Algorithmic Game Theory

Introduction to Mechanism Design for Single Parameter Environments

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Generalizations to single-parameter environments

Single-parameter mechanisms

- In many cases, we do not have a single item to sell, but multiple items
- But still, the valuation of a bidder could be determined by a single number (e.g., value per unit)
- Note: the valuation function may depend on various other parameters, but we assume only a single parameter is private information to the bidder
  - The other parameters may be publicly known information
- We can treat all these settings in a unified manner
- Our focus: **Direct revelation** mechanisms
  - The mechanism asks each bidder to submit the parameter that completely determines her valuation function

Examples of single-parameter environments

- **Single-item auctions:**
  - One item for sale
  - each bidder is asked to submit his value for acquiring the item
- **k-item unit-demand auctions**
  - k identical items for sale
  - each bidder submits his value per unit and can win at most one unit
- **Knapsack auctions**
  - k identical items, each bidder has a value for obtaining a certain number of units
- **Single-minded auctions**
  - a set of (non-identical) items for sale
  - each bidder is interested in acquiring a specific subset of items (known to the mechanism)
  - Each bidder submits his value for the set she desires

Examples of single-parameter environments

- Sponsored search auctions
  - multiple advertising slots available, arranged from top to bottom
  - each bidder interested in acquiring as high a slot as possible
  - each bidder submits his value per click
- Public project mechanisms
  - deciding whether to build a public project (e.g., a park)
  - each bidder submits his value for having the project built

In all these settings, we can have multiple winners in the auction

Some Notation

- Suppose we have n players
- Let v_i be the parameter that is private information to player i
  - Usually v_i corresponds to value per unit, or value obtained at the desirable outcome, or maximum amount willing to pay (dependent on the context)

General form of direct-revelation mechanisms for single-parameter problems:

- Input: The bidding vector $\mathbf{b} = (b_1,\ldots, b_n)$ by the players
  - each b_i may differ from v_i
- Allocation rule: Choose an allocation $\mathbf{x}(\mathbf{b}) = (x_1(\mathbf{b}), x_2(\mathbf{b}),\ldots, x_n(\mathbf{b}))$
  - $x_i(\mathbf{b})$ = number of units received by pl. i or more generally the decision on what is allocated to i
- Payment rule: $\mathbf{p}(\mathbf{b}) = (p_1(\mathbf{b}), p_2(\mathbf{b}),\ldots, p_n(\mathbf{b}))$
  - $p_i(\mathbf{b})$ = payment for bidder i

Some Notation

- We will use (**x**, **p**) to refer to a mechanism with allocation function **x**, and payment function **p**
- Final utility of bidder i in a mechanism M = (**x**, **p**):
  - $u_i(\mathbf{b}) = v_i\, x_i(\mathbf{b}) - p_i(\mathbf{b})$
  - Quasi-linear form of utility functions
- For simplicity, we often write $(x_1, x_2,\ldots, x_n)$ instead of $(x_1(\mathbf{b}), x_2(\mathbf{b}),\ldots, x_n(\mathbf{b}))$
- We focus on mechanisms that satisfy Individual Rationality:
  - If a bidder i is a non-winner ($x_i(\mathbf{b}) = 0$), then $p_i(\mathbf{b}) = 0$
  - For winners, the payment rule satisfies $p_i(\mathbf{b}) \in [0, b_i\, x_i(\mathbf{b})]$ for every bidding vector **b** and every i
  - The auctioneer can never ask a bidder for a payment higher than her declared total value for what she won

Examples of single-parameter environments

Describing the feasible allocations

- **Single-item auctions:**
  - $x_i \in \{0, 1\}$ for every i, and $\Sigma_i x_i = 1$
- **k-item unit-demand auctions**
  - k identical items for sale
  - $x_i \in \{0, 1\}$, $\Sigma_i x_i <= k$
- **Knapsack auctions**
  - k identical items for sale
  - For each bidder, demand of w_i units
  - $x_i \in \{0, 1\}$ for every i, $\Sigma_i w_i x_i <= k$
- **Public project mechanisms**
  - Deciding whether to build a public project (e.g., a park)
  - Only 2 feasible allocations: (0, 0, ..., 0) or (1, 1, ..., 1)

