Algorithmic Game Theory

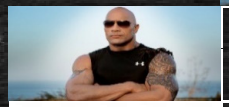
CoReLab (NTUA)

Lecture 2:
Nash Equilibria in Bimatrix Games

2-player 0-sum game: Rock-Paper-Scissors

- 2-players: "row" and "column"
- Payoff of row player = loss of column player
- Nash Equilibrium:

$$\boldsymbol{x} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \boldsymbol{y} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

	0,0	-1,1	1,-1
	1,-1	0,0	-1,1
	-1,1	1,-1	0,0

Nash Equilibrium: A set of strategies such that no player can increase their payoff by deviating unilaterally from their strategy.

2-player 0-sum game

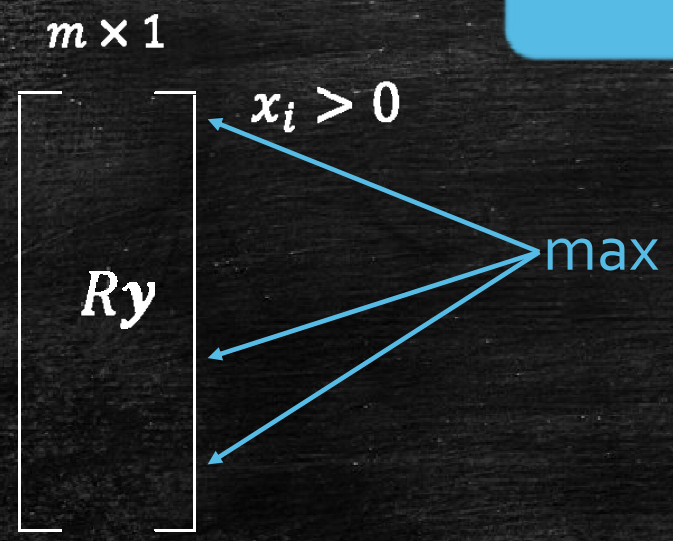
- Row player has m (pure) strategies and column player has n.
- Strategy $\longrightarrow$ probability vector over the pure strategies

Row player: $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \in \Delta_m$ Column player: $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \Delta_n$

- A 2-player game is a tuple (R, C) of playoff matrices of size $m \times n$ for the row and the column player respectively.
  The game is zero-sum if $R = -C$.

2-player 0-sum game

$(\boldsymbol{x}, \boldsymbol{y})$ is a Nash Equilibrium $\Leftrightarrow$ $\begin{array}{l} \boldsymbol{x}^T R\boldsymbol{y} \geq \boldsymbol{x}'^T R\boldsymbol{y} \;\; \forall \boldsymbol{x}' \in \Delta_m \\ \boldsymbol{x}^T C\boldsymbol{y} \geq \boldsymbol{x}^T C\boldsymbol{y}' \;\; \forall \boldsymbol{y}' \in \Delta_n \end{array}$



- If row player knew $\boldsymbol{y}$, she would choose $\boldsymbol{x}$ so that its positive coordinates correspond to the max values of $R\boldsymbol{y}$.

$$\boldsymbol{x}^T R\boldsymbol{y} \geq \boldsymbol{e}_i^T R\boldsymbol{y} \;\; \forall i \in [m]$$
$$\boldsymbol{x}^T R\boldsymbol{y} \geq \boldsymbol{x}^T R\boldsymbol{e}_j \;\; \forall j \in [n]$$

2-player 0-sum game: Example

$$R = -C = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

- If row player chooses $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then $\boldsymbol{x}^T C = \begin{bmatrix} -3x_1 + 2x_2 \\ x_1 - x_2 \end{bmatrix} = -\boldsymbol{x}^T R$.
- Then the column player responds optimally and gets $\max(-3x_1 + 2x_2, x_1 - x_2)$ so the row player gets $-\max(-3x_1 + 2x_2, x_1 - x_2) = \min(3x_1 - 2x_2, -x_1 + x_2)$.
- Row player chooses $\boldsymbol{x}$ to `maximize` $\min(3x_1 - 2x_2, -x_1 + x_2)$.

2-player 0-sum game: Example

We can formulate $\max_{x_1,x_2} \min(3x_1 - 2x_2, -x_1 + x_2)$ as a LP:

$$
\begin{aligned}
& \max z \\
\text{s.t.} \quad & 3x_1 - 2x_2 \geq z \\
& -x_1 + x_2 \geq z \\
& x_1 + x_2 = 1 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

- The solution of this LP $\boldsymbol{x} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$ and the solution of its dual LP $\boldsymbol{y} = \begin{bmatrix} 2/7 \\ 5/7 \end{bmatrix}$ is a Nash Equilibrium where the payoff of the row player is $1/7$.

Quick LP recap!

2-player 0-sum game

(Primal) LP for the row player:

$$\max z$$

s.t. $\boldsymbol{x}^T R \geq \underbrace{[z \quad \cdots \quad z]^T}_{n}$

$$\sum_{i=1}^{m} x_i = 1$$

$$x_i \geq 0 \; \forall i \in [m]$$

Solution: $\max_{\boldsymbol{x}} \min_{\boldsymbol{y}} \boldsymbol{x}^T R \boldsymbol{y}$

(Dual) LP for the column player:

$$\max \; z'$$

s.t. $C\boldsymbol{y} \geq \underbrace{[z' \quad \cdots \quad z']}_{m}$

$$\sum_{i=1}^{n} y_j = 1$$

$$\mathrm{y_j} \geq 0 \; \forall j \in [n]$$

Solution: $\max_{\boldsymbol{y}} \min_{\boldsymbol{x}} \boldsymbol{x}^T C \boldsymbol{y} = -\min_{\boldsymbol{y}} \max_{\boldsymbol{x}} \boldsymbol{x}^T R \boldsymbol{y}$

2-player 0-sum game

- If $(\boldsymbol{x}, z)$ and $(\boldsymbol{y}, z')$ are optimal solutions for the row and column player LPs respectively, then $(\boldsymbol{x}, \boldsymbol{y})$ is a N.E. with $z = -z'$ (and vise versa).

- $(\boldsymbol{x}, z)$ is feasible $\Rightarrow \boldsymbol{x}^T R \geq \underbrace{[z \quad \cdots \quad z]^T}_{n} \xRightarrow[\sum_{i=1}^{n} y_j = 1]{} \boldsymbol{x}^T R \boldsymbol{y} \geq z$

  Therefore if the row player plays $\boldsymbol{x}$ against $\boldsymbol{y}$ she gains at least z.

- $(\boldsymbol{y}, z')$ is feasible $\Rightarrow C\boldsymbol{y} \geq \underbrace{[z' \quad \cdots \quad z']}_{m} \xRightarrow[\sum_{i=1}^{n} x'_j = 1]{} \boldsymbol{x}'^T R \boldsymbol{y} \leq -z' \quad \forall x'$

  Therefore the row player gains at most $-z' = z$ if the column player plays $\boldsymbol{y}$.

Thus, $\boldsymbol{x}$ is best response and the same holds for $\boldsymbol{y}$.

2-player 0-sum game

- There is always a N.E.
- It doesn't matter who plays first (Minmax Theorem)
- A pair of any two optimal solutions of the LPs is a N.E.
- No matter what strategies the players follow, the payoffs in a N.E. are the same (a.k.a. value of the game).
- The set of equilibria and the sets of optimal strategies are all convex.

Nash's Theorem

- Let there be n players, each player p with a set of strategies S_p (set of mixed strategies denoted by Δ_{S_p}).
- There is a utility function for every player $u_p: \Delta_{S_1} \times \cdots \times \Delta_{S_n} \to \mathbb{R}$.
- A mixed strategy profile $\vec{\boldsymbol{x}} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$ is a N.E. if for every player p:

$$u_p(\vec{\boldsymbol{x}}) \geq u_p(\boldsymbol{x}'_p; \vec{\boldsymbol{x}}_{-p})$$

Nash's Theorem

Nash's Theorem (1950)	Every (finite) game has a Nash Equilibrium.
Brouwer's Theorem (1911)	Every continuous function from a closed compact convex set to itself has a fixed point.
Sperner's Lemma (1950)	Every proper coloring of a triangulation has a panchromatic triangle.
Parity Argument (1990)	If a directed graph has an unbalanced node, then it must have another.

Nash's Theorem

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- The set of mixed strategy profiles $\Delta_{S_1} \times \cdots \times \Delta_{S_n} = \Delta$ is closed, compact and convex.
- We need to find a continuous $f: \Delta \to \Delta$.

Take $\vec{y} = f(\vec{x})$ where $y_p(s_p) = \frac{x_p(s_p) + G_{p,s_p}(\vec{x})}{1 + \sum_{s'_p \in S_p} G_{p,s'_p}(\vec{x})}$

and $G_{p,s_p}(\vec{x}) = u_p(s_p; \vec{x}_{-p}) \dot{-} u_p(\vec{x})$ is the gain of player p if she switched to strategy s_p.

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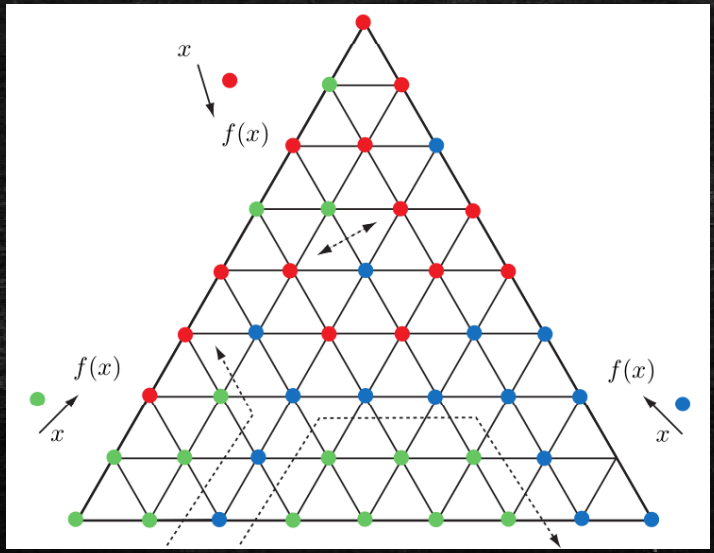

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