομιλία:	2 μονάδες
Quiz:	1 μονάδα

Computational Complexity

Graduate Course

Antonis Antonopoulos

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Bibliography

Textbooks

1. C. Papadimitriou, **Computational Complexity**, Addison Wesley, 1994
2. S. Arora, B. Barak, **Computational Complexity: A Modern Approach**, Cambridge University Press, 2009
3. O. Goldreich, **Computational Complexity: A Conceptual Perspective**, Cambridge University Press, 2008

Lecture Notes

1. L. Trevisan, **Lecture Notes in Computational Complexity**, 2002, UC Berkeley
2. J. Katz, **Notes on Complexity Theory**, 2011, University of Maryland
3. Jin-Yi Cai, **Lectures in Computational Complexity**, 2003, University of Wisconsin Madison

Contents

- **Computational Complexity**: Quantifying the amount of computational resources required to solve a given task. *Classify* computational problems according to their inherent difficulty in complexity classes, and prove relations among them.

- **Structural Complexity**: "The study of the relations between various complexity classes and the global properties of individual classes. [...] The goal of structural complexity is a *thorough understanding of the relations between the various complexity classes and the internal structure of these complexity classes.*" [J. Hartmanis]

Decision Problems

- Have answers of the form "*yes*" or "*no*".
- Encoding: each instance x of the problem is represented as a *string* of an alphabet Σ ($|\Sigma| \geq 2$).
- Decision problems have the form "Is x in L?", where L is a *language*, $L \subseteq \Sigma^*$.

- So, for an encoding of the input, using the alphabet Σ, we associate the following language with the decision problem Π:

$$L(\Pi) = \{x \in \Sigma^* \mid x \text{ is a representation of a "yes" instance of the problem } \Pi\}$$

Example

- Given a number x, is this number prime? ($x \overset{?}{\in}$ PRIMES)
- Given graph G and a number k, is there a clique with k (or more) nodes in G?

Search Problems

- Have answers of the form of an **object**.
- **Relation** $R(x, y)$ connecting instances x with answers (objects) y we wish to find for x.
- Given instance x, find a y such that $(x, y) \in R$.

Search Problems

- Have answers of the form of an **object**.
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- Given instance x, find a y such that $(x, y) \in R$.

Example

FACTORING: Given integer N, find its prime decomposition:

$$N = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$$

Optimization Problems

- For each instance x there is a **set of Feasible Solutions** $F(x)$.
- To each $s \in F(x)$ we map a positive integer $c(x)$, using **the objective function** $c(s)$.
- We search for the solution $s \in F(x)$ which minimizes (or maximizes) the objective function $c(s)$.

Optimization Problems

- For each instance x there is a **set of Feasible Solutions $F(x)$**.
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Example

- The **Traveling Salesperson Problem** (TSP):
  Given a finite set $C = \{c_1, \dots, c_n\}$ of cities and a distance $d(c_i, c_j) \in \mathbb{Z}^+, \forall (c_i, c_j) \in C^2$, we ask for a permutation π of C, that minimizes this quantity:

$$\sum_{i=1}^{n-1} d(c_{\pi(i)}, c_{\pi(i+1)}) + d(c_{\pi(n)}, c_{\pi(1)})$$

A Model Discussion

- There are many computational models (RAM, Turing Machines etc).
- The **Church-Turing Thesis** states that all computation models are equivalent. That is, every computation model can be simulated by a Turing Machine.

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A Model Discussion

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- In Complexity Theory, we consider **efficiently computable** the problems which are solved (aka the languages that are decided) in **polynomial number of steps** (*Edmonds-Cobham Thesis*).

$\boxed{\textbf{Efficiently Computable} \equiv \textbf{Polynomial-Time Computable}}$

Summary

- Computational Complexity classifies problems into classes, and studies the relations and the structure of these classes.
- We have decision problems with boolean answer, or function/optimization problems which output an object as an answer.
- Given some nice properties of polynomials, we identify polynomial-time algorithms as efficient algorithms.

Contents

Definition

A Turing Machine M is a quintuple $M = (Q, \Sigma, \delta, q_0, F)$:

- $Q = \{q_0, q_1, q_2, q_3, \ldots, q_n, q_{\text{yes}}, q_{\text{no}}\}$ is a finite set of states.
- Σ is the alphabet. The tape alphabet is $\Gamma = \Sigma \cup \{\sqcup\}$.
- $q_0 \in Q$ is the initial state.
- $F \subseteq Q$ is the set of final states.
- $\delta : (Q \setminus F) \times \Gamma \to Q \times \Gamma \times \{S, L, R\}$ is the transition function.

- A TM is a “programming language” with a single data structure (a tape), and a cursor, which moves left and right on the tape.

- Function δ is the **program** of the machine.

Turing Machines and Languages

Definition

Let $L \subseteq \Sigma^*$ be a language and M a TM such that, for every string $x \in \Sigma^*$:

- If $x \in L$, then $M(x) =$ "yes"
- If $x \notin L$, then $M(x) =$ "no"

Then we say that ***M* decides *L***.

- Alternatively, we say that $M(x) = L(x)$, where $L(x) = \chi_L(x)$ is the *characteristic function* of L (if we consider 1 as "yes" and 0 as "no").
- If L is decided by some TM M, then L is called a **recursive language**.

Definition

If for a language L there is a TM M, which if $x \in L$ then $M(x) =$ "yes", and if $x \notin L$ then $M(x) \uparrow$, we call L **recursively enumerable**.

*By $M(x) \uparrow$ we mean that M does not halt on input x (it runs forever).

Theorem

If L is recursive, then it is recursively enumerable.

Proof: *Exercise*

Definition

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If L is recursive, then it is recursively enumerable.

Proof: *Exercise*

Definition

If f is a function, $f : \Sigma^* \to \Sigma^*$, we say that a TM M computes f if, for any string $x \in \Sigma^*$, $M(x) = f(x)$. If such M exists, f is called a **recursive function**.

- Turing Machines can be thought as algorithms for solving string related problems.

Multitape Turing Machines

- We can extend the previous Turing Machine definition to obtain a Turing Machine with multiple tapes:

Definition

A k-tape Turing Machine M is a quintuple $M = (Q, \Sigma, \delta, q_0, F)$:

- $Q = \{q_0, q_1, q_2, q_3, \dots, q_n, q_{\text{halt}}, q_{\text{yes}}, q_{\text{no}}\}$ is a finite set of states.
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- $\delta : (Q \setminus F) \times \Gamma^k \to Q \times (\Gamma \times \{S, L, R\})^k$ is the transition function.

Bounds on Turing Machines

- We will characterize the “performance” of a Turing Machine by the amount of *time* and *space* required on instances of size n, when these amounts are expressed as a function of n.

Definition

Let $T : \mathbb{N} \to \mathbb{N}$. We say that machine M operates within time $T(n)$ if, for any input string x, the time required by M to reach a final state is at most $T(|x|)$. Function T is a **time bound** for M.

Definition

Let $S : \mathbb{N} \to \mathbb{N}$. We say that machine M operates within space $S(n)$ if, for any input string x, M visits at most $S(|x|)$ locations on its work tapes (excluding the input tape) during its computation. Function S is a **space bound** for M.

Multitape Turing Machines

Theorem

Given any k-tape Turing Machine M operating within time $T(n)$, we can construct a TM M' operating within time $\mathcal{O}\left(T^2(n)\right)$ such that, for any input $x \in \Sigma^*$, $M(x) = M'(x)$.

Proof: See Th.2.1 (p.30) in [1].

Multitape Turing Machines

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Given any k-tape Turing Machine M operating within time $T(n)$, we can construct a TM M' operating within time $\mathcal{O}\left(T^2(n)\right)$ such that, for any input $x \in \Sigma^*$, $M(x) = M'(x)$.

Proof: See Th.2.1 (p.30) in [1].

This is a strong evidence of the robustness of our model:
Adding a bounded number of strings does not increase their computational capabilities, and affects their efficiency only polynomially.

Linear Speedup

Theorem

Let M be a TM that decides $L \subseteq \Sigma^*$, that operates within time $T(n)$. Then, for every $\varepsilon > 0$, there is a TM M' which decides the same language and operates within time $T'(n) = \varepsilon T(n) + n + 2$.

Proof: See Th.2.2 (p.32) in [1].

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Proof: See Th.2.2 (p.32) in [1].

- If, for example, T is linear, i.e. something like cn, then this theorem states that the constant c can be made arbitrarily close to 1. So, it is fair to start using the $\mathcal{O}(\cdot)$ notation in our time bounds.
- A similar theorem holds for space:

Linear Speedup

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Theorem

Let M be a TM that decides $L \subseteq \Sigma^*$, that operates within space $S(n)$. Then, for every $\varepsilon > 0$, there is a TM M' which decides the same language and operates within space $S'(n) = \varepsilon S(n) + 2$.

Nondeterministic Turing Machines

- We will now introduce an **unrealistic** model of computation:

Definition

A Turing Machine M is a quintuple $M = (Q, \Sigma, \delta, q_0, F)$:

- $Q = \{q_0, q_1, q_2, q_3, \ldots, q_n, q_{\text{halt}}, q_{\text{yes}}, q_{\text{no}}\}$ is a finite set of states.
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- $\delta : (Q \setminus F) \times \Gamma \to Pow(Q \times \Gamma \times \{S, L, R\})$ is the transition **relation**.

Nondeterministic Turing Machines

- In this model, an input is accepted if **there is** *some sequence* of nondeterministic choices that results in "yes".
- An input is rejected if there is *no sequence* of choices that lead to acceptance.
- Observe the similarity with recursively enumerable languages.

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Definition

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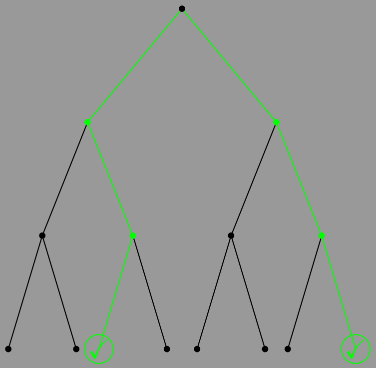
Definition

We say that M operates within bound $T(n)$, if for every input $x \in \Sigma^*$ and every sequence of nondeterministic choices, M reaches a final state within $T(|x|)$ steps.

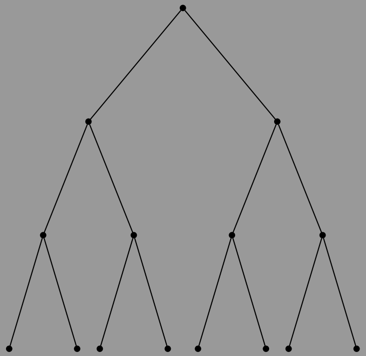
- The above definition requires that M does not have computation paths longer than $T(n)$, where $n = |x|$ the length of the input.
- The amount of time charged is the *depth* of the computation tree. **computation tree**.

Examples of Nondeterministic Computations

Example



Accepting computation



Rejecting Computation

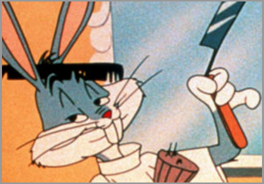
- Without loss of generality, the computation trees are binary, full and complete. (*why*?)

Summary

- A recursive language is decided by a TM.
- A recursive enumerable language is accepted by a TM that halts only if $x \in L$.
- Multiple tape TMs can be simulated by a one-tape TM with quadratic overhead.
- Linear speedup justifies the $\mathcal{O}(\cdot)$ notation.
- Nondeterministic TMs move in "parallel universes", making different choices simultaneously.
- A Deterministic TM computation is a *path*.
- A Nondeterministic TM computation is a *tree*, i.e. exponentially many paths ran simultaneously.

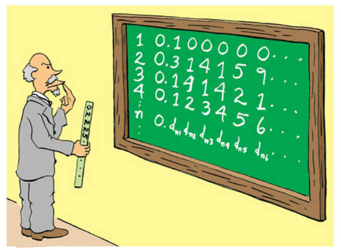
Contents

Diagonalization



Suppose there is a town with just one barber, who is male. In this town, the barber shaves all those, and only those, men in town who do not shave themselves. Who shaves the barber?

Diagonalization is a technique that was used in many different cases:



George showed it wouldn't fit in.

Diagonalization

Theorem

The functions from $\mathbb{N}$ to $\mathbb{N}$ are uncountable.

Proof: Let, for the sake of contradiction that are countable: $\phi_1, \phi_2, \ldots$. Consider the following function: $f(x) = \phi_x(x) + 1$. This function must appear somewhere in this enumeration, so let $\phi_y = f(x)$. Then $\phi_y(x) = \phi_x(x) + 1$, and if we choose y as an argument, then $\phi_y(y) = \phi_y(y) + 1$. $\square$

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- Using the same argument:

Theorem

The functions from $\{0,1\}^*$ to $\{0,1\}$ are uncountable.

Machines as strings

- It is obvious that we can represent a Turing Machine as a string: *just write down the description and encode it using an alphabet, e.g.* $\{0, 1\}$.
- We denote by $\llcorner M \lrcorner$ the TM M's representation as a string.
- Also, if $x \in \Sigma^*$, we denote by M_x the TM that x represents.

Keep in mind that:

- **Every string represents *some* TM.**
- **Every TM is represented by *infinitely many* strings.**

- There exists (*at least*) an uncomputable function from $\{0, 1\}^*$ to $\{0, 1\}$, since the set of all TMs is countable.

The Universal Turing Machine

- So far, our computational models are specified to solve a single problem.
- Turing observed that there is a TM that can simulate any other TM M, given M's description as input.

Theorem

There exists a TM $\mathcal{U}$ such that for every $x, w \in \Sigma^*$, $\mathcal{U}(x, w) = M_w(x)$. Also, if M_w halts within T steps on input x, then $\mathcal{U}(x, w)$ halts within $CT \log T$ steps, where C is a constant independent of x, and depending only on M_w's alphabet size number of tapes and number of states.

Proof: See section 3.1 in [1], and Th. 1.9 and section 1.7 in [2].

The Halting Problem

- Consider the following problem: "*Given the description of a TM M, and a string x, will M halt on input x?*" This is called the HALTING PROBLEM.
- **We want to compute this problem ! ! !** (Given a computer program and an input, will this program enter an infinite loop?)
- In language form: $\mathrm{H} = \{\llcorner M \lrcorner; x \mid M(x) \downarrow\}$, where " $\downarrow$ " means that the machine halts, and " $\uparrow$ " that it runs forever.

Theorem

H *is recursively enumerable.*

Proof: See Th.3.1 (p.59) in [1]

- In fact, H is not just a recursively enumerable language:
  If we had an algorithm for deciding H, then we would be able to derive an algorithm for deciding any r.e. language (**RE**-complete).

The Halting Problem

- But....

Theorem

H *is not recursive.*

Proof: See Th.3.1 (p.60) in [1]

- Suppose, for the sake of contradiction, that there is a TM M_H that decides H.
- Consider the TM D:
  $\boxed{D(\llcorner M \lrcorner) : \textbf{ if } M_H(\llcorner M \lrcorner ; \llcorner M \lrcorner) = \text{“yes” } \textbf{then} \uparrow \textbf{ else } \text{“yes”}}$
- What is $D(\llcorner D \lrcorner)$?

The Halting Problem

- But....

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Proof:

See Th.3.1 (p.60) in [1]

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- What is $D(\llcorner D \lrcorner)$?
- If $D(\llcorner D \lrcorner) \uparrow$, then M_H accepts the input, so $\llcorner D \lrcorner ; \llcorner D \lrcorner \in$ H, so $D(D) \downarrow$.
- If $D(\llcorner D \lrcorner) \downarrow$, then M_H rejects $\llcorner D \lrcorner ; \llcorner D \lrcorner$, so $\llcorner D \lrcorner ; \llcorner D \lrcorner \notin$ H, so $D(D) \uparrow$. $\square$

- Recursive languages are a *proper* subset of recursive enumerable ones.
- Recall that the complement of a language L is defined as:

$$\overline{L} = \{x \in \Sigma^* \mid x \notin L\} = \Sigma^* \setminus L$$

Theorem

1. *If L is recursive, so is $\overline{L}$.*
2. *L is recursive if and only if L and $\overline{L}$ are recursively enumerable.*

Proof: Exercise

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- Recall that the complement of a language L is defined as:

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Theorem

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Proof: Exercise

- Let $E(M) = \{x \mid (q_0, \triangleright, \varepsilon) \stackrel{M*}{\rightarrow} (q, y \sqcup x\sqcup, \varepsilon\}$
- $E(M)$ is the language *enumerated* by M.

Theorem

L is recursively enumerable iff there is a TM M such that $L = E(M)$.

More Undecidability

- The HALTING PROBLEM, our first undecidable problem, was the first, but not the only undecidable problem. Its spawns a wide range of such problems, via *reductions*.
- To show that a problem A is undecidable we establish that, if there is an algorithm for A, then there would be an algorithm for H, which is absurd.

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- To show that a problem A is undecidable we establish that, if there is an algorithm for A, then there would be an algorithm for H, which is absurd.

Theorem

The following languages are not recursive:

1. $\{\llcorner M \lrcorner \mid M \text{ halts on all inputs}\}$
2. $\{\llcorner M \lrcorner; x \mid \text{There is a } y \text{ such that } M(x) = y\}$
3. $\{\llcorner M \lrcorner; x \mid \text{The computation of } M \text{ uses all states of } M\}$
4. $\{\llcorner M \lrcorner; x; y \mid M(x) = y\}$

Rice's Theorem

- The previous problems lead us to a more general conclusion:

Any non-trivial property of languages of Turing Machines is undecidable

- If a TM M accepts a language L, we write $L = L(M)$.

Theorem (Rice's Theorem)

Suppose that $\mathcal{C}$ is a proper, non-empty subset of the set of all recursively enumerable languages. Then, the following problem is undecidable:

Given a Turing Machine M, is $L(M) \in \mathcal{C}$?

Rice's Theorem

Proof:

See Th.3.2 (p.62) in [1]

- We can assume that $\emptyset \notin \mathcal{C}$ (*why?*).
- Since $\mathcal{C}$ is nonempty, $\exists\, L \in \mathcal{C}$, accepted by the TM M_L.
- Let M_H the TM accepting the HALTING PROBLEM for an arbitrary input x. For each $x \in \Sigma^*$, we construct a TM M as follows:

  $\boxed{M(y)\ :\ \textbf{if } M_H(x) = \text{“yes” } \textbf{then } M_L(y) \textbf{ else } \uparrow}$

- We claim that: $L(M) \in \mathcal{C}$ if and only if $x \in$ H.

Rice's Theorem

Proof:

See Th.3.2 (p.62) in [1]

- We can assume that $\emptyset \notin \mathcal{C}$ (*why?*).
- Since $\mathcal{C}$ is nonempty, $\exists\, L \in \mathcal{C}$, accepted by the TM M_L.
- Let M_H the TM accepting the HALTING PROBLEM for an arbitrary input x. For each $x \in \Sigma^*$, we construct a TM M as follows:

$$\boxed{M(y) : \textbf{ if } M_H(x) = \text{“yes” } \textbf{then } M_L(y) \textbf{ else } \uparrow}$$

- We claim that: $L(M) \in \mathcal{C}$ if and only if $x \in \mathrm{H}$.
  Proof of the claim:
  - If $x \in \mathrm{H}$, then $M_H(x) = $ “yes”, and so M will accept y or never halt, depending on whether $y \in L$. Then the language accepted by M is exactly L, which is in $\mathcal{C}$.
  - If $M_H(x) \uparrow$, M never halts, and thus M accepts the language $\emptyset$, which is not in $\mathcal{C}$. $\square$

Summary

- TMs are encoded by strings.
- The Universal TM $\mathcal{U}(x, \llcorner M \lrcorner)$ can simulate any other TM M along with an input x.
- The Halting Problem is recursively enumerable, but not recursive.
- Many other problems can be proved undecidable, by a *reduction* from the Halting Problem.
- Rice's theorem states that *any non-trivial property of TMs is an undecidable problem*.

Contents

Parameters used to define complexity classes:

- Model of Computation (Turing Machine, RAM, Circuits)
- Mode of Computation (Deterministic, Nondeterministic, Probabilistic)
- Complexity Measures (*Time, Space, Circuit Size-Depth*)
- Other Parameters (Randomization, Interaction)

Our first complexity classes

Definition

Let $L \subseteq \Sigma^*$, and $T, S : \mathbb{N} \to \mathbb{N}$:

- We say that $L \in \mathbf{DTIME}[T(n)]$ if there exists a TM M deciding L, which operates within the *time* bound $\mathcal{O}(T(n))$, where $n = |x|$.
- We say that $L \in \mathbf{DSPACE}[S(n)]$ if there exists a TM M deciding L, which operates within *space* bound $\mathcal{O}(S(n))$, that is, for any input x, requires space at most $S(|x|)$.
- We say that $L \in \mathbf{NTIME}[T(n)]$ if there exists a *nondeterministic* TM M deciding L, which operates within the time bound $\mathcal{O}(T(n))$.
- We say that $L \in \mathbf{NSPACE}[S(n)]$ if there exists a *nondeterministic* TM M deciding L, which operates within space bound $\mathcal{O}(S(n))$.

Our first complexity classes

- The above are **Complexity Classes**, in the sense that they are sets of languages.
- All these classes are parameterized by a function T or S, so they are *families* of classes (for each function we obtain a complexity class).

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Definition (Complementary complexity class)

For any complexity class $\mathcal{C}$, $co\mathcal{C}$ denotes the class: $\{\overline{L} \mid L \in \mathcal{C}\}$, where $\overline{L} = \Sigma^* \setminus L = \{x \in \Sigma^* \mid x \notin L\}$.

- We want to define "reasonable" complexity classes, in the sense that we want to "compute more problems", given more computational resources.

Constructible Functions

- Can we use all computable functions to define Complexity Classes?

Constructible Functions

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Theorem (Gap Theorem)

For any computable functions r and a, there exists a computable function f such that $f(n) \geq a(n)$, and

$$\mathbf{DTIME}[f(n)] = \mathbf{DTIME}[r(f(n))]$$

- That means, for $r(n) = 2^{2^n}$, the incementation from $f(n)$ to $2^{2^{f(n)}}$ does not allow the computation of any new function!
- So, we must use some restricted families of functions:

Constructible Functions

Definition (Time-Constructible Function)

A nondecreasing function $T : \mathbb{N} \to \mathbb{N}$ is **time constructible** if $T(n) \geq n$ and there is a TM M that computes the function $x \mapsto \llcorner T(|x|) \lrcorner$ in time $T(n)$.

Definition (Space-Constructible Function)

A nondecreasing function $S : \mathbb{N} \to \mathbb{N}$ is **space-constructible** if $S(n) > \log n$ and there is a TM M that computes $S(|x|)$ using $S(|x|)$ space, given x as input.

Constructible Functions

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- The restriction $T(n) \geq n$ is to allow the machine to read its input.
- The restriction $S(n) > \log n$ is to allow the machine to "remember" the index of the cell of the input tape that it is currently reading.

Constructible Functions

- Also, if $f_1(n), f_2(n)$ are time/space-constructible functions, so are $f_1 + f_2, f_1 \cdot f_2$ and $f_1^{f_2}$.

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Constructible Functions

- Also, if $f_1(n), f_2(n)$ are time/space-constructible functions, so are $f_1 + f_2, f_1 \cdot f_2$ and $f_1^{f_2}$.
- If we use only *constructible* functions, we can prove **Hierarchy Theorems**, stating that with more resources we can compute more languages:

Theorem (Hierarchy Theorems)

Let t_1, t_2 be time-constructible functions, and s_1, s_2 be space-constructible functions. Then:

1. *If* $t_1(n) \log t_1(n) = o(t_2(n))$, *then* $\mathbf{DTIME}(t_1) \subsetneq \mathbf{DTIME}(t_2)$.
2. *If* $t_1(n+1) = o(t_2(n))$, *then* $\mathbf{NTIME}(t_1) \subsetneq \mathbf{NTIME}(t_2)$.
3. *If* $s_1(n) = o(s_2(n))$, *then* $\mathbf{DSPACE}(s_1) \subsetneq \mathbf{DSPACE}(s_2)$.
4. *If* $s_1(n) = o(s_2(n))$, *then* $\mathbf{NSPACE}(s_1) \subsetneq \mathbf{NSPACE}(s_2)$.

Simplified Case of Deterministic Time Hierarchy Theorem

Theorem

$\mathbf{DTIME}[n] \subsetneq \mathbf{DTIME}[n^{1.5}]$

Simplified Case of Deterministic Time Hierarchy Theorem

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Proof (*Diagonalization*):

See Th.3.1 (p.69) in [2]

Let D be the following machine:

On input x, run for $|x|^{1.4}$ steps $\mathcal{U}(M_x, x)$;
If $\mathcal{U}(M_x, x) = b$, then return $1 - b$;
Else return 0;

- Clearly, $L = L(D) \in \mathbf{DTIME}[n^{1.5}]$
- We claim that $L \notin \mathbf{DTIME}[n]$:
  Let $L \in \mathbf{DTIME}[n] \Rightarrow \exists\, M : \; M(x) = D(x) \; \forall x \in \Sigma^*$, and M works for $\mathcal{O}(|x|)$ steps.
  The time to simulate M using $\mathcal{U}$ is $c|x| \log |x|$, for some c.

Simplified Case of Deterministic Time Hierarchy Theorem

Proof (*cont'd*):
$\exists n_0 : n^{1.4} > cn \log n \; \forall n \geq n_0$
There exists a x_M, s.t. $x_M = \llcorner M \lrcorner$ and $|x_M| > n_0$ (*why?*) Then,
$D(x_M) = 1 - M(x_M)$ (while we have also that $D(x) = M(x), \; \forall x$)

Simplified Case of Deterministic Time Hierarchy Theorem

Proof (*cont'd*):

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Contradiction!! $\square$

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$\mathbf{D(x_M) = 1 - M(x_M)}$ (while we have also that $D(x) = M(x), \ \forall x$)
Contradiction!! $\square$

- So, we have the hierachy:

$$\mathbf{DTIME}[n] \subsetneq \mathbf{DTIME}[n^2] \subsetneq \mathbf{DTIME}[n^3] \subsetneq \cdots$$

- We will later see that the class containing the problems we can efficiently solve (recall the Edmonds-Cobham Thesis) is the class $\mathbf{P} = \bigcup_{c \in \mathbb{N}} \mathbf{DTIME}[n^c]$.

- Hierarchy Theorems tell us how classes of the same kind relate to each other, when we vary the complexity bound.
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Theorem

Suppose that $T(n), S(n)$ are time-constructible and space-constructible functions, respectively. Then:

1. $\mathbf{DTIME}[T(n)] \subseteq \mathbf{NTIME}[T(n)]$
2. $\mathbf{DSPACE}[S(n)] \subseteq \mathbf{NSPACE}[S(n)]$
3. $\mathbf{NTIME}[T(n)] \subseteq \mathbf{DSPACE}[T(n)]$
4. $\mathbf{NSPACE}[S(n)] \subseteq \mathbf{DTIME}[2^{\mathcal{O}(S(n))}]$

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Corollary

$$\mathbf{NTIME}[T(n)] \subseteq \bigcup_{c>1} \mathbf{DTIME}[c^{T(n)}]$$

Proof:

See Th.7.4 (p.147) in [1]

1. Trivial
2. Trivial
3. We can simulate the machine for each nondeterministic choice, using at most $T(n)$ steps in each simulation.
   There are *exponentially* many simulations, but we can simulate them one-by-one, *reusing the same space*.
4. Recall the notion of a configuration of a TM: For a k-tape machine, is a $2k-2$ tuple: $(q, i, w_2, u_2, \ldots, w_{k-1}, u_{k-1})$
   How many configurations are there?
   - $|Q|$ choices for the state
   - $n+1$ choices for i, and
   - Fewer than $|\Sigma|^{(2k-2)S(n)}$ for the remaining strings

   So, the total number of configurations on input size n is at most $nc_1^{S(n)} = 2^{\mathcal{O}(S(n))}$.

Proof (*cont'd*):

Definition (Configuration Graph of a TM)

The configuration graph of M on input x, denoted $G(M,x)$, has as **vertices** all the possible configurations, and there is an **edge** between two vertices C and C' if and only if C' can be reached from C in one step, according to M's transition function.

Proof (*cont'd*):

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- So, we have reduced this simulation to REACHABILITY* problem (also known as S-T CONN), for which we know there is a poly-time ($\mathcal{O}\left(n^2\right)$) algorithm.
- So, the simulation takes $\left(2^{\mathcal{O}(S(n))}\right)^2 \sim 2^{\mathcal{O}(S(n))}$ steps. $\square$

*REACHABILITY: Given a graph G and two nodes $v_1, v_n \in V$, is there a path from v_1 to v_n?

The essential Complexity Hierarchy

Definition

$$\mathbf{L} = \mathbf{DSPACE}[\log n]$$

$$\mathbf{NL} = \mathbf{NSPACE}[\log n]$$

$$\mathbf{P} = \bigcup_{c \in \mathbb{N}} \mathbf{DTIME}[n^c]$$

$$\mathbf{NP} = \bigcup_{c \in \mathbb{N}} \mathbf{NTIME}[n^c]$$

$$\mathbf{PSPACE} = \bigcup_{c \in \mathbb{N}} \mathbf{DSPACE}[n^c]$$

$$\mathbf{NPSPACE} = \bigcup_{c \in \mathbb{N}} \mathbf{NSPACE}[n^c]$$

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$$\mathbf{EXPSPACE} = \bigcup_{c \in \mathbb{N}} \mathbf{DSPACE}[2^{n^c}]$$

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$$\mathbf{L} \subseteq \mathbf{NL} \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE} \subseteq \mathbf{NPSPACE} \subseteq \mathbf{EXP} \subseteq \mathbf{NEXP}$$

Certificate Characterization of NP

Definition

Let $R \subseteq \Sigma^* \times \Sigma^*$ a binary relation on strings.

- R is called **polynomially decidable** if there is a DTM deciding the language $\{x; y \mid (x, y) \in R\}$ in polynomial time.
- R is called **polynomially balanced** if $(x, y) \in R$ implies $|y| \leq |x|^k$, for some $k \geq 1$.

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Theorem

*Let $L \subseteq \Sigma^*$ be a language. $L \in$ **NP** if and only if there is a polynomially decidable and polynomially balanced relation R, such that:*

$$L = \{x \mid \exists y \, R(x, y)\}$$

- This y is called **succinct certificate**, or **witness**.
- So, an **NP** **Search Problem** is the problem of *computing* witnesses.

Proof:

See Pr.9.1 (p.181) in [1]

($\Leftarrow$) If such an R exists, we can construct the following NTM deciding L:
"On input x, *guess* a y, such that $|y| \leq |x|^k$, and then test (in poly-time) if $(x, y) \in R$. If so, accept, else reject." Observe that an accepting computation exists if and only if $x \in L$.

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($\Leftarrow$) If such an R exists, we can construct the following NTM deciding L:
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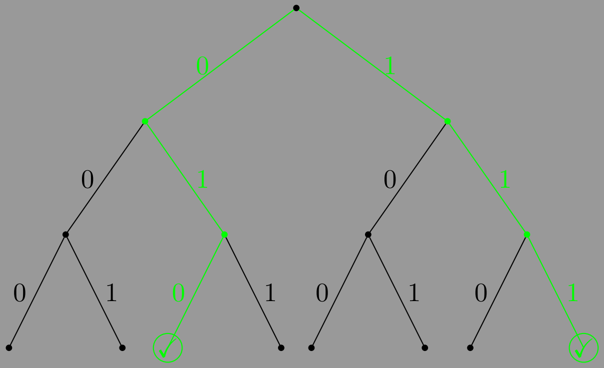
($\Rightarrow$) If $L \in$ **NP**, then $\exists$ an NTM N that decides L in time $|x|^k$, for some k. Define the following R:
"$(x, y) \in R$ if and only if y is an **encoding** of an accepting computation of $N(x)$."
R is polynomially balanced and decidable (*why?*), so, given by assumption that N decides L, we have our conclusion. $\square$

Certificate Characterization of NP

Example (Encoding of a computation path)



- 010 and 111 encode accepting paths.

Can creativity be automated?

As we saw:

- Class **P**: Efficient **Computation**
- Class **NP**: Efficient **Verification**
- So, if we can efficiently verify a mathematical proof, can we create it efficiently?

Can creativity be automated?

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If $P = NP$...

- For every mathematical statement, and given a page limit, we would (quickly) generate a proof, if one exists.
- Given detailed constraints on an engineering task, we would (quickly) generate a design which meets the given criteria, if one exists.
- Given data on some phenomenon and modeling restrictions, we would (quickly) generate a theory to explain the data, if one exists.

See "A. Wigderson: Knowledge, Creativity and **P** versus **NP**"

Complementary complexity classes

- Deterministic complexity classes are in general closed under complement (*co*$\mathbf{L} = \mathbf{L}$, *co*$\mathbf{P} = \mathbf{P}$, *co*$\mathbf{PSPACE} = \mathbf{PSPACE}$).
- Complementaries of non-deterministic complexity classes are very interesting:
- The class *co*$\mathbf{NP}$ contains all the languages that have **succinct disqualifications** (the analogue of *succinct certificate* for the class $\mathbf{NP}$). The "no" instance of a problem in *co*$\mathbf{NP}$ has a short proof of its being a "no" instance.
- So:

$$\boxed{\mathbf{P} \subseteq \mathbf{NP} \cap co\mathbf{NP}}$$

- Note the *similarity* and the *difference* with $\mathbf{R} = \mathbf{RE} \cap co\mathbf{RE}$.

Quantifier Characterization of Complexity Classes

Definition

We denote as $\mathcal{C} = (Q_1/Q_2)$, where $Q_1, Q_2 \in \{\exists, \forall\}$, the class $\mathcal{C}$ of languages L satisfying:

- $x \in L \Rightarrow Q_1 y\, R(x, y)$
- $x \notin L \Rightarrow Q_2 y\, \neg R(x, y)$

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- $x \in L \Rightarrow Q_1 y\, R(x, y)$
- $x \notin L \Rightarrow Q_2 y\, \neg R(x, y)$

- $\mathbf{P} = (\forall/\forall)$
- $\mathbf{NP} = (\exists/\forall)$
- $co\mathbf{NP} = (\forall/\exists)$

Savitch's Theorem

- REACHABILITY $\in$ **NL**.

See Ex.2.10 (p.48) in [1]

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Theorem (Savitch's Theorem)

REACHABILITY $\in \mathbf{DSPACE}[\log^2 n]$

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Theorem (Savitch's Theorem)

REACHABILITY $\in \mathbf{DSPACE}[\log^2 n]$

Proof:

See Th.7.4 (p.149) in [1]

$REACH(x, y, i)$: "*There is a path from x to y, of length $\leq i$*".

- We can solve REACHABILITY if we can compute $REACH(x, y, n)$, for any nodes $x, y \in V$, since any path in G can be at most n long.
- If $i = 1$, we can check whether $REACH(x, y, i)$.
- If $i > 1$, we use recursion:

Proof (*cont'd*):

```
def REACH(s,t,k)
    if k==1:
        if (s==t or (s,t) in edges): return true
    if k>1:
        for u in vertices:
            if (REACH(s,u, floor(k/2)) and
            (REACH(u,t,ceil(k/2))): return true
    return false
```

Proof (*cont'd*):

```
def REACH(s,t,k)
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    return false
```

- We generate all nodes u one after the other, *reusing* space.
- The algorithm has recursion depth of $\lceil \log n \rceil$.
- For each recursion level, we have to store s, t, k and u, that is, $\mathcal{O}(\log n)$ space.
- Thus, the total space used is $\mathcal{O}\left(\log^2 n\right)$. $\square$

Savitch's Theorem

Corollary

$\mathbf{NSPACE}[S(n)] \subseteq \mathbf{DSPACE}[S^2(n)]$, for any space-constructible function $S(n) \geq \log n$.

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Proof:

- Let M be the nondeterministic TM to be simulated.
- We run the algorithm of Savitch's Theorem proof on the configuration graph of M on input x.
- Since the configuration graph has $c^{S(n)}$ nodes, $\mathcal{O}\left(S^2(n)\right)$ space suffices. $\square$

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Corollary

$$\mathbf{PSPACE} = \mathbf{NPSPACE}$$

NL-Completeness

- In Complexity Theory, we "connect" problems in a complexity class with partial ordering relations, called **reductions**, which formalize the notion of "*a problem that is at least as hard as another*".
- A reduction must be computationally weaker than the class in which we use it.

NL-Completeness

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- A reduction must be computationally weaker than the class in which we use it.

Definition

A language L_1 is **logspace reducible** to a language L_2, denoted $L_1 \leq_m^\ell L_2$, if there is a function $f : \Sigma^* \to \Sigma^*$, computable by a DTM in $\mathcal{O}(\log n)$ space, such that for all $x \in \Sigma^*$:

$$x \in L_1 \Leftrightarrow f(x) \in L_2$$

We say that a language A is **NL**-complete if it is in **NL** and for every $B \in \mathbf{NL}$, $B \leq_m^\ell A$.

NL-Completeness

Theorem

REACHABILITY is **NL**-*complete.*

NL-Completeness

Theorem

REACHABILITY is **NL***-complete.*

Proof:

See Th.4.18 (p.89) in [2]

- We 've argued why REACHABILITY $\in$ **NL**.
- Let $L \in$ **NL**, that is, it is decided by a $\mathcal{O}(\log n)$ NTM N.
- Given input x, we can construct the *configuration graph* of $N(x)$.
- We can assume that this graph has a *single* accepting node.
- We can construct this in logspace: Given configurations C, C' we can in space $\mathcal{O}\left(|C| + |C'|\right) = \mathcal{O}(\log |x|)$ check the graph's adjacency matrix if they are connected by an edge.
- It is clear that $x \in L$ if and only if the produced instance of REACHABILITY has a "yes" answer. $\square$

Certificate Definition of NL

- We want to give a characterization of **NL**, similar to the one we gave for **NP**.
- A certificate may be polynomially long, so a logspace machine may not have the space to store it.
- So, we will assume that the certificate is provided to the machine on a separate tape that is **read once**.

Certificate Definition of NL

Definition

A language L is in **NL** if there exists a deterministic TM M with an additional special read-once input tape, such that for every $x \in \Sigma^*$:

$$x \in L \Leftrightarrow \exists y, |y| \in poly(|x|), M(x, y) = 1$$

where by $M(x, y)$ we denote the output of M where x is placed on its input tape, and y is placed on its special read-once tape, and M uses at most $\mathcal{O}(\log |x|)$ space on its read-write tapes for every input x.

- What if remove the read-once restriction and allow the TM's head to move back and forth on the certificate, and read each bit multiple times?

Immerman-Szelepscényi

Theorem (The Immerman-Szelepscényi Theorem)

$\overline{\text{REACHABILITY}} \in \mathbf{NL}$

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Proof:

See Th.4.20 (p.91) in [2]

- It suffices to show a $\mathcal{O}(\log n)$ verification algorithm A such that: $\forall\ (G, s, t)$, $\exists$ a polynomial certificate u such that: $A((G, s, t), u) =$ "yes" iff t is not reachable from s.
- A has read-once access to u.
- G's vertices are identified by numbers in $\{1, \ldots, n\} = [n]$
- C_i: "*The set of vertices reachable from s in $\leq i$ steps.*"
- Membership in C_i is easily certified:
- $\forall i \in [n]$: $v_0, \ldots, v_k$ along the path from s to v, $k \leq i$.
- The certificate is at most polynomial in n.

The Immerman-Szelepscényi Theorem

Proof (*cont'd*):

- We can check the certificate using read-once access:
  1. $v_0 = s$
  2. for $j > 0$, $(v_{j-1}, v_j) \in E(G)$
  3. $v_k = v$
  4. Path ends within at most i steps
- We now construct two types of certificates:
  1. A certificate that a vertex $v \notin C_i$, given $|C_i|$.
  2. A certificate that $|C_i| = c$, for some c, given $|C_{i-1}|$.
- Since $C_0 = \{s\}$, we can provide the 2nd certificate to convince the verifier for the sizes of $C_1, \ldots, C_n$
- C_n is the set of vertices *reachable* from s.

The Immerman-Szelepscényi Theorem

Proof (*cont'd*):

- Since the verifier has been convinced of $|C_n|$, we can use the 1st type of certificate to convince the verifier that $t \notin C_n$.

The Immerman-Szelepscényi Theorem

Proof (*cont'd*):

- Since the verifier has been convinced of $|C_n|$, we can use the 1st type of certificate to convince the verifier that $t \notin C_n$.

- **Certifying that $v \notin C_i$, given $|C_i|$**
  The certificate is the list of certificates that $u \in C_i$, for every $u \in C_i$.
  The verifier will check:
  1. Each certificate is valid
  2. Vertex u, given a certificate for u, is larger than the previous.
  3. No certificate is provided for v.
  4. The total number of certificates is exactly $|C_i|$.

The Immerman-Szelepscényi Theorem

Proof (*cont'd*):

Certifying that $v \notin C_i$, given $|C_{i-1}|$

The certificate is the list of certificates that $u \in C_{i-1}$, for every $u \in C_{i-1}$

The verifier will check:

1. Each certificate is valid
2. Vertex u, given a certificate for u, is larger than the previous.
3. No certificate is provided for v *or for a neighbour of* v.
4. The total number of certificates is exactly $|C_{i-1}|$.

The Immerman-Szelepscényi Theorem

Proof (*cont'd*):

Certifying that $v \notin C_i$, given $|C_{i-1}|$

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The verifier will check:

1. Each certificate is valid
2. Vertex u, given a certificate for u, is larger than the previous.
3. No certificate is provided for v *or for a neighbour of* v.
4. The total number of certificates is exactly $|C_{i-1}|$.

Certifying that $|C_i| = c$, given $|C_{i-1}|$

The certificate will consist of n certificates, for vertices 1 to n, in ascending order.

The verifier will check all certificates, and count the vertices that have been certified to be in C_i. If $|C_i| = c$, it accepts. □

The Immerman-Szelepscényi Theorem

Corollary

For every space constructible $S(n) > \log n$:

$$\mathbf{NSPACE}[S(n)] = co\mathbf{NSPACE}[S(n)]$$

Proof:

- Let $L \in \mathbf{NSPACE}[S(n)]$. We will show that $\exists\ S(n)$ space-bounded NTM $\overline{M}$ deciding $\overline{L}$:
- $\overline{M}$ on input x uses the above certification procedure on the *configuration graph* of M. $\square$

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Corollary

$$\mathbf{NL} = co\mathbf{NL}$$

What about Undirected Reachability?

- UNDIRECTED REACHABILITY captures the phenomenon of configuration graphs with both directions.
- H. Lewis and C. Papadimitriou defined the class **SL** (**S**ymmetric **L**ogspace) as the class of languages decided by a **Symmetric Turing Machine** using logarithmic space.
- Obviously,

$$\boxed{\mathbf{L} \subseteq \mathbf{SL} \subseteq \mathbf{NL}}$$

- As in the case of **NL**, UNDIRECTED REACHABILITY is **SL**-complete.
- But in 2004, Omer Reingold showed, using expander graphs, a deterministic logspace algorithm for UNDIRECTED REACHABILITY, so:

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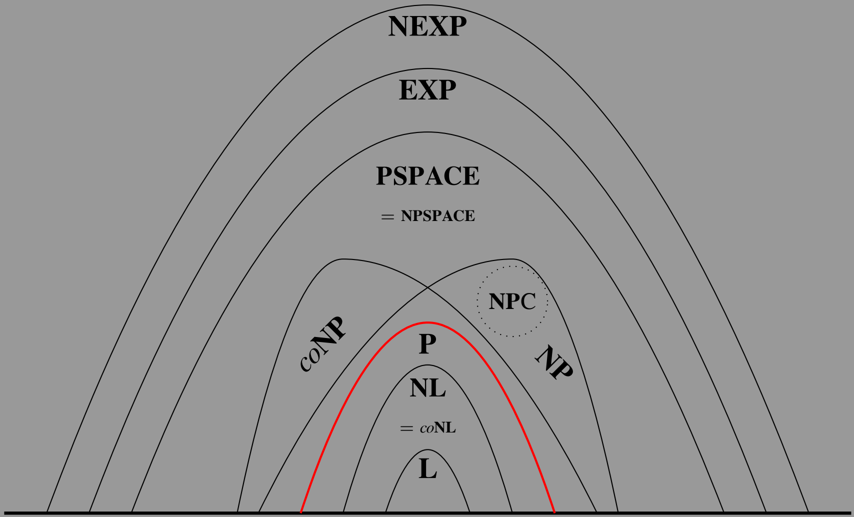
Theorem (Reingold, 2004)

$$\mathbf{L} = \mathbf{SL}$$

Our Complexity Hierarchy Landscape



Karp Reductions

Definition

A language L_1 is **Karp reducible** to a language L_2, denoted by $L_1 \leq_m^p L_2$, if there is a function $f : \Sigma^* \to \Sigma^*$, computable by a polynomial-time DTM, such that for all $x \in \Sigma^*$:

$$x \in L_1 \Leftrightarrow f(x) \in L_2$$

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$$x \in L_1 \Leftrightarrow f(x) \in L_2$$

Definition

Let $\mathcal{C}$ be a complexity class.

- We say that a language A is **$\mathcal{C}$-hard** (or $\leq_m^p$-hard for $\mathcal{C}$) if for every $B \in \mathcal{C}$, $B \leq_m^p A$.
- We say that a language A is **$\mathcal{C}$-complete**, if it is $\mathcal{C}$-hard, and also $A \in \mathcal{C}$.

Karp reductions vs logspace redutions

Theorem

A logspace reduction is a polynomial-time reduction.

Proof:

See Th.8.1 (p.160) in [1]

- Let M the logspace reduction TM.
- M has $2^{\mathcal{O}(\log n)}$ possible configurations.
- The machine is deterministic, so *no configuration can be repeated* in the computation.
- So, the computation takes $\mathcal{O}\left(n^k\right)$ time, for some k. $\square$

Circuits and CVP

Definition (Boolean circuits)

For every $n \in \mathbb{N}$ an n-input, single output Boolean Circuit C is a directed acyclic graph with n sources and *one* sink.

- All nonsource vertices are called *gates* and are labeled with one of $\wedge$ (and), $\vee$ (or) or $\neg$ (not).
- The vertices labeled with $\wedge$ and $\vee$ have *fan-in* (i.e. number or incoming edges) 2.
- The vertices labeled with $\neg$ have *fan-in* 1.
- For every vertex v of C, we assign a value as follows: for some input $x \in \{0,1\}^n$, if v is the i-th input vertex then $val(v) = x_i$, and otherwise $val(v)$ is defined recursively by applying v's logical operation on the values of the vertices connected to v.
- The *output* $C(x)$ is the value of the output vertex.

Circuits and CVP

Definition (CVP)

Circuit Value Problem (CVP): Given a circuit C and an assignment x to its variables, determine whether $C(x) = 1$.

- CVP $\in$ $\mathbf{P}$.

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Example

REACHABILITY $\leq_m^{\ell}$ CVP: Graph $G \to$ circuit $R(G)$:

- The gates are of the form:
  - $g_{i,j,k}$, $1 \leq i,j \leq n$, $0 \leq k \leq n$.
  - $h_{i,j,k}$, $1 \leq i,j,k \leq n$
- $g_{i,j,k}$ is **true** iff there is a path from i to j without intermediate nodes bigger than k.
- $h_{i,j,k}$ is **true** iff there is a path from i to j without intermediate nodes bigger than k, and k is used.

Circuits and CVP

Example

- Input gates: $g_{i,j,0}$ is **true** iff ($i = j$ or $(i, j) \in E(G)$).
- For $k = 1, \ldots, n$: $h_{i,j,k} = \left(g_{i,k,k-1} \wedge g_{k,j,k-1}\right)$
- For $k = 1, \ldots, n$: $g_{i,j,k} = \left(g_{i,j,k-1} \vee h_{i,j,k}\right)$
- The output gate $g_{1,n,n}$ is **true** iff there is a path from 1 to n using no intermediate paths above n (sic).
- We also can compute the reduction in logspace: go over all possible i, j, k's and output the appropriate edges and sorts for the variables $(1, \ldots, 2n^3 + n^2)$.

Composing Reductions

Theorem

If $L_1 \leq_m^\ell L_2$ and $L_2 \leq_m^\ell L_3$, then $L_1 \leq_m^\ell L_3$.

Proof:

See Prop.8.2 (p.164) in [1]

- Let R, R' be the aforementioned reductions.
- We have to prove that $R'(R(x))$ is a logspace reduction.
- But $R(x)$ may by longer than $\log |x|$...

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- We have to prove that $R'(R(x))$ is a logspace reduction.
- But $R(x)$ may by longer than $\log |x|$...
- We simulate $M_{R'}$ by remembering the head position i of the input string of $M_{R'}$, i.e. the output string of M_R.
- If the head moves to the right, we increment i and simulate M_R long enough to take the i^{th} bit of the output.
- If the head stays in the same position, we just remember the i^{th} bit.
- If the head moves to the left, we decrement i and **start M_R from the beginning**, until we reach the desired bit. □

Closure under reductions

- Complete problems are the **maximal elements** of the reductions partial ordering.
- Complete problems capture the essence and difficulty of a complexity class.

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If $A \leq B$ and $B \in \mathcal{C}$, then $A \in \mathcal{C}$.

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- **P**, **NP**, *co***NP**, **L**, **NL**, **PSPACE**, **EXP** are closed under Karp and logspace reductions.
- If an **NP**-complete language is in **P**, then **P** $=$ **NP**.
- If L is **NP**-complete, then $\bar{L}$ is *co***NP**-complete.

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- If an $\mathbf{NP}$-complete language is in $\mathbf{P}$, then $\mathbf{P} = \mathbf{NP}$.
- If L is $\mathbf{NP}$-complete, then $\bar{L}$ is $co\mathbf{NP}$-complete.
- If a $co\mathbf{NP}$-complete problem is in $\mathbf{NP}$, then $\mathbf{NP} = co\mathbf{NP}$.

P-Completeness

Theorem

If two classes $\mathcal{C}$ and $\mathcal{C}'$ are both closed under reductions and there is an $L \subseteq \Sigma^*$ complete for both $\mathcal{C}$ and $\mathcal{C}'$, then $\mathcal{C} = \mathcal{C}'$.

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Theorem

If two classes $\mathcal{C}$ and $\mathcal{C}'$ are both closed under reductions and there is an $L \subseteq \Sigma^*$ complete for both $\mathcal{C}$ and $\mathcal{C}'$, then $\mathcal{C} = \mathcal{C}'$.

- Consider the **Computation Table** T of a poly-time TM $M(x)$:
  $\boxed{T_{ij} \text{ represents the contents of tape position } j \text{ at step } i.}$
- But how to remember the head position and state?
  At the i^{th} step: if the state is q and the head is in position j, then $T_{ij} \in \Sigma \times Q$.
- We say that the table is **accepting** if $T_{|x|^k-1,j} \in (\Sigma \times \{q_{yes}\})$, for some j.
- Observe that T_{ij} depends only on the contents of the **same** or **adjacent** positions at time $i - 1$.

P-Completeness

Theorem

CVP is **P***-complete.*

P-Completeness

Theorem

CVP is **P***-complete.*

Proof:

See Th. 8.1 (p.168) in [1]

- We have to show that for any $L \in \mathbf{P}$ there is a reduction R from L to CVP.
- $R(x)$ must be a variable-free circuit such that $x \in L \Leftrightarrow R(x) = 1$.
- T_{ij} depends only on $T_{i-1,j-1}, T_{i-1,j}, T_{i-1,j+1}$.
- Let $\Gamma = \Sigma \cup (\Sigma \times Q)$.
- Encode $s \in \Gamma$ as $(s_1, \ldots, s_m)$, where $m = \lceil \log |\Gamma| \rceil$.
- Then the computation table can be seen as a table of binary entries $S_{ij\ell}$, $1 \leq \ell \leq m$.
- $S_{ij\ell}$ depends only on the $3m$ entries $S_{i-1,j-1,\ell'}, S_{i-1,j,\ell'}, S_{i-1,j+1,\ell'}$, where $1 \leq \ell' \leq m$.

P-Completeness

Proof (*cont'd*):

- So, there are m Boolean Functions $f_1, \ldots, f_m : \{0,1\}^{3m} \to \{0,1\}$ s.t.:
$$S_{ij\ell} = f_\ell(\overrightarrow{S}_{i-1,j-1}, \overrightarrow{S}_{i-1,j}, \overrightarrow{S}_{i-1,j+1})$$
- Thus, there exists a Boolean Circuit C with $3m$ inputs and m outputs computing T_{ij}.
- C depends only on M, and has constant size.
- $R(x)$ will be $(|x|^k - 1) \times (|x|^k - 2)$ copies of C.
- The input gates are fixed.
- $R(x)$'s output gate will be the first bit of $C_{|x|^k-1,1}$.
- The circuit C is fixed, so we can generate indexed copies of C, using $\mathcal{O}(\log |x|)$ space for indexing. $\square$

CIRCUIT SAT & SAT

Definition (CIRCUIT SAT)

Given Boolen Circuit C, is there a truth assignment x appropriate to C, such that $C(x) = 1$?

Definition (SAT)

Given a Boolean Expression ϕ in CNF, is it satisfiable?

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Example

CIRCUIT SAT $\leq_m^\ell$ SAT:

- Given $C \rightarrow$ Boolean Formula $R(C)$, s.t. $C(x) = 1 \Leftrightarrow R(C)(x) = T$.
- Variables of $C \rightarrow$ variables of $R(C)$.
- Gate g of $C \rightarrow$ variable g of $R(C)$.

CIRCUIT SAT & SAT

Example

- Gate g of $C \rightarrow$ clauses in $R(C)$:
  - g **variable** gate: add $(\neg g \vee x) \wedge (g \vee \neg x)$ $\quad \equiv g \Leftrightarrow x$
  - g **TRUE** gate: add (g)
  - g **FALSE** gate: add $(\neg g)$
  - g **NOT** gate & $pred(g) = h$: add $(\neg g \vee \neg h) \wedge (g \vee h)$ $\quad \equiv g \Leftrightarrow \neg h$
  - g **OR** gate & $pred(g) = \{h, h'\}$: add $(\neg h \vee g) \wedge (\neg h' \vee g) \wedge (h \vee h' \vee \neg g)$ $\quad \equiv g \Leftrightarrow (h \vee h')$
  - g **AND** gate & $pred(g) = \{h, h'\}$: add $(\neg g \vee h) \wedge (\neg g \vee h') \wedge (\neg h \vee \neg h' \vee g)$ $\quad \equiv g \Leftrightarrow (h \wedge h')$
  - g **output** gate: add (g)
- $R(C)$ is satisfiable if and only if C is.
- The construction can be done within $\log |x|$ space. $\square$

Bounded Halting Problem

- We can define the time-bounded analogue of HP:

Definition (Bounded Halting Problem (BHP))

Given the code $\llcorner M \lrcorner$ of an NTM M, and input x and a string 0^t, decide if M accepts x in t steps.

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Theorem

BHP is $\mathbf{NP}$*-complete.*

Proof:

- BHP $\in \mathbf{NP}$.
- Let $A \in \mathbf{NP}$. Then, $\exists$ NTM M deciding A in time $p(|x|)$, for some $p \in poly(|x|)$.
- The reduction is the function $R(x) = \langle \llcorner M \lrcorner, x, 0^{p(|x|)} \rangle$. $\square$

Cook's Theorem

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Proof:

See Th.8.2 (p.171) in [1]

- SAT $\in$ **NP**.
- Let $L \in$ **NP**. We will show that $L \leq_m^\ell$ CIRCUIT SAT $\leq_m^\ell$ SAT.
- Since $L \in$ **NP**, there exists an NPTM M deciding L in n^k steps.
- Let $(c_1, \ldots, c_{n^k}) \in \{0,1\}^{n^k}$ a certificate for M (recall the binary encoding of the computation tree).

Cook's Theorem

Proof (*cont'd*):

- If we fix a certificate, then the computation is *deterministic* (the language's Verifier $V(x, y)$ is a DPTM).
- So, we can define the **computation table** $T(M, x, \overrightarrow{c})$.
- As before, all non-top row and non-extreme column cells T_{ij} will depend *only* on $T_{i-1,j-1}, T_{i-1,j}, T_{i-1,j+1}$ and the nondeterministic choice c_{i-1}.
- We now fixed a circuit C with $3m + 1$ input gates.
- Thus, we can construct in $\log |x|$ space a circuit $R(x)$ with variable gates $c_1, \dots c_{n^k}$ corresponding to the **nondeterministic choices** of the machine.
- $R(x)$ is satisfiable if and only if $x \in L$. $\square$

NP-completeness: Web of Reductions

- Many **NP**-complete problems stem from Cook's Theorem via reductions:
  - 3SAT, MAX2SAT, NAESAT
  - IS, CLIQUE, VERTEX COVER, MAX CUT
  - $\text{TSP}_{(D)}$, 3COL
  - SET COVER, PARTITION, KNAPSACK, BIN PACKING
  - INTEGER PROGRAMMING (IP): Given m inequalities oven n variables $u_i \in \{0, 1\}$, is there an assignment satisfying all the inequalities?
- Always remember that these are **decision versions** of the corresponding **optimization problems**.

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- Always remember that these are **decision versions** of the corresponding **optimization problems**.
- But 2SAT, 2COL $\in$ **P**.

NP-completeness: Web of Reductions

Example

SAT $\leq_m^\ell$ IP:

- Every clause can be expressed as an inequality, eg:

$$(x_1 \vee \bar{x}_2 \vee \bar{x}_3) \longrightarrow x_1 + (1 - x_2) + (1 - x_3) \geq 1$$

NP-completeness: Web of Reductions

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- This method is generalized by the notion of *Constraint Satisfaction Problems*.
- A **Constraint Satisfaction Problem** (CSP) generalizes SAT by allowing clauses of arbitrary form (instead of ORs of literals).

3SAT is the subcase of *qCSP*, where arity $q = 3$ and the constraints are ORs of the involved literals.

Quantified Boolean Formulas

Definition (Quantified Boolean Formula)

A **Quantified Boolean Formula** F is a formula of the form:

$$F = \exists x_1 \forall x_2 \exists x_3 \cdots Q_n x_n \ \phi(x_1, \ldots, x_n)$$

where ϕ is *plain* (quantifier-free) boolean formula.

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Example

$$F = \exists x_1 \forall x_2 \exists x_3 \left[(x_1 \vee \neg x_2) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \right]$$

The above is a True QBF ($(1, 0, 0)$ and $(1, 1, 1)$ satisfy it).

Quantified Boolean Formulas

Theorem

TQBF *is* **PSPACE**-*complete.*

Quantified Boolean Formulas

Theorem

TQBF *is* **PSPACE**-*complete.*

Proof: See Th. 19.1 (p.456) in [1] – Th.4.13 (p.84) in [2]

- TQBF $\in$ **PSPACE**:
  - Let ϕ be a QBF, with n variables and length m.
  - Recursive algorithm $A(\phi)$:
  - If $n = 0$, then there are only constants, hence $\mathcal{O}(m)$ time/space.
  - If $n > 0$:
    $A(\phi) = A(\phi|_{x_1=0}) \vee A(\phi|_{x_1=1})$, if $Q_1 = \exists$, and
    $A(\phi) = A(\phi|_{x_1=0}) \wedge A(\phi|_{x_1=1})$, if $Q_1 = \forall$.
  - Both recursive computations can be run on *the same space*.
  - So $space_{n,m} = space_{n-1,m} + \mathcal{O}(m) \Rightarrow space_{n,m} = \mathcal{O}(n \cdot m)$.

Quantified Boolean Formulas

Proof (*cont'd*):

- Now, let M a TM with space bound $p(n)$.
- We can create the configuration graph of $M(x)$, having size $2^{\mathcal{O}(p(n))}$.
- M accepts x iff there is a path of length at most $2^{\mathcal{O}(p(n))}$ from the initial to the accepting configuration.
- Using Savitch's Theorem idea, for two configurations C and C' we have:
$REACH(C, C', 2^i) \Leftrightarrow$
$\Leftrightarrow \exists C'' \left[REACH(C, C'', 2^{i-1}) \wedge REACH(C'', C', 2^{i-1}) \right]$

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- But, this is a bad idea: Doubles the size each time.
- Instead, we use additional variables:
  $\exists C'' \forall D_1 \forall D_2 \left[(D_1 = C \wedge D_2 = C'') \vee (D_1 = C'' \wedge D_2 = C')\right] \Rightarrow REACH(D_1, D_2, 2^{i-1})$

Quantified Boolean Formulas

Proof (*cont'd*):

- The base case of the recursion is $C_1 \to C_2$, and can be encoded as a quantifier-free formula.
- The size of the formula in the i^{th} step is $space_i \leq space_{i-1} + \mathcal{O}(p(n)) \Rightarrow \mathcal{O}(p^2(n))$. □

*Logical Characterizations

- **Descriptive complexity** is a branch of computational complexity theory and of finite model theory that characterizes complexity classes by the *type of logic* needed to express the languages in them.

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Theorem (Fagin's Theorem)

The set of all properties expressible in Existential Second-Order Logic is precisely **NP**.

Theorem

The class of all properties expressible in Horn Existential Second-Order Logic with Successor is precisely **P**.

- HORNSAT is **P**-complete.

Summary 1/2

- We define complexity classes using a computation model/mode and complexity measures.
- Time/Space constructible functions are used as complexity measures.
- Classes of the same kind form *proper hierarchies*.
- **NP** is the class of *easily verifiable* problems: given a *certificate*, one can efficiently verify that it is correct.
- Savitch's Theorem implies that **PSPACE** = **NPSPACE**.

Summary 2/2

- Reductions relate problems with respect to hardness.
- Complete problems reflect the difficulty of the class.
- REACHABILITY is **NL**-complete.
- Immerman-Szelepscényi's Theorem implies that $\mathbf{NL} = co\mathbf{NL}$.
- Circuit Value Problem (CVP) is **P**-complete under logspace reductions.
- CIRCUIT SAT and SAT are **NP**-complete.
- True Quantified Boolean Formula (TQBF) is **PSPACE**-complete.

Contents

Oracle TMs and Oracle Classes

Definition

A Turing Machine $M^?$ with *oracle* is a multi-string deterministic TM that has a special string, called **query string**, and three special states: $q_?$ (**query state**), and q_{YES}, q_{NO} (*answer states*). Let $A \subseteq \Sigma^*$ be an arbitrary language. The computation of oracle machine M^A proceeds like an ordinary TM except for transitions from the query state: *From the $q_?$ moves to either q_{YES}, q_{NO}, depending on whether the current query string is in A or not.*

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A Turing Machine $M^?$ with *oracle* is a multi-string deterministic TM that has a special string, called **query string**, and three special states: $q_?$ (**query state**), and q_{YES}, q_{NO} (*answer states*). Let $A \subseteq \Sigma^*$ be an arbitrary language. The computation of oracle machine M^A proceeds like an ordinary TM except for transitions from the query state: *From the $q_?$ moves to either q_{YES}, q_{NO}, depending on whether the current query string is in A or not.*

- The answer states allow the machine to use this answer to its further computation.
- The computation of $M^?$ with oracle A on iput x is denoted as $M^A(x)$.

Oracle TMs and Oracle Classes

Definition

Let $\mathcal{C}$ be a time complexity class (deterministic or nondeterministic). Define $\mathcal{C}^A$ to be the *class* of all languages decided by machines of the same sort and time bound as in $\mathcal{C}$, only that the machines have now oracle access to A. Also, we define: $\mathcal{C}_1^{\mathcal{C}_2} = \bigcup_{L \in \mathcal{C}_2} \mathcal{C}_1^L$.

For example, $\mathbf{P}^{\mathbf{NP}} = \bigcup_{L \in \mathbf{NP}} \mathbf{P}^L$. Note that $\mathbf{P}^{\mathrm{SAT}} = \mathbf{P}^{\mathbf{NP}}$.

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Theorem

There exists an oracle A for which $\mathbf{P}^A = \mathbf{NP}^A$.

Proof: Th.14.4 (p.340) in [1]

Take A to be a **PSPACE**-complete language.Then:

$$\mathbf{PSPACE} \subseteq \mathbf{P}^A \subseteq \mathbf{NP}^A \subseteq \mathbf{PSPACE}^A = \mathbf{PSPACE}^{\mathbf{PSPACE}} \subseteq \mathbf{PSPACE}. \quad \square$$

Oracle TMs and Oracle Classes

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There exists an oracle B for which $\mathbf{P}^B \neq \mathbf{NP}^B$.

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Proof:

Th.14.5 (p.340-342) in [1]

- We will find a language $L \in \mathbf{NP}^B \setminus \mathbf{P}^B$.
- Let $L = \{1^n \mid \exists x \in B \text{ with } |x| = n\}$.
- $L \in \mathbf{NP}^B$ *(why?)*
- We will define the oracle $B \subseteq \{0,1\}^*$ such that $L \notin \mathbf{P}^B$:

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- $L \in \mathbf{NP}^B$ (*why?*)
- We will define the oracle $B \subseteq \{0,1\}^*$ such that $L \notin \mathbf{P}^B$:
- Let $M_1^?, M_2^?, \ldots$ an enumeration of all PDTMs with oracle, such that every machine appears *infinitely many* times in the enumeration.
- We will define B iteratively: $B_0 = \emptyset$, and $B = \bigcup_{i \geq 0} B_i$.
- In i^{th} stage, we have defined B_{i-1}, the set of all strings in B with length $< i$.
- Let also X the set of **exceptions**.

Proof (*cont'd*):

- We simulate $M_i^B(1^i)$ for $i^{\log i}$ steps.
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- **If** $|x| < i$, we look for x in B_{i-1}.
- $\rightarrow$ **If** $x \in B_{i-1}$, M_i^B goes to q_{YES}
  $\rightarrow$ **Else** M_i^B goes to q_{NO}
- **If** $|x| \geq i$, M_i^B goes to q_{NO} ,and $x \rightarrow X$.

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- **If** $|x| \geq i$, M_i^B goes to q_{NO} ,and $x \rightarrow X$.
- Suppose that after at most $i^{\log i}$ steps the machine *rejects*.
  - Then we define $B_i = B_{i-1} \cup \{x \in \{0,1\}^* : |x| = i, x \notin X\}$ so $1^i \in L$, and $L(M_i^B) \neq L$.
  - Why $\{x \in \{0,1\}^* : |x| = i, x \notin X\} \neq \emptyset$? ?
- If the machine *accepts*, we define $B_i = B_{i-1}$, so that $1^i \notin L$.
- If the machine fails to halt in the allotted time, we set $B_i = B_{i-1}$, but we know that the same machine will appear in the enumeration with an index sufficiently large. $\square$

A First Barrier: The Limits of Diagonalization

- As we saw, an oracle can transfer us to an alternative computational "*universe*".
  (We saw a universe where $\mathbf{P} = \mathbf{NP}$, and another where $\mathbf{P} \neq \mathbf{NP}$)
- Diagonalization is a technique that relies in the facts that:
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- Diagonalization is a technique that relies in the facts that:
  - **TMs are (effectively) represented by strings.**
  - **A TM can simulate another without much overhead in time/space.**
- So, diagonalization or any other proof technique relies only on these two facts, holds also for *every* oracle.
- Such results are called **relativizing results**.
  E.g., $\mathbf{P}^A \subseteq \mathbf{NP}^A$, for every $A \in \{0, 1\}^*$.
- The above two theorems indicate that **P** vs. **NP** is a **nonrelativizing** result, so diagonalization and any other relativizing method doesn't suffice to prove it.

Cook Reductions

- A problem A is **Cook-Reducible** to a problem B, denoted by $A \leq_T^p B$, if there is an oracle DTM M^B which in polynomial time decides A (*making at most polynomial many queries to* B).
- That is: $A \in \mathbf{P}^B$.

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Theorem

$\mathbf{P}, \mathbf{PSPACE}$ *are closed under* $\leq_T^p$.

- Is **NP** closed under $\leq_T^p$? (*cf. Problem Sets!*)

*Random Oracles

- We proved that:
  - $\exists A \subseteq \Sigma^* : \mathbf{P}^A = \mathbf{NP}^A$
  - $\exists B \subseteq \Sigma^* : \mathbf{P}^B \neq \mathbf{NP}^B$
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- Now, consider the set $\mathcal{U} = Pow(\Sigma^*)$, and the sets:

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Theorem (Bennet, Gill)

$$\mathbf{Pr}_{B \subseteq \Sigma^*}\left[\mathbf{P}^B \neq \mathbf{NP}^B\right] = 1$$

See H. Vollmer & K.W. Wagner, "Measure one Results in Computational Complexity Theory"

The Polynomial Hierarchy

Polynomial Hierarchy Definition

- $\Delta_0^p = \Sigma_0^p = \Pi_0^p = \mathbf{P}$
- $\Delta_{i+1}^p = \mathbf{P}^{\Sigma_i^p}$
- $\Sigma_{i+1}^p = \mathbf{NP}^{\Sigma_i^p}$
- $\Pi_{i+1}^p = co\mathbf{NP}^{\Sigma_i^p}$
- $$\mathbf{PH} \equiv \bigcup_{i \geqslant 0} \Sigma_i^p$$

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- $\Sigma_0^p = \mathbf{P}$
- $\Delta_1^p = \mathbf{P}$, $\Sigma_1^p = \mathbf{NP}$, $\Pi_1^p = co\mathbf{NP}$
- $\Delta_2^p = \mathbf{P}^{\mathbf{NP}}$, $\Sigma_2^p = \mathbf{NP}^{\mathbf{NP}}$, $\Pi_2^p = co\mathbf{NP}^{\mathbf{NP}}$

$$
\begin{tikzcd}
\vdots & & \vdots \\
& \Delta_3^p = \mathbf{P}^{\mathbf{NP}^{\mathbf{NP}}} \arrow[ul] \arrow[ur] & \\
\Pi_2^p = co\mathbf{NP}^{\mathbf{NP}} \arrow[ur] & & \Sigma_2^p = \mathbf{NP}^{\mathbf{NP}} \arrow[ul] \\
& \Delta_2^p = \mathbf{P}^{\mathbf{NP}} \arrow[ul] \arrow[ur] & \\
\Pi_1^p = co\mathbf{NP} \arrow[ur] & & \Sigma_1^p = \mathbf{NP} \arrow[ul] \\
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\end{tikzcd}
$$

- $\Sigma_i^p, \Pi_i^p \subseteq \Sigma_{i+1}^p$
- $A, B \in \Sigma_i^p \Rightarrow$ $A \cup B \in \Sigma_i^p$, $A \cap B \in \Sigma_i^p$
- $A \in \Pi_i^p \Rightarrow$ $\overline{A} \in \Sigma_i^p$
- $A, B \in \Delta_i^p \Rightarrow$ $A \cup B$, $A \cap B$ and $\overline{A} \in \Delta_i^p$

Theorem

Let L be a language , and $i \geq 1$. $L \in \Sigma_i^p$ iff there is a polynomially balanced relation R such that the language $\{x; y : (x, y) \in R\}$ is in Π_{i-1}^p and

$$L = \{x : \exists y, s.t. : (x, y) \in R\}$$

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Proof (by Induction):

Th.17.8 (p.425) in [1]

- $\boxed{\text{For } i = 1\text{:}}$
  $\{x; y : (x, y) \in R\} \in \mathbf{P}$,so $L = \{x | \exists y : (x, y) \in R\} \in \mathbf{NP}$ ✓

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- $\boxed{\text{For } i > 1:}$
  If $\exists R \in \Pi_{i-1}^p$, we must show that $L \in \Sigma_i^p \Rightarrow$
  $\exists$ NTM with Σ_{i-1}^p oracle: NTM(x) guesses a y and asks Σ_{i-1}^p oracle whether $(x, y) \notin R$.

Proof (*cont'd*):

If $L \in \Sigma_i^p$, we must show the existence of R:

- $L \in \Sigma_i^p \Rightarrow \exists$ NTM M^K, $K \in \Sigma_{i-1}^p$, which decides L.
- $K \in \Sigma_{i-1}^p \Rightarrow \exists S \in \Pi_{i-2}^p : (z \in K \Leftrightarrow \exists w : (z, w) \in S)$.
- We must describe a relation R (we know: $x \in L \Leftrightarrow$ accepting computation of $M^K(x)$)
- Query Steps: "yes"$\rightarrow z_i$ has a certificate w_i st $(z_i, w_i) \in S$.
- So, $R(x, y) =$"$(x, y) \in R$ *iff y records an accepting computation of* $M^?$ *on x , together with a certificate* w_i *for each* **yes** *query* z_i *in the computation.*"
- We must show $\{x; y : (x, y) \in R\} \in \Pi_{i-1}^p$:
  - Check that all steps of $M^?$ are legal (*poly time*).
  - Check that $(z_i, w_i) \in S$ (*in* Π_{i-2}^p, *and thus in* Π_{i-1}^p).
  - For all "no" queries z_i', check $z_i' \notin K$ (*another* Π_{i-1}^p). $\square$

Corollary

Let L be a language , and $i \geq 1$. $L \in \Pi_i^p$ iff there is a polynomially balanced relation R such that the language $\{x; y : (x, y) \in R\}$ is in Σ_{i-1}^p and

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Let L be a language , and $i \geq 1$. $L \in \Sigma_i^p$ iff there is a polynomially balanced, polynomially-time decicable $(i + 1)$-ary relation R such that:

$$L = \{x : \exists y_1 \forall y_2 \exists y_3 ... Q y_i, s.t. : (x, y_1, ..., y_i) \in R\}$$

where the i^{th} quantifier Q is $\forall$, if i is even, and $\exists$, if i is odd.

Remark

$$\Sigma_i^p = (\underbrace{\exists\forall\exists\cdots Q_i}_{i \text{ quantifiers}} / \underbrace{\forall\exists\forall\cdots Q_i'}_{i \text{ quantifiers}}) \qquad \Pi_i^p = (\underbrace{\forall\exists\forall\cdots Q_i}_{i \text{ quantifiers}} / \underbrace{\exists\forall\exists\cdots Q_i'}_{i \text{ quantifiers}})$$

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Theorem

If for some $i \geq 1$, $\Sigma_i^p = \Pi_i^p$, then for all $j > i$:

$$\Sigma_j^p = \Pi_j^p = \Delta_j^p = \Sigma_i^p$$

Or, the polynomial hierarchy *collapses* to the i^{th} level.

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Proof: Th.17.9 (p.427) in [1]

- It suffices to show that: $\Sigma_i^p = \Pi_i^p \Rightarrow \Sigma_{i+1}^p = \Sigma_i^p$.
- Let $L \in \Sigma_{i+1}^p \Rightarrow \exists R \in \Pi_i^p$: $L = \{x | \exists y : (x, y) \in R\}$
- $\Pi_i^p = \Sigma_i^p \Rightarrow R \in \Sigma_i^p$
- $(x, y) \in R \Leftrightarrow \exists z : (x, y, z) \in S$, $S \in \Pi_{i-1}^p$.
- So, $x \in L \Leftrightarrow \exists y; z : (x, y, z) \in S$, $S \in \Pi_{i-1}^p$, hence $L \in \Sigma_i^p$. $\square$

Corollary

If **P**=**NP**, or even **NP**=co**NP**, the Polynomial Hierarchy collapses to the first level.

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QSAT_i Definition

Given expression ϕ, with Boolean variables partitioned into i sets X_i,is ϕ satisfied by the overall truth assignment of the expression:

$$\exists X_1 \forall X_2 \exists X_3 \ldots . QX_i \phi$$

where Q is $\exists$ if i is *odd*, and $\forall$ if i is even.

Theorem

For all $i \geq 1$ QSAT_i is Σ_i^p-complete.

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Proof: Th.17.11 (p.429) in [1]

- Let L is **PH**-complete.
- Since $L \in \mathbf{PH}$, $\exists i \geq 0 : L \in \Sigma_i^p$.
- But any $L' \in \Sigma_{i+1}^p$ reduces to L.
- Since PH is closed under reductions, we imply that $L' \in \Sigma_i^p$, so $\Sigma_i^p = \Sigma_{i+1}^p$. $\square$

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Theorem

$\mathbf{PH} \subseteq \mathbf{PSPACE}$

- $\mathbf{PH} \stackrel{?}{=} \mathbf{PSPACE}$ (**Open**). If it was, then **PH** has complete problems, so it collapses to some finite level.

Relativized Results

Let's see how the inclusion of the Polynomial Hierarchy to Polynomial Space, and the inclusions of each level of **PH** to the next relativizes:

- $\mathbf{PH}^A \neq \mathbf{PSPACE}^A$ relative to *some* oracle $A \subseteq \Sigma^*$.
  (Yao 1985, Håstad 1986)
- $\mathbf{Pr}_A[\mathbf{PH}^A \neq \mathbf{PSPACE}^A] = 1$
  (Cai 1986, Babai 1987)
- $(\forall i \in \mathbb{N})\ \Sigma_i^{p,A} \subsetneq \Sigma_{i+1}^{p,A}$ relative to *some* oracle $A \subseteq \Sigma^*$.
  (Yao 1985, Håstad 1986)
- $\mathbf{Pr}_A[(\forall i \in \mathbb{N})\ \Sigma_i^{p,A} \subsetneq \Sigma_{i+1}^{p,A}] = 1$
  (Rossman-Servedio-Tan, 2015)

Self-Reducibility of SAT

- For a Boolean formula ϕ with n variables and m clauses.
- It is easy to see that:

$$\boxed{\phi \in \mathrm{SAT} \Leftrightarrow (\phi|_{x_1=0} \in \mathrm{SAT}) \vee (\phi|_{x_1=1} \in \mathrm{SAT})}$$

- Thus, we can **self-reduce** SAT to instances of smaller size.
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Example

$$\phi(x_1, x_2)$$

$$\phi|_{x_1=0} \qquad \phi|_{x_1=1}$$

$$\phi|_{x_1=0,x_2=0} \quad \phi|_{x_1=0,x_2=1} \quad \phi|_{x_1=1,x_2=0} \quad \phi|_{x_1=1,x_2=1}$$

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FSAT: Given a Boolean expression ϕ, if ϕ is satisfiable then return a satisfying truth assignment for ϕ. Otherwise return "no".

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- What about the opposite?
- If SAT $\in$ **P**, we can use the self-reducibility property to fix variables one-by-one, and retrieve a solution.
- We only need $2n$ calls to the *alleged* poly-time algorithm for SAT.

What about TSP?

- We can solve TSP using a hypothetical algorithm for the **NP**-complete decision version of TSP:

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- We can solve TSP using a hypothetical algorithm for the **NP**-complete decision version of TSP:
- We can find the cost of the optimum tour by **binary search** (in the interval $[0, 2^n]$).
- When we find the optimum cost C, we fix it, and start changing intercity distances one-by one, by setting each distance to $C + 1$.
- We then ask the **NP**-oracle if there still is a tour of optimum cost at most C:
  - If there is, then this edge is not in the optimum tour.
  - If there is not, we know that this edge is in the optimum tour.
- After at most n^2 (polynomial) oracle queries, we can extract the optimum tour, and thus have the solution to TSP.

The Classes $\mathbf{P}^{\mathbf{NP}}$ and $\mathbf{FP}^{\mathbf{NP}}$

- $\mathbf{P}^{\mathrm{SAT}}$ is the class of languages decided in pol time with a SAT oracle (*Polynomial number of adaptive queries*).
- SAT is **NP**-complete $\Rightarrow \mathbf{P}^{\mathrm{SAT}}=\mathbf{P}^{\mathbf{NP}}$.
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Definition (Reductions for Function Problems)

A function problem A reduces to B if there exists $R, S \in \mathbf{FL}$ such that:

- $x \in A \Rightarrow R(x) \in B$.
- If z is a correct output of $R(x)$, then $S(z)$ is a correct output of x.

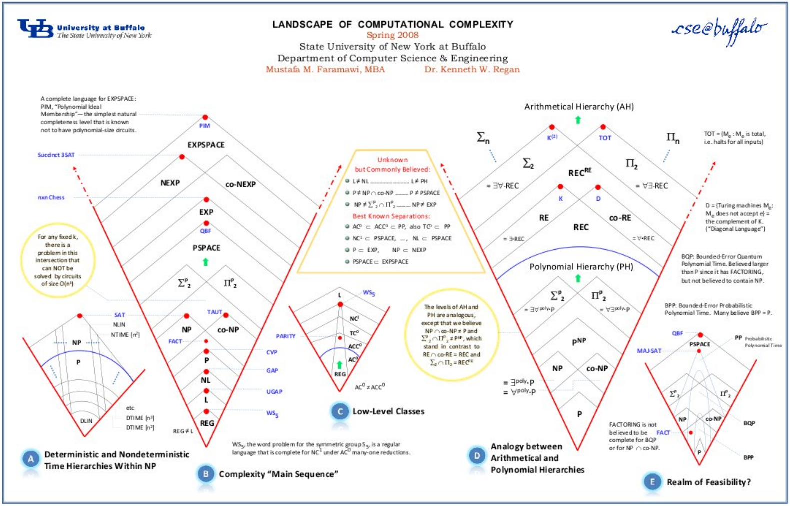
Theorem

TSP *is* $\mathbf{FP}^{\mathbf{NP}}$*-complete.*

Summary

- Oracle TMs have one-step oracle access to some language.
- There exist oracles $A, B \subseteq \Sigma^*$ such that $\mathbf{P}^A = \mathbf{NP}^A$ and $\mathbf{P}^B \neq \mathbf{NP}^B$.
- Relativizing results hold for *every* oracle.
- A Cook reduction $A \leq_T^p B$ is a poly-time TM deciding A, by using B as an oracle.
- The Polynomial Hierarchy can be viewed as:
  - Oracle hierarchy of consecutive **NP** oracles.
  - Quantifier hierarchy of alternating quantifiers.
- If for some $i \geq 1$ $\Sigma_i^p = \Pi_i^p$, or there is a **PH**-complete problem, then PH collapses to some finite level.
- Optimization problems with decision version in **NP** (such as TSP) are in $\mathbf{FP}^{\mathbf{NP}}$.

The Complexity Universe



Contents

Problems...

- After years of efforts, there are problems in **NP** without a polynomial-time algorithm or a completeness proof.
- Famous examples: FACTORING_D, GI (Graph Isomorphism). (where FACTORING_D is the problem of *deciding* if a given number has a factor $\leq k$).
- So, are there **NP** problems that are neither in **P** nor **NP**-complete?

Degrees

- The **$\leq_T^p$-degree** of a language A consists of all languages L such that $L \equiv_T^p A$ (that is, $L \leq_T^p A \ \wedge \ A \leq_T^p L$).

Degrees

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- There are three possibilities:
  - $\mathbf{P} = \mathbf{NP}$, thus all languages in **NP** are $\leq^p_T$-complete for **NP**, so **NP** contains *exactly one* $\leq^p_T$-degree.
  - $\mathbf{P} \neq \mathbf{NP}$, and **NP** contains *two different* degrees: **P** and **NP**-complete languages.
  - $\mathbf{P} \neq \mathbf{NP}$, and **NP** contains more degrees, so there exists a language in $\mathbf{NP} \setminus \mathbf{P}$ that is not **NP**-complete.

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  - $\mathbf{P} \neq \mathbf{NP}$, and **NP** contains more degrees, so there exists a language in $\mathbf{NP} \setminus \mathbf{P}$ that is not **NP**-complete.
- We will show that the second case cannot happen.

Enumerations

- Recall that any string can potentially encode a TM.
(*We map all the invalid encodings to the “empty” TM M_0, which reject all strings.*)
- A TM M is encoded by infinitely many strings.
- So, there exists a function $e(x)$ such that:
  1. For every $x \in \Sigma^*$, $e(x)$ represents a TM.
  2. Every TM is represented by at least one $e(x)$.
  3. The code of the TM $e(x)$ can be easily decoded.
- Such a function is called an **enumeration** of TMs (Deterministic or Nondeterministic).

Enumerations

- When we consider classes like **P** or **NP**, we can easily enumerate only these machines, a *subclass* of all DTMs (NTMs):

Enumerations

- When we consider classes like **P** or **NP**, we can easily enumerate only these machines, a *subclass* of all DTMs (NTMs):
- Recall that if a function is **time-constructible**, there exists a DTM halting after exactly $t(n)$ moves. Such a machine is called a $t(n)$**-clock machine**.
- For any DTM M_1, we can attach a $t(n)$-clock machine M_2 and obtain a "product" machine $M_3 = \langle M_1, M_2 \rangle$, which halts if either M_1 or M_2 halts, and accepts only if M_1 accepts.

Enumerations

- Consider the functions $p_i(n) = n^i, i \geq 1$.
- If $\{M_x\}$ is an enumeration of DTMs, let $M_{\langle x,i \rangle}$ be the machine M_x attached with a $p_i(n)$-clock machine.
- Then, $\{M_{\langle x,i \rangle}\}$ is an **enumeration of all polynomial-time clocked machines**, and it is an enumeration of languages in **P**, such that:
  - Every machine $M_{\langle x,i \rangle}$ accepts a language in **P**.
  - Every language in **P** is accepted by at least a machine in the enumeration (in fact, by infinite number of machines).

Enumerations

- The same holds for **NP**.
  (*enumerate all poly-time alarm clocked NTMs*)
- We can do the same trick with **space**, using a **yardstick**, a DTM that halts after visiting *exactly* $s(n)$ memory cells.
- We can also enumerate all the functions in **FP**, and all polynomial-time *oracle* DTMs or NTMs.

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- We can also enumerate all the functions in **FP**, and all polynomial-time *oracle* DTMs or NTMs.
- This list will **not** contain *all* the poly-time bounded machines!
  (Reminder: It is undecidable to determine whether a given TM halts in polynomial time for all inputs.)

Ladner’s Theorem

Theorem (Ladner)

If **P** $\neq$ **NP**, *there exists a language in* **NP**, *which is neither in* **P** *nor* **NP**-*complete.*

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Proof (*Blowing holes in* SAT): Th. 14.1 (p.330) in [1]

- Idea: We will construct a language A by taking an **NP**-complete language, and "blow holes" to it, so that it is no longer **NP**-complete, neither in **P**.
- Let $\{M_i\}$ an enumeration of all polynomial-time *clocked* TMs.
- Let $\{F_i\}$ an enumeration of all polynomial-time *clocked* functions.

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- Let $\{M_i\}$ an enumeration of all polynomial-time *clocked* TMs.
- Let $\{F_i\}$ an enumeration of all polynomial-time *clocked* functions.
- Define A as follows:

$$A = \{x \mid x \in \text{SAT} \ \wedge \ f(|x|) \text{ is even}\}$$

Ladner's Theorem

Proof (*cont'd*):

- If $f \in \mathbf{FP}$, then $A \in \mathbf{NP}$: Guess a truth assignment, compute $f(|x|)$ and verify.
- We define f by a polynomial-time TM M_f computing it.
- Let also M_{SAT} be the machine that decides SAT, and $f(0) = f(1) = 2$.

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- We define f by a polynomial-time TM M_f computing it.
- Let also M_{SAT} be the machine that decides SAT, and $f(0) = f(1) = 2$.
- On input 1^n, M_f operates in two stages, each lasting for exactly n steps:
- $\boxed{\text{First Stage}}$

  M_f computes $f(0), f(1), \ldots$ until it runs out of time.
  - Let $f(x) = k$ the last value of f it was able to compute.
  - Then M_f outputs either k or $k+1$, to be determined in the next stage:

Ladner's Theorem

Proof (*cont'd*):

- $\boxed{\text{Second Stage}}$

  If $k = 2i$:
  - M_f tries to find a $z \in \{0,1\}^*$ such that $M_i(z)$ outputs the *wrong* answer to "$z \in A$" question ($M_i(z) \neq A(z)$):
    - Simulate $M_i(z), M_{\mathrm{SAT}}(z), f(|z|)$ for all z in lexicographic order.
    - If such a string is found in the allotted time, output $k+1$, else output k.

  If $k = 2i - 1$:
  - M_f tries to find a string z such that $F_i(z)$ is an *incorrect* Karp reduction from SAT to A ($M_{\mathrm{SAT}}(z) \neq A(F_i(z))$):
    - Simulate $F_i(z), M_{\mathrm{SAT}}(z), M_{\mathrm{SAT}}(F_i(z)), f(|F_i(z)|)$ for all z in lexicographic order.
    - If such a string is found in the allotted time, output $k+1$, else output k.
- M_f runs in polynomial time.
- $f(n+1) \geq f(n)$.

Ladner’s Theorem

Proof (*cont’d*):

- **We claim that $A \notin$ P:**

Ladner's Theorem

Proof (*cont'd*):

- **We claim that $A \notin \mathbf{P}$:**
- Suppose that $A \in \mathbf{P}$. Then, there is an i s.t. $L(M_i) = A$.
- Then, the second stage of M_f with $k = 2i$ will never find a z satisfying the desired property.
- $f(n) = 2i$ for all $n \geq n_0$, for some n_0.
- So, $f(n)$ is even for all but finitely many n.
- A coincides with SAT on all but finitely many input sizes.
- Then SAT $\in \mathbf{P}$, contradiction!

Ladner’s Theorem

Proof (*cont’d*):

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Ladner's Theorem

Proof (*cont'd*):

- **We claim that A is not NP-complete:**
- Suppose that A is **NP**-complete, then there is a reduction F_i from SAT to A.
- Then, the second stage of M_f with $k = 2i - 1$ will never find a z satisfying the desired property.
- So, $f(n)$ is odd on all but finitely many input sizes.
- Then A is a finite language, hence in **P**, contradiction! $\square$

- Using the same technique, we can prove an analog of *Post's problem* in Recursion Theory:

Theorem

If $\mathbf{P} \neq \mathbf{NP}$, *there exist* $A, B \in \mathbf{NP}$ *such that* $A \not\leq_T^p B$ *and* $B \not\leq_T^p A$.

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- Ladner’s Theorem (*generalized by Schöning*) implies also that:

Corollary

If $\mathbf{P} \neq \mathbf{NP}$*, then for every language* $B \in \mathbf{NP} \setminus \mathbf{P}$*, there exists a set* $A \in \mathbf{NP} \setminus \mathbf{P}$ *such that* $A \leq_T^p B$ *and* $B \not\leq_T^p A$*.*

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So, if $\mathbf{P} \neq \mathbf{NP}$, then $\mathbf{NP}$ contains *infinitely many* distinct $\leq_T^p$-degrees.

Polynomial-Time Isomorphism

- All **NP**-complete problems are related through reductions.
- Many reductions can be converted to stronger relations:

Definition

Two languages $A, B \subseteq \Sigma^*$ are *polynomial-time isomorphic* if there exists a function $h : \Sigma^* \to \Sigma^*$ such that:

1. h is a bijection.
2. For all $x \in \Sigma^*$: $x \in A \Leftrightarrow h(x) \in B$.
3. Both h and h^{-1} are polynomial-time computable.

Functions h and h^{-1} are then called *polynomial-time isomorphisms*.

- Which reductions are polynomial-time isomorphisms?

Padding Functions

Definition

Let $L \subseteq \Sigma^*$ be a language. We say that function $pad : \Sigma^* \times \Sigma^* \to \Sigma^*$ is a *padding function* for L if it has the following properties:

1. It is computable in logarithmic space.
2. Forall $x, y \in \Sigma^*$, $pad(x, y) \in L \Leftrightarrow x \in L$.
3. Forall $x, y \in \Sigma^*$, $|pad(x, y)| > |x| + |y|$
4. There is a logarithmic-space algorithm, which, given $pad(x, y)$ recovers y.

- Such languages are called *paddable*.
- Function *pad* is essentially a length-increasing reduction from L to itself that "encodes" another string y into the instance of L.

Padding Functions Examples

Example (SAT)

Let x an instance with n variables and m clauses. Let $y \in \Sigma^*$:
$pad(x, y)$ is an instance of SAT containing all clauses of x, plus $m + |y|$ more clauses, and $|y| + 1$ more variables.

- The first m clauses are copies of x_{n+1} clause.
- The last $m + i^{th}$ $(i = 1, \cdots, |y|)$ are either $\neg x_{n+i+1}$ (if $y(i) = 0$) or x_{n+i+1} (if $y(i) = 1$).

Is that a padding function?

1. It is log-space computable.
2. It doesn't affect x's satisfiability.
3. It is length increasing.
4. Given $pad(x, y)$ we can find where the "added" part begins.

Padding Functions

- We would like to have this kind of implication: $(A \leq_m^p B) \wedge (B \leq_m^p A) \stackrel{?}{\Rightarrow} (A \text{ isomorphic to } B)$.
- But, unfortunately, this is **not** sufficient.
- We finally want to have a polynomial-time version of Schröder-Bernstein Theorem:

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- We finally want to have a polynomial-time version of Schröder-Bernstein Theorem:

Theorem (Schröder-Bernstein)

If there exists a 1-1 mapping from a set A to a set B, and a 1-1 mapping from B to A, then there is a bijection between A and B.

- To achieve this analogy, we need to "enhance" our reductions with the previous features (1-1, length increasing, and polynomial time computable and invertible).

Padding Functions

- We can use padding function to transform regular reductions to "desired" ones:

Theorem

Let R be a reduction from A to B, and pad a padding function for B. Then, the function mapping $x \in \Sigma^*$ to $pad(R(x), x)$ is a length-increasing 1-1 reduction. Furthermore, there exists R^{-1}, computable in logarithmic space, which given $pad(R(x), x)$ recovers x.

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Theorem (Polynomial-time version of Schröder-Bernstein Theorem)

Let A and B be paddable languages. If $A \leq_m^p B$ and $B \leq_m^p A$, then A and B are polynomial-time isomorphic.

Padding Functions

Corollary

*The following **NP**-complete languages are pol. isomorphic:*
SAT, VERTEX COVER, HAMILTON PATH, CLIQUE, MAX CUT, TRIPARTITE MATCHING, KNAPSACK

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Definition (Berman-Hartmanis Conjecture)

All **NP**-complete languages are polynomial-time isomorphic to each other!

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Definition (Berman-Hartmanis Conjecture)

All **NP**-complete languages are polynomial-time isomorphic to each other!

- Berman-Hartmanis Conjecture $\Rightarrow \mathbf{P} \neq \mathbf{NP}$ (*why?*)

Translation Results

Theorem

If $\mathbf{NEXP} \neq \mathbf{EXP}$, *then* $\mathbf{P} \neq \mathbf{NP}$.

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If **NEXP** $\neq$ **EXP**, *then* **P** $\neq$ **NP**.

Proof:

- We will prove that if **P** $=$ **NP**, then **NEXP** $=$ **EXP**.
- Let $L \in \mathbf{NTIME}[2^{n^c}]$ and M a TM deciding it. We define:

$$L_p = \{x\$^{2^{|x|^c}} \mid x \in L\}$$

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- L_p is in **NP**: Simulate $M(x)$ for $2^{|x|^c}$ steps and output the answer. The running time of this machine is polynomial in its input size.
- By our assumption, $L_p \in \mathbf{P}$.
- We can use the machine in **P** to decide L in **EXP**: on input x, pad it using $2^{|x|^c}$ \$'s, and use the machine in **P** to decide L_p.
- The running time is $2^{|x|^c}$, so $L \in \mathbf{EXP}$.

$\square$

Separation Results

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- Let $L \in \mathbf{DTIME}[2^{n^2}]$.
- We define:

$$L_p = \{x\$^\ell \mid x \in L \wedge |x\$^\ell| = |x|^2\}$$

- $L_p \in \mathbf{DTIME}[2^n]$
- From our assumption: $L_p \in \mathbf{PSPACE} \Rightarrow L_p \in \mathbf{DSPACE}[n^k]$, for some $k \in \mathbb{N}$.

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Proof (*cont'd*):

- We can convert this n^k-space-bounded machine to another, deciding L:
- Given x, add $\ell = |x|^2 - |x|$ \$'s, and simulate the n^k-space-bounded machine on the padded input.
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- We used $|x|^{2k}$ space, so $L \in \mathbf{PSPACE} \Rightarrow \mathbf{DTIME}[2^{n^2}] \subseteq \mathbf{PSPACE}$.
- But, $\mathbf{E} \subsetneq \mathbf{DTIME}[2^{n^2}]$, and so $\mathbf{E} \neq \mathbf{PSPACE}$. $\square$

Density of Languages

Definition

Let $L \subseteq \Sigma^*$ be a language. We define as its **density** the following function from $\mathbb{N} \rightarrow \mathbb{N}$:

$$dens_L(n) = |\{x \in L : |x| \leq n\}|$$

- $dens_L(n)$ is the *number of strings* in L of length up to n.

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Theorem

If $A, B \subseteq \Sigma^*$ are polynomial-time isomorphic, then $dens_A$ and $dens_B$ are polynomially related.

Proof:

- All $x \in A$ with $|x| \leq n$ are mapped to $y \in B$ with $|y| \leq p(n)$, where p is the polynomial bound of the isomorphism.
- The mapping is 1-1, so $dens_A(n) \leq dens_B(p(n))$. $\square$

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Definition

A language L is *sparse* if there exists a polynomial q such that for every $n \in \mathbb{N} : dens_L(n) \leq q(n)$.

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- Let $A \subseteq \Sigma^*$ with padding function $p : \Sigma^* \times \Sigma^* \to \Sigma^*$.
- Suppose that A is sparse: $\exists q \, \forall n \in \mathbb{N} : dens_A(n) \leq q(n)$.
- Since $p \in \mathbf{FP}$, $\exists \, r \in poly(n) : |p(x, y)| \leq r(|x| + |y|)$.

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- Fix a $x \in A$, since p is 1-1 :

$$2^n \leq |\{p(x, y) : |y| \leq n\}| \leq dens_A(r(|x| + n)) \leq q(r(|x| + n))$$

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- Thus, $2^n / q(r(|x| + n)) \leq 1$. Contradiction! $\square$

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If the Berman-Hartmanis conjecture is true, then all **NP***-complete and all co***NP***-complete languages are not sparse.*

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- Berman-Hartmanis conjecture is true$\Rightarrow$ every **NP**-complete language A is polynomial-time isomorphic to SAT.
- Let f be this isomorphism, and pad_{SAT} a padding function for SAT.
- Define $p_A(x, y) := f^{-1}(pad_{\text{SAT}}(f(x), y))$

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- Define $p_A(x, y) := f^{-1}(pad_{\text{SAT}}(f(x), y))$
- Then $x \in A \Leftrightarrow f(x) \in \text{SAT} \Leftrightarrow pad_{\text{SAT}}(f(x), y) \in \text{SAT} \Leftrightarrow f^{-1}(pad_{\text{SAT}}(f(x), y)) \in A$.
- pad_{SAT} and f are polynomial time *computable* and *invertible*.

Sparse Languages

Proof (*cont'd*):

- So, p_A is a padding function for A, hence A is paddable.
- By the previous theorem, A is *not sparse*.

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Theorem (Mahaney)

If $\mathbf{P} \neq \mathbf{NP}$, *all* **NP**-*complete languages are not sparse.*

Sparse Languages

Theorem (Mahaney)

For any sparse $S \neq \emptyset$, $\mathrm{SAT} \leq_m^p S$ if and only if $\mathbf{P} = \mathbf{NP}$.

Proof: (Ogihara-Watanabe)

- ($\Leftarrow$) trivial.
- ($\Rightarrow$) Let LSAT the language:

$$LSAT = \{\langle \phi, \sigma \rangle \mid \phi \text{ boolean formula, and } \exists \tau, \tau \preceq \sigma : \ \phi|_\tau = T\}$$

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- Note that $\langle \phi, 1^n \rangle \in$ LSAT $\Leftrightarrow \phi \in$ SAT, so LSAT is **NP**-complete.
- Also, if $\sigma_1 \preceq \sigma_2$ and $\langle \phi, \sigma_1 \rangle \in$ LSAT, then $\langle \phi, \sigma_2 \rangle \in$ LSAT.
- So, LSAT $\leq_m^p S$, and let f be the **reduction**.

Sparse Languages

Proof (*cont'd*):

- Consider the self-reducibility tree of ϕ as a **partial assignments tree**:

$$\phi(x_1, x_2, x_3)$$

$$\phi|_{x_1=0} \qquad \phi|_{x_1=1}$$

$$\phi|_{x_1x_2=00} \qquad \phi|_{x_1x_2=01} \qquad \phi|_{x_1x_2=10} \qquad \phi|_{x_1x_2=11}$$

$$\phi|_{000} \quad \phi|_{001} \quad \phi|_{010} \quad \phi|_{011} \quad \phi|_{100} \quad \phi|_{101} \quad \phi|_{110} \quad \phi|_{111}$$

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- We will use the reduction f as a subroutine to an algorithm for SAT.
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- The algorithm will work on the p.a. tree by pruning some nodes at each level:
  - Start from root.
  - If the next level has $> q(n)$ nodes, **prune** until the nodes will be $\leq q(n)$.
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- At the end, there will be n levels with at most $q(n)$ nodes each, so the tree is polynomial.

Sparse Languages

Proof (*cont'd*):

Pruning Procedure

- Remove Duplicates:
  - If $f(\langle\phi,\sigma_1\rangle) = f(\langle\phi,\sigma_2\rangle)$ and $\sigma_1 \preceq \sigma_2$, then we throw away σ_2.

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- For leftmost nodes removal, if the level contains more than q nodes, there will be at least one σ s.t. $f(\langle\phi, \sigma\rangle) \notin S$ (S has $\leq q(n)$ strings). Then ϕ will *not* have a satisfying t.a. smaller than σ, so all partial t.a.'s to the left of σ can be pruned. $\square$

Summary

- Classes like **NP**, **PSPACE** or **FP** can be *effectively enumerated*.
- If $\mathbf{P} \neq \mathbf{NP}$, there exist problems in **NP** which are not **NP**-complete neither in **P**.
- We can obtain polynomial-time isomorphisms between languages, given they are interreducible and paddable.
- Berman-Hartmanis Conjecture postulates that all NP-complete languages are polynomial-time isomorphic to each other.
- We can use padding to *translate upwards* equalities between complexity classes.
- If $\mathbf{P} \neq \mathbf{NP}$, then a *sparse* set *cannot* be $\leq_m^p$-hard for **NP**.