Allocation rules and truthful mechanisms

- Can we understand how to derive truthful mechanisms?
- Actually, we can rephrase this as:
  - Suppose we are given an allocation rule **x**
  - Can we tell if **x** can be combined with a pricing rule **p**, so that (**x**, **p**) is a truthful mechanism?
- This would allow us to focus only on designing the allocation algorithm appropriately
- Consider the single-item auction
  - Allocation rule 1: Give the item to the highest bidder
  - Allocation rule 2: Give the item to the 2^{nd} highest bidder
- For rule 1, we have seen how to turn it into a truthful mechanism (Vickrey auction)
- For rule 2?
  - We have not seen how to do this, but we have also not proved that it cannot be done

Allocation rules and truthful mechanisms

- Consider a mechanism with allocation rule **x**
- Fix a player i, and fix a profile $\mathbf{b}_{-i}$ for the other players
- Allocation to player i at a profile $\mathbf{b} = (z, \mathbf{b}_{-i})$ is given by $x_i(\mathbf{b})$
- Keeping $\mathbf{b}_{-i}$ fixed, we can view the allocation to player i as a function of his bid
    - $x_i = x_i(z, \mathbf{b}_{-i})$, if bidder i bids z

- Definition: An allocation rule is monotone if for every bidder i, and every profile $\mathbf{b}_{-i}$, the allocation $x_i(z, \mathbf{b}_{-i})$ to i is non-decreasing in z
- I.e., bidding higher can only get you more stuff

Monotonicity of allocation rules

Examples

- Back to the single-item auction
- The allocation rule that gives the item to the highest bidder is monotone
  - If a bidder wins at profile **b**, she continues to be a winner if she raises her own bid (keeping $\mathbf{b}_{-i}$ fixed)
  - If she was not a winner at **b**, then by raising her bid, she will either remain a non-winner or she will become a winner
- The allocation rule that gives the item to the 2nd highest bidder is not monotone
  - If I am a winner and raise my bid, I may become the highest bidder and will stop being a winner

Myerson's lemma

[Myerson '81]

- Theorem: For every single-parameter environment,
  - An allocation rule **x** can be turned into a truthful mechanism if and only if it is monotone
  - If **x** is monotone, then there is a unique payment rule **p**, so that (**x**, **p**) is a truthful mechanism
    - Subject to the constraint that if $b_i = 0$, then $p_i = 0$

- One of the classic results in mechanism design
- In fact, in many cases we can also compute the payments by a simple formula

Myerson's lemma

- Allocation rule **x** is truthful =>
  Allocation rule **x** is monotone: forall z, y, (**x**(z) – **x**(y))(z – y) ≥ 0

If z is the true value:

$$\boldsymbol{x}(z) \cdot z - \boldsymbol{p}(z) \geq \boldsymbol{x}(y) \cdot z - \boldsymbol{p}(y) \tag{1}$$

If y is the true value:

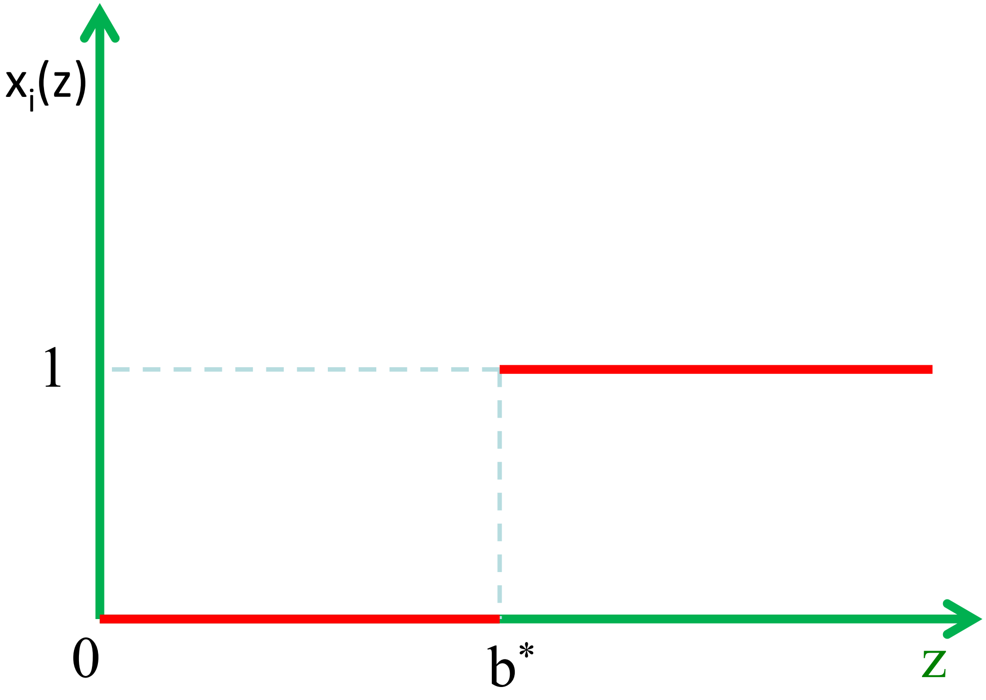
$$\boldsymbol{x}(y) \cdot y - \boldsymbol{p}(y) \geq \boldsymbol{x}(z) \cdot y - \boldsymbol{p}(z) \tag{2}$$

Summing up (1) and (2):

$$\boldsymbol{x}(z) \cdot z + \boldsymbol{x}(y) \cdot y \geq \boldsymbol{x}(y) \cdot z + \boldsymbol{x}(z) \cdot y \Leftrightarrow$$
$$(\boldsymbol{x}(z) - \boldsymbol{x}(y)) \cdot z \geq (\boldsymbol{x}(z) - \boldsymbol{x}(y)) \cdot y \Leftrightarrow$$
$$(\boldsymbol{x}(z) - \boldsymbol{x}(y)) \cdot (z - y) \geq 0$$

Myerson's lemma and payment formula

- For the payment rule, we need to look for each bidder at the allocation function $x_i(z, \mathbf{b}_{-i})$
- For the single-item truthful auction:
  - Fix $\mathbf{b}_{-i}$ and let $b^* = \max_{j \neq i} b_j$



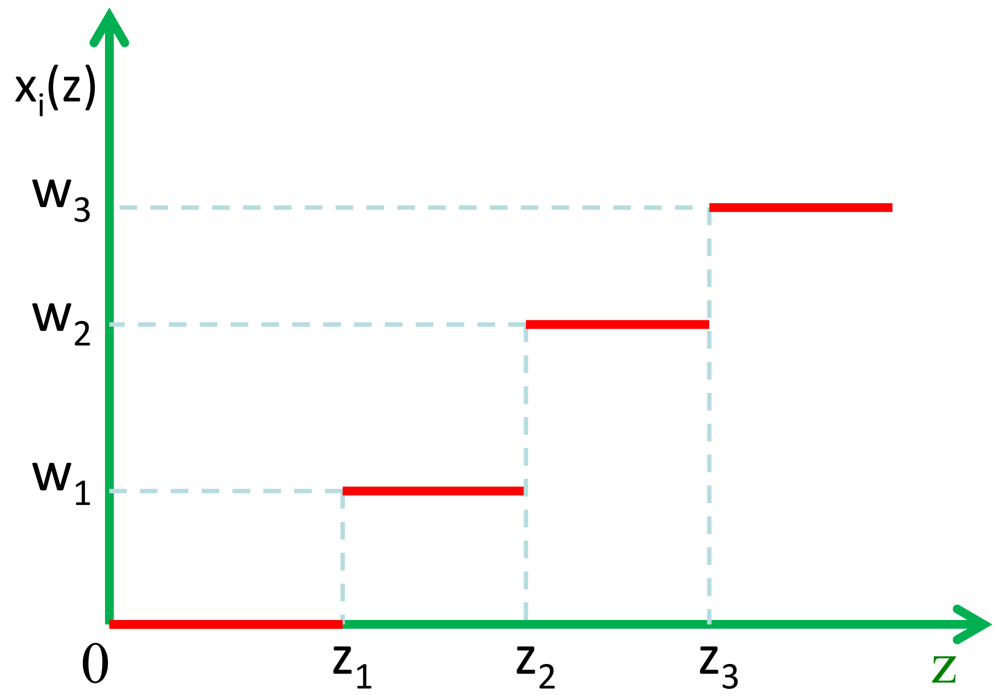
Facts:

- For any fixed $\mathbf{b}_{-i}$, the allocation function is piecewise linear with 1 jump
- The Vickrey payment is precisely the value at which the jump happens
- The jump changes the allocation from 0 to 1 unit

Myerson's lemma and payment formula

For most scenarios of interest

- The allocation is piecewise linear with multiple jumps
- The jump determines how many extra units the bidder wins



- Suppose bidder i bids b_i
- Look at the jumps of $x_i(z, b_{-i})$ in the interval $[0, b_i]$
- Suppose we have k jumps
- Jump at z_1: w_1
- Jump at z_2: $w_2 - w_1$
- Jump at z_3: $w_3 - w_2$
- ...
- Jump at z_k: $w_k - w_{k-1}$

Myerson's lemma and payment formula

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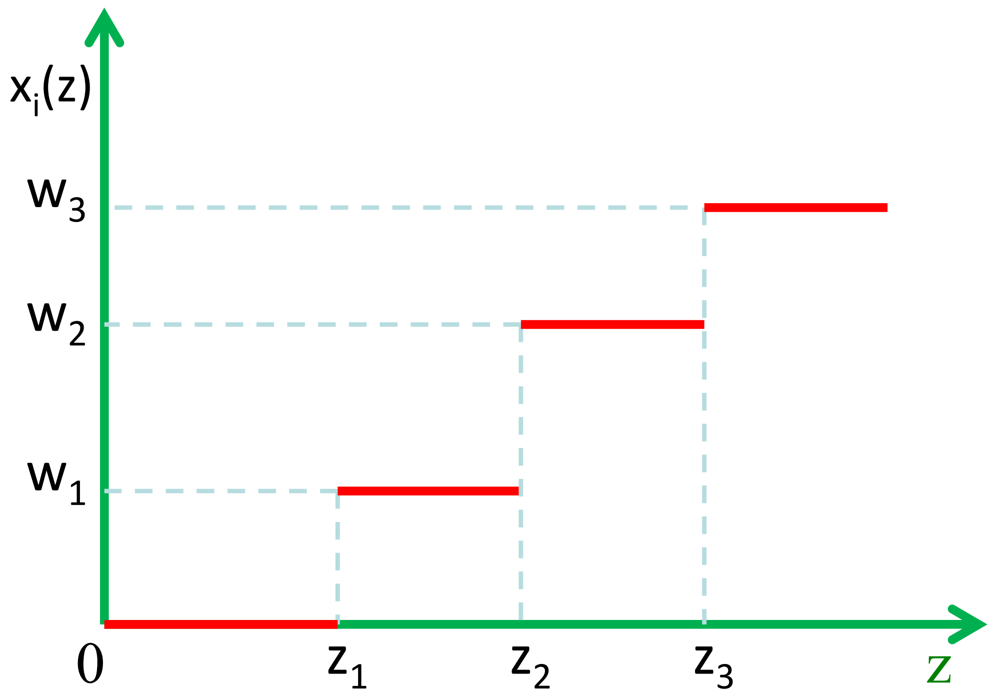

Payment formula

- For each bidder i at a profile b, find all the jump points within [0, b_i]
- $p_i(b) = \sum_j z_j \cdot$ [jump at z_j]
  $= \sum_j z_j \cdot [w_j - w_{j-1}]$
- The formula can also be generalized for monotone but not piecewise linear functions

Myerson's lemma

- Allocation rule **x** is truthful (and thus, monotone) => find appropriate payments **p**

If z is the true value:

$$\boldsymbol{x}(z) \cdot z - \boldsymbol{p}(z) \geq \boldsymbol{x}(y) \cdot z - \boldsymbol{p}(y) \tag{1}$$

If y is the true value:

$$\boldsymbol{x}(y) \cdot y - \boldsymbol{p}(y) \geq \boldsymbol{x}(z) \cdot y - \boldsymbol{p}(z) \tag{2}$$

Combining (1) and (2), we get:

$$z(\boldsymbol{x}(z) - \boldsymbol{x}(y)) \leq \boldsymbol{p}(y) - \boldsymbol{p}(z) \leq y(\boldsymbol{x}(z) - \boldsymbol{x}(y))$$

Assuming that y tends to z from above, in the limit, we get:

$$\boldsymbol{p}'(z) = z \cