Algorithmic Game Theory
CoReLab (NTUA)

Lecture 3:

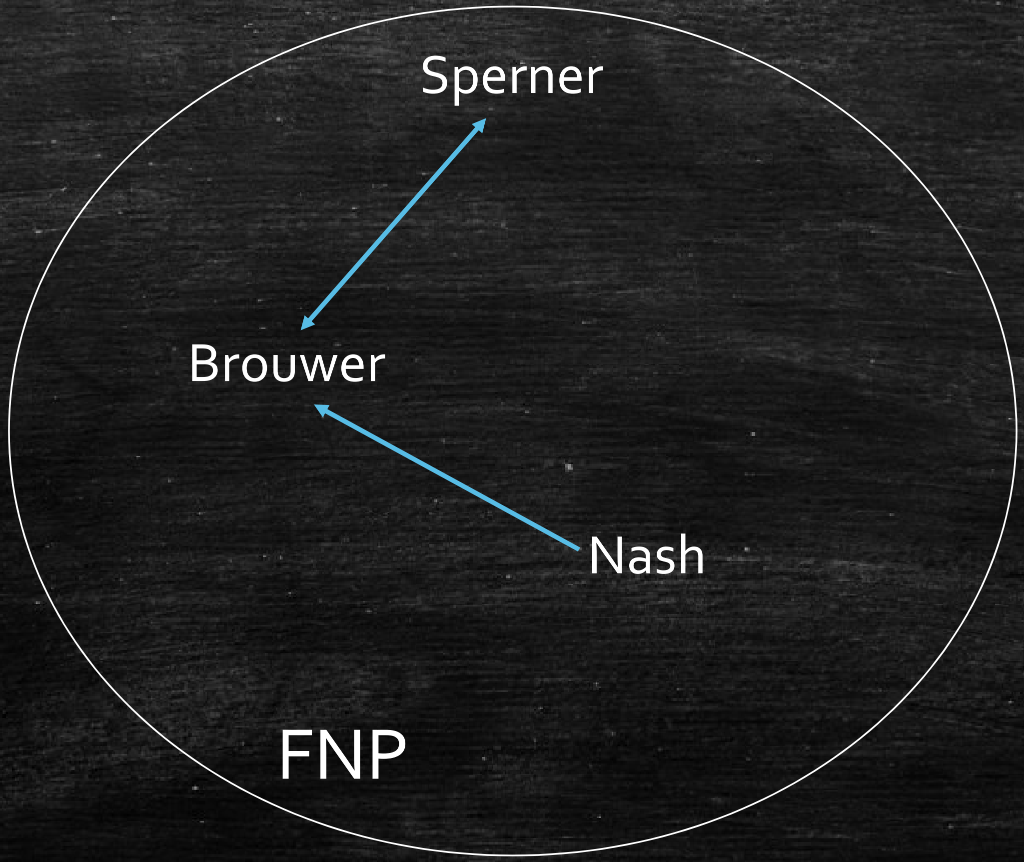
(In)tractability of Nash Equilibria
PPAD completeness
LMM

So far

NE in 2-player zero sum ↔ LP Duality

Nash's Theorem (1950)	Every (finite) game has a Nash Equilibrium.
⬆	
Brouwer's Theorem (1911)	Every continuous function from a closed compact convex set to itself has a fixed point.
⬆	
Sperner's Lemma (1950)	Every proper coloring of a triangulation has a panchromatic triangle.
⬆	
Parity Argument (1990)	If a directed graph has an unbalanced node, then it must have another.

What we know



General 2-player games

A slightly more ambitious attempt would be to face general 2- player games and provide efficient algorithms or prove hardness results.

An other direction would be to face 3-player zero sum games….

but in fact these games can only be harder.(!)

Nash vs NP

The problem resisted polynomial algorithms for a long time which altered the research direction towards hardness results.

The first idea would be to prove Nash an FNP-complete problem.

But accepting an FSAT $\longrightarrow$ Nash

reduction directly implies NP=coNP. (!)

Nash vs TFNP

What prevented our previous attempt was the fact that Nash problem always has solution.

So the next idea would be to prove it complete for this class.

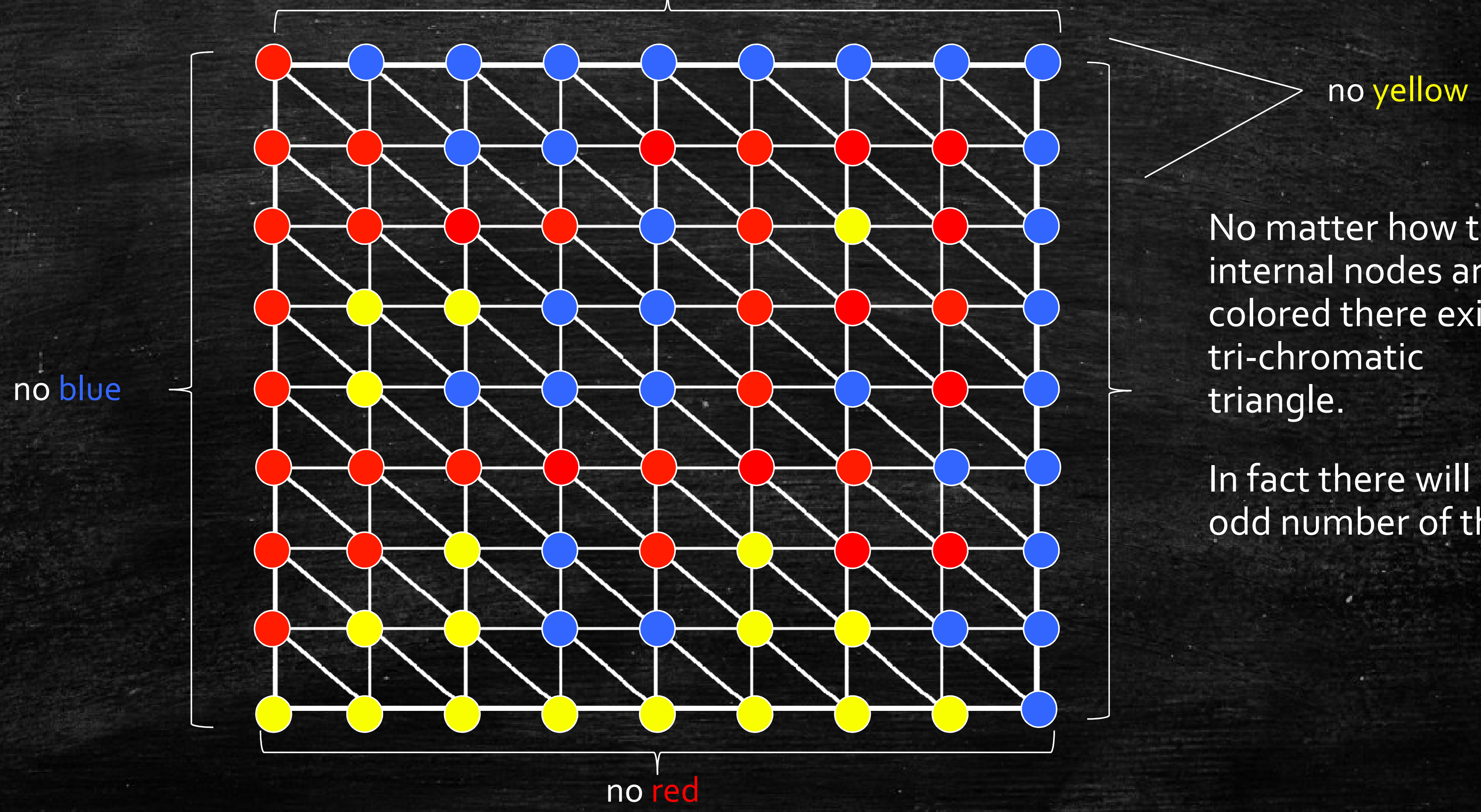
But no complete problem is known for TFNP.

Complexity Theory of Total Search Problems

In order to overcome the obstacles we face we need to work as follows:

1. Identify the combinatorial structure that makes our problems total.
2. Define a new complexity class inspired from our observation.
3. Check the 'tightness' of our class – in other words that our problems are complete for the class.

Sperner's Lemma revisited



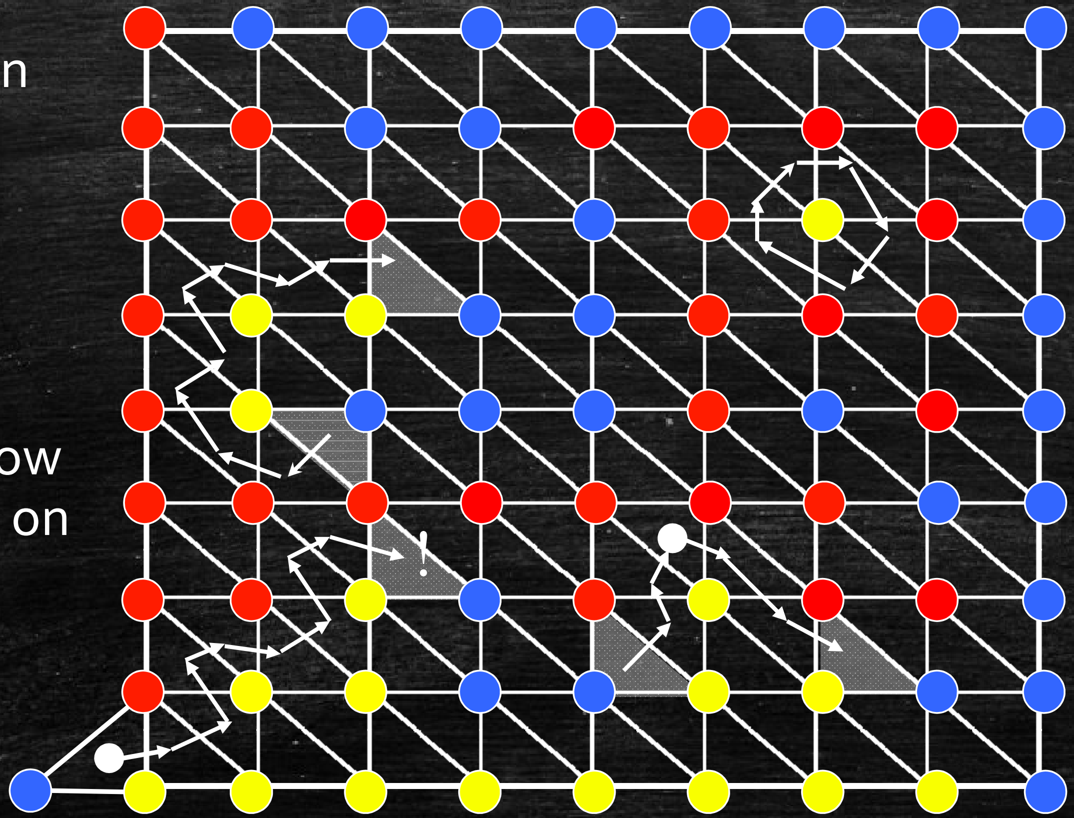
No matter how the internal nodes are colored there exists a tri-chromatic triangle.

In fact there will be an odd number of them.

Proof of Sperner's Lemma

1. We introduce an artificial vertex on the bottom left

2. We define a directed walk crossing red-yellow doors having red on our left

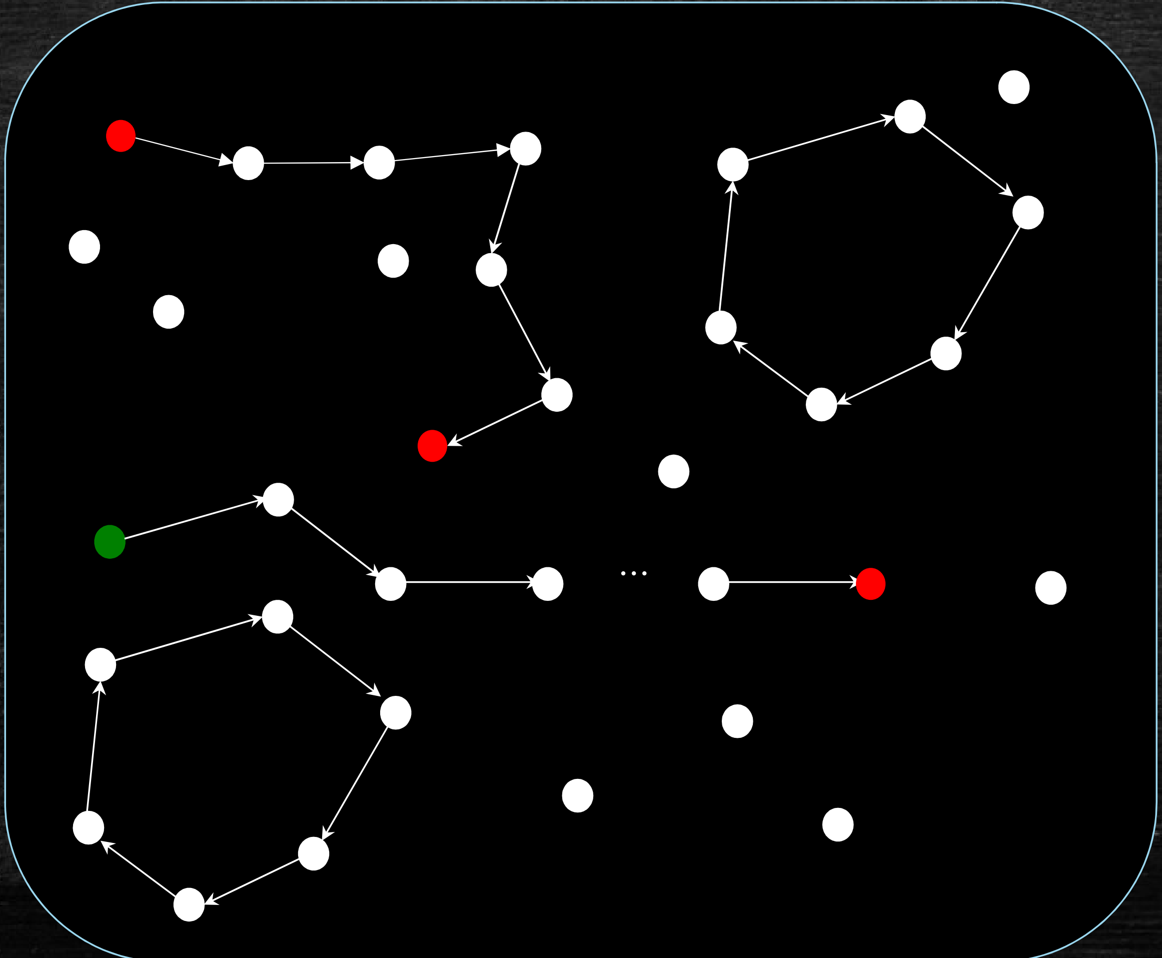


- Claim: The walk can't get out nor can it loop into itself

- It follows that there is an odd number of tri-chromatic triangles

Parity Argument
Graph Representation

- Every vertex has in and out degree at most 1
- Each vertex with degree 1 is an acceptable solution (except the artificial one)

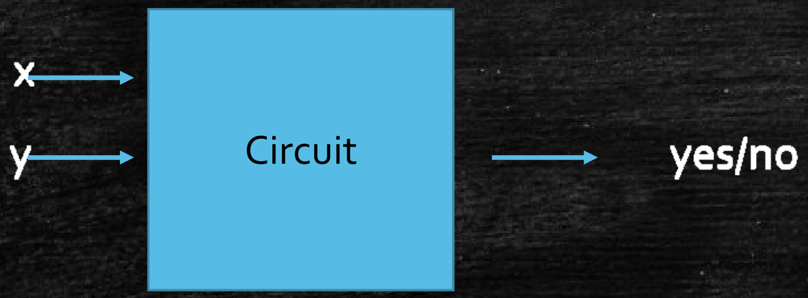


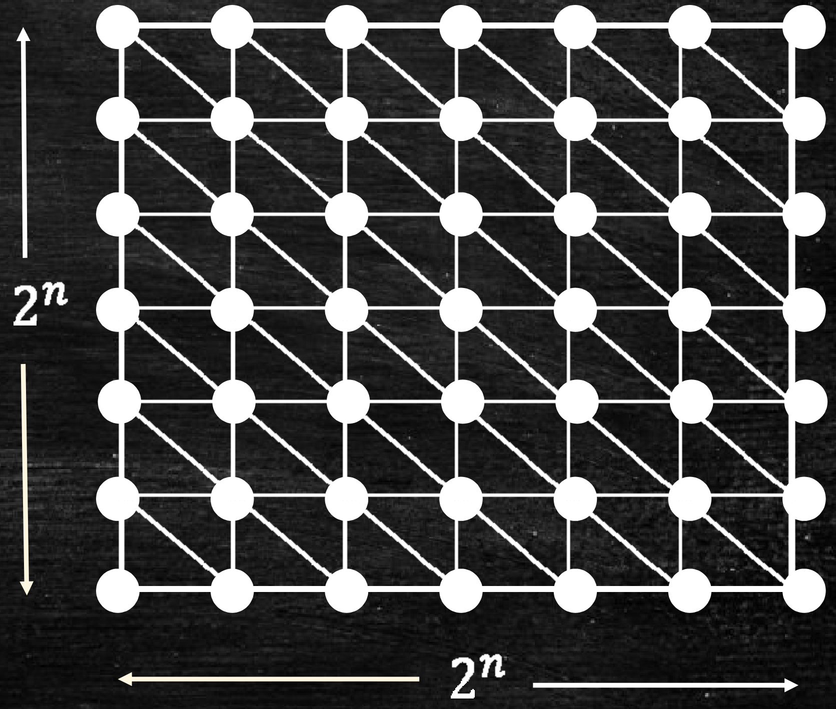
- By the parity argument there is always an even number of solutions
- Notice that if we insist in finding the pair of our green node the problem is beyond FNP!

Why Sperner is hard?

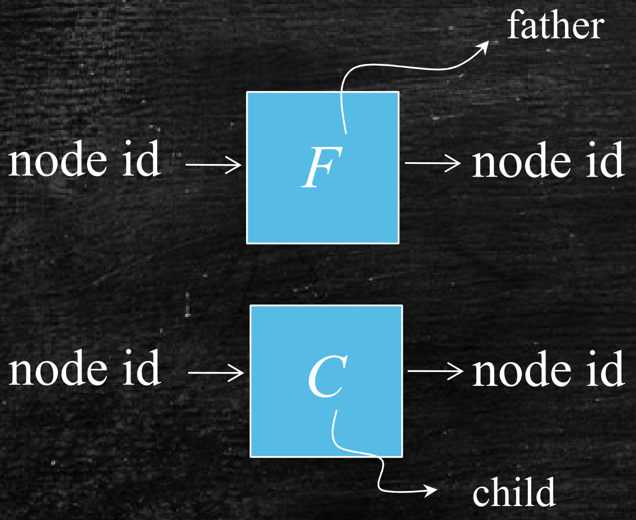
We have to work with a graph of exponential size!

Input: 2 n-bit numbers

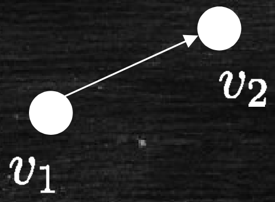




The PPAD Class [Papadimitriou '94]



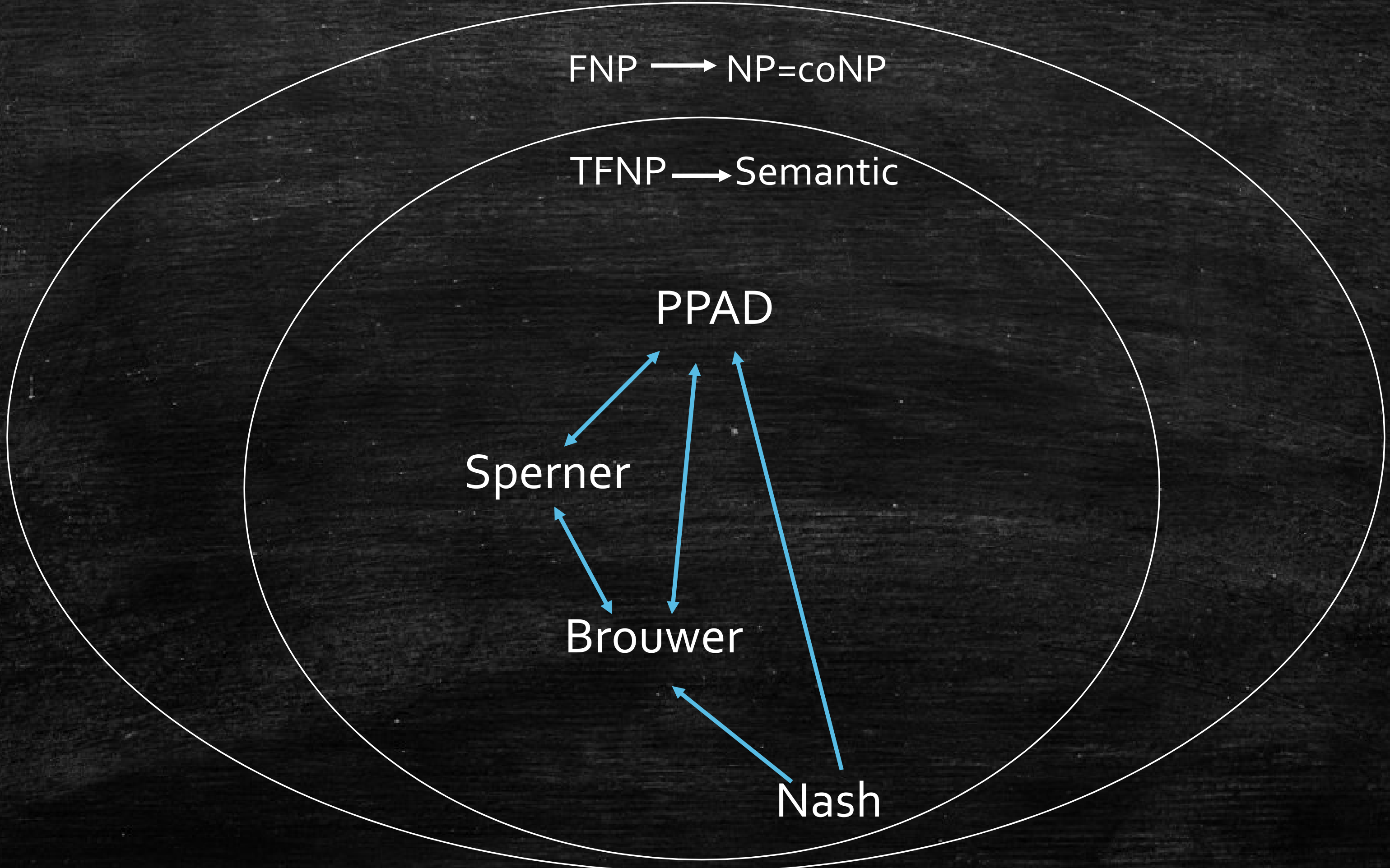
$$F(v_2) = v_1 \wedge C(v_1) = v_2$$



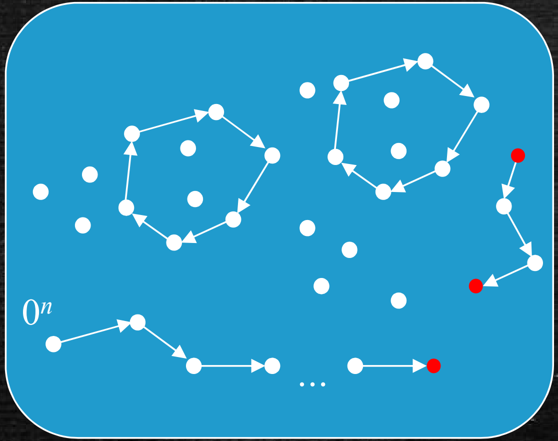
END OF THE LINE → Given F and C: If 0^n is an unbalanced node, find another unbalanced node. Otherwise say "yes".

PPAD = *{ Search problems in FNP reducible to END OF THE LINE}*

What we know



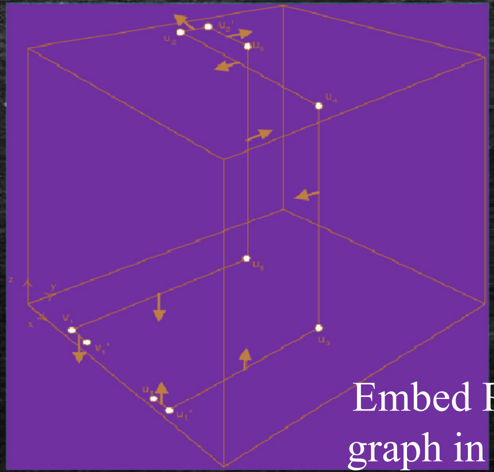
2-Nash PPAD-complete
[Daskalakis,Goldberg,Papadimitriou 2006]



Generic PPAD

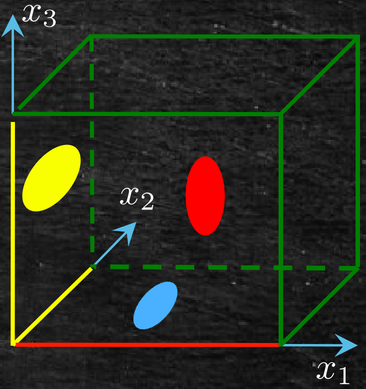
[Pap '94]

[DGP '05]



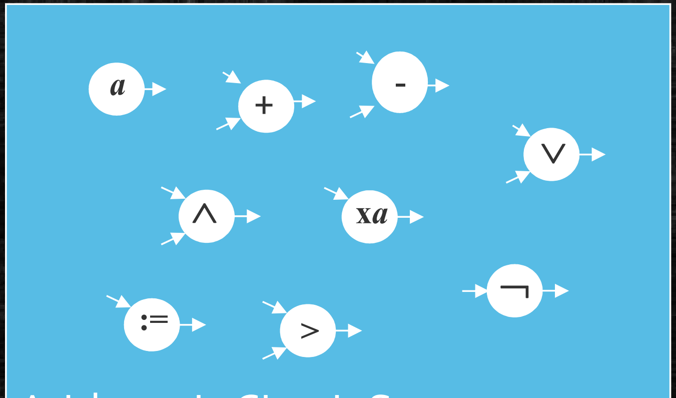
Embed PPAD graph in $[0,1]^3$

[DGP '05]



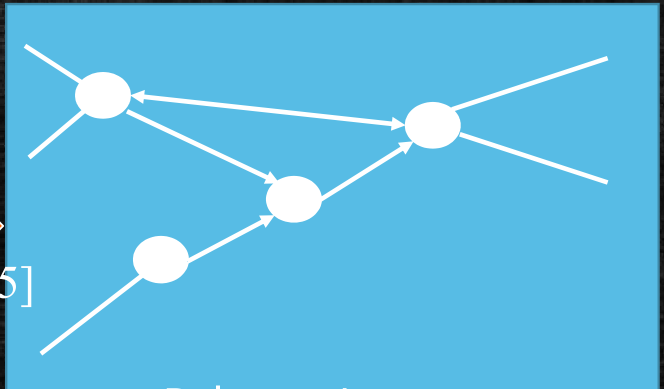
3D-SPERNER

[DGP '05]



Arithmetic Circuit Sat

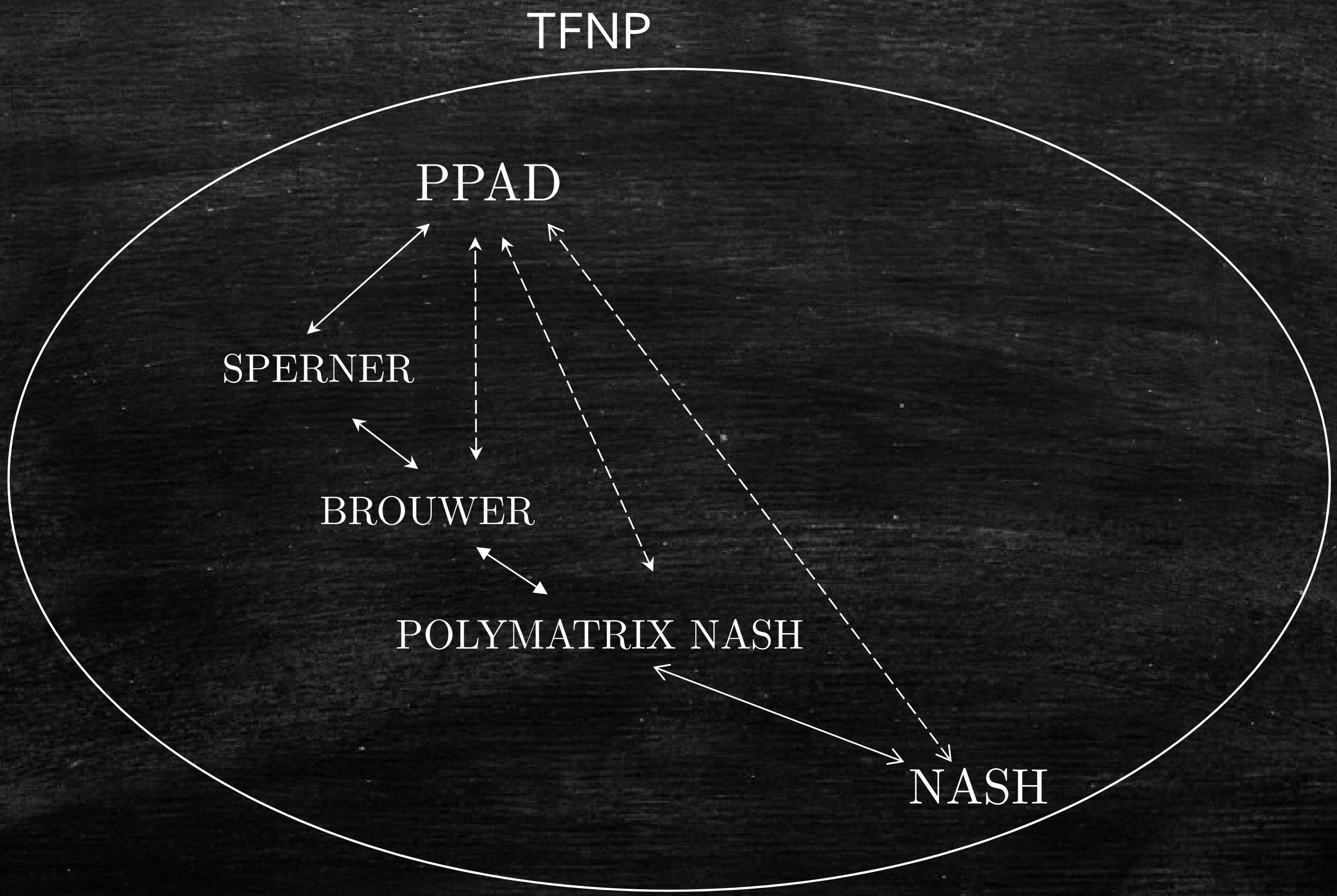
[DGP '05]



Polymatrix game

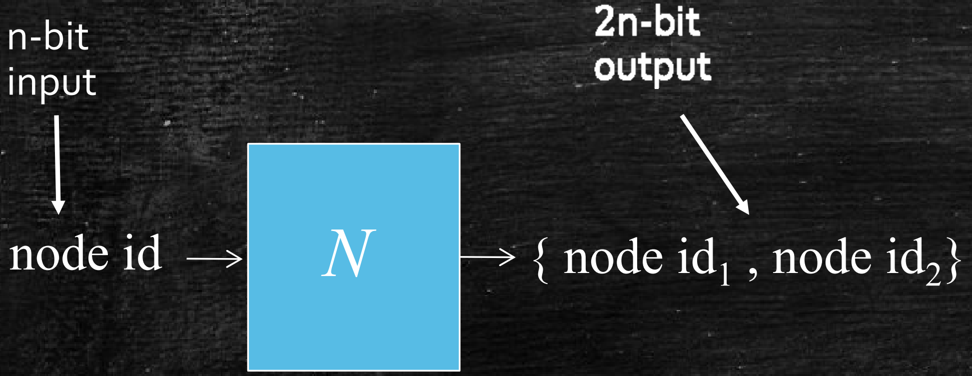
[DGP '05]

PPAD completeness of 2-Nash

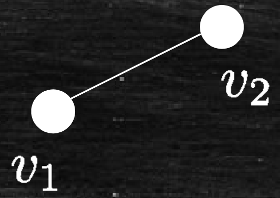


Arguments of existence and respective complexity classes
PPA[Papadimitriou ’94]

‘If a node has odd degree then there must be an other.’



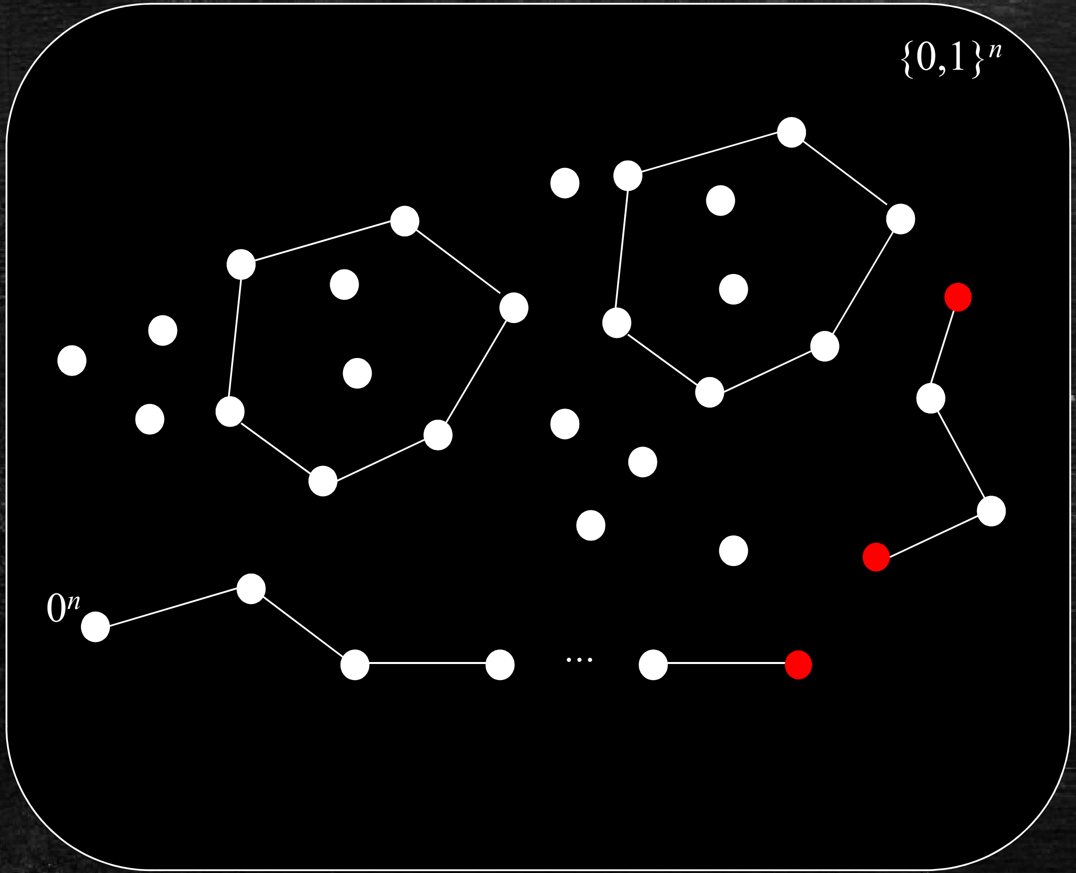
$$v_1 \in N(u_2) \ \& \ u_2 \in N(u_1)$$



ODD DEGREE NODE ⟶ Given N: If 0^n has odd degree, find another node with odd degree. Otherwise say “yes”.

PPA = { *Search problems in FNP reducible to* ODD DEGREE NODE}

PPA
Graph Representation



Exponentially large graph

Every node has degree at most 2

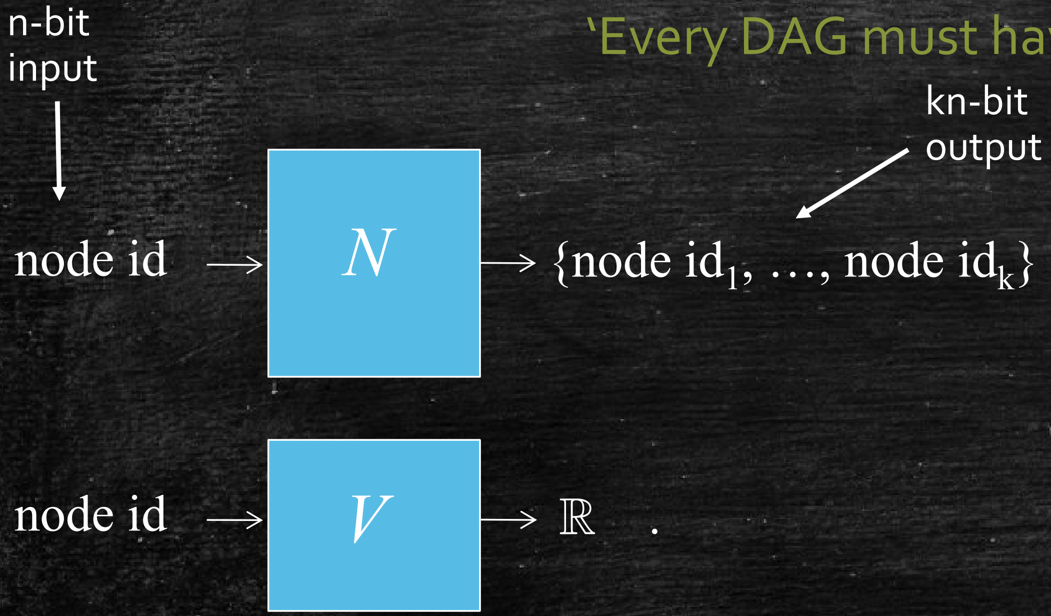
PLS [JPY ‘89]

‘Every DAG must have a sink.’

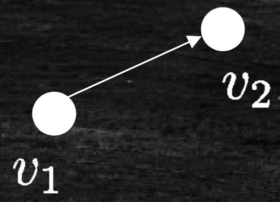
- **Local Max Cut** is a well known PLS-complete problem.
- **Spoiler! PNE** in **Congestion Games** is also PLS-complete.

PLS [JPY '89]

'Every DAG must have a sink.'



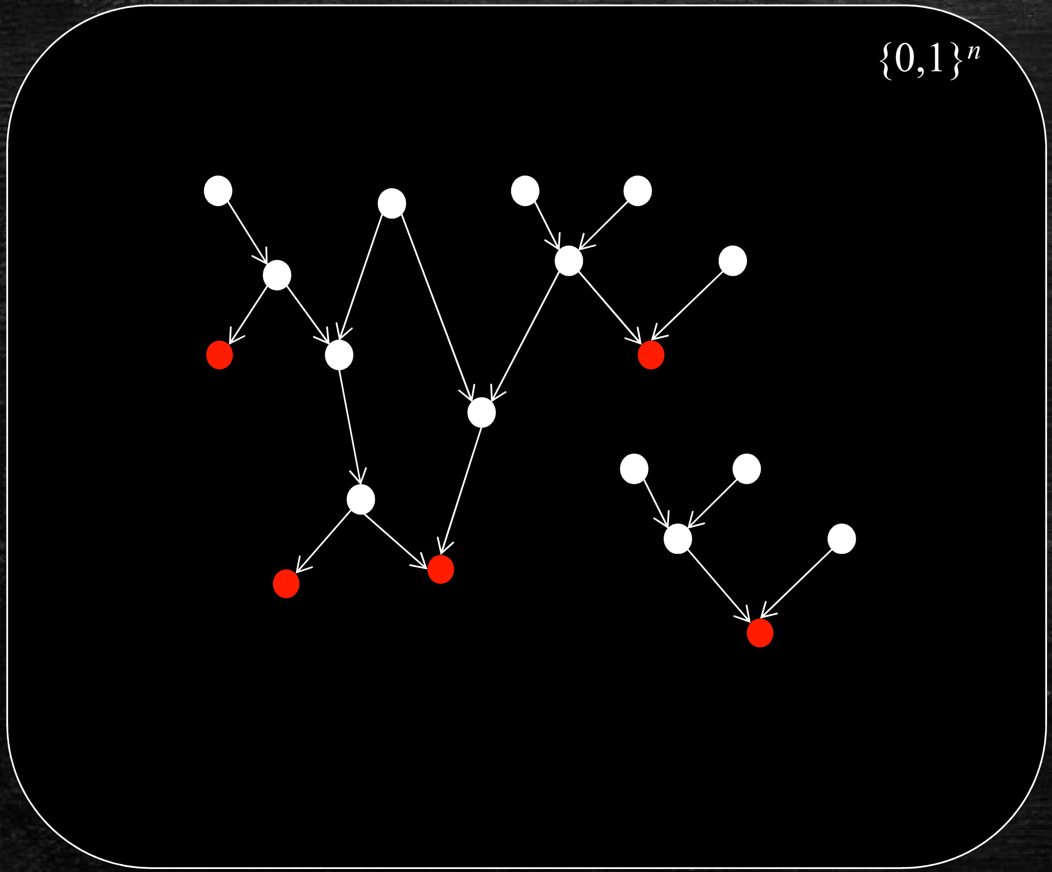
$$v_2 = N(v_1) \,\&\, V(v_2) > V(v_1)$$



FIND SINK ⟶ Given N, V: Find x s.t. $V(x) \geq V(y)$, for all $y \in N(x)$.

PLS = { *Search problems in FNP reducible to* FIND SINK }

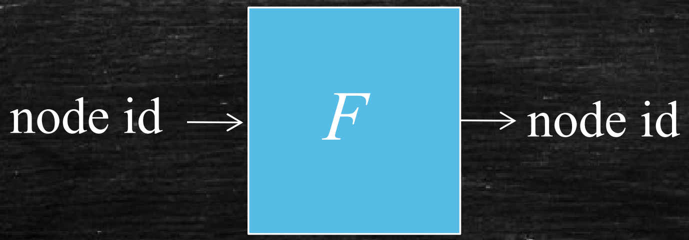
PLS
Graph Representation



Exponentially large directed acyclic graph

PPP

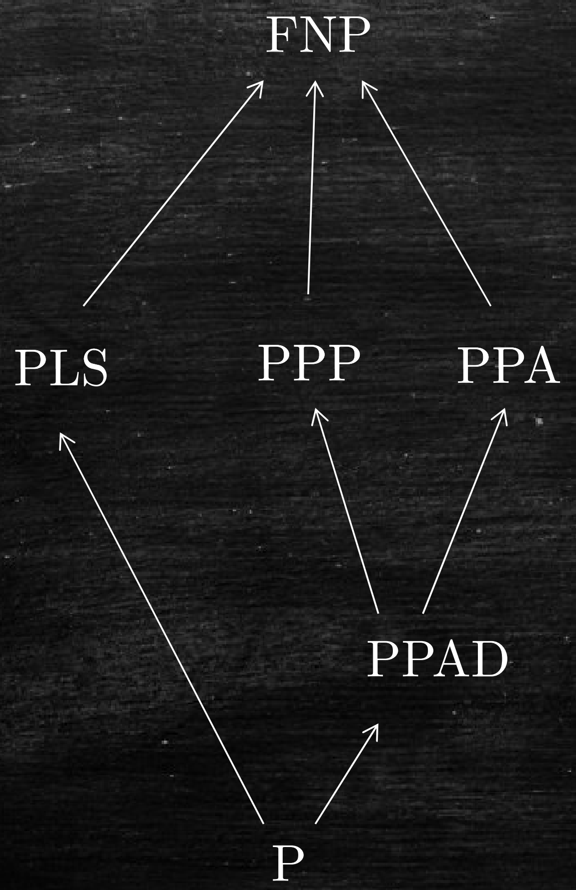
"If a function maps n elements to n-1 elements, then there is a collision."



COLLISION ⟶ Given F: Find x s.t. $F(x) = 0^n$; or find $x \neq y$ s.t. $F(x) = F(y)$.

PPP = { *Search problems in FNP reducible to* COLLISION }

Inclusions



2 −player Symmetric Games

A bimatix game represented by two matrices (A, B) is called **Symmetric** if $B = A^T$ (i.e., the two players have the same set of strategies and their utilities remain the same if their roles are reversed).

A strategy profile x is a **Symmetric Nash Equilibrium** if both players playing x results in a Nash Equilibrium.

Looking at Symmetric Games is no loss of generality!

Reduction from Nash to Symmetric Nash

Fix any bimatrix game represented by the matrices A, B (w.l.o.g. with positive entries).
Now consider the Symmetric Game defined by the matrices below:

$$C_1 = \begin{bmatrix} 0 & A \\ B^T & 0 \end{bmatrix} \begin{matrix} \}\, x \\ \}\, y \end{matrix} \qquad C_2 = \begin{matrix} \overbrace{}^{x} & \overbrace{}^{y} \\ \end{matrix} \begin{bmatrix} 0 & B \\ A^T & 0 \end{bmatrix}$$

Let (x, y) be a Symmetric NE.
In order (x, y) to be a best response to itself, x must be a best response to y and vice versa.

Recap

GOOD NEWS

- NE in 2-player zero sum ↔ LP Duality

BAD NEWS

- NE in general 2-player games PPAD complete

  (Lemke-Howson exponential running time algorithm)

SOLUTION

In order to sidestep the probable intractability of NE we are going to relax our equilibrium concept!

Approximate Nash Equilibrium

For any $\varepsilon > 0$ a pair of mixed strategies x, y is called an ε -Nash equilibrium if:

i. For every mixed strategy x' of the row player, $(x', Ay) \leq (x, Ay) + \varepsilon$

ii. For every mixed strategy y' of the column player, $(x, By') \leq (x, By) + \varepsilon$

Lipton Markakis Mehta ’03

Main result

(Assuming all entries of A, B between 0,1)

For any NE x^*, y^* and for any $\varepsilon > 0$, there exists, for every $k \geq {}^{12 \ln n}/_{\varepsilon^2}$ a pair of k −uniform strategies x', y' such that:

1. x', y' is an ε-NE
2. $|(x', Ay') - (x^*, Ay^*)| < \varepsilon$ (row player gets almost the same payoff as in the NE)
3. $|(x', By') - (x^*, By^*)| < \varepsilon$ (column player gets almost the same payoff as in the NE)

Lipton Markakis Mehta '03
Proof Sketch via Probabilistic Method

- Given $x^*, y^*, \varepsilon > 0$ fix $k \geq {}^{12 \ln n}/_{\varepsilon^2}$

- Form multiset X sampling k times independently, from the pure strategies of the row player according to the distribution x^*.
  Respectively, form Y from the pure strategies of the column player.

- Let x' be the k −uniform strategy related with multiset X and y' the k −uniform strategy related with multiset Y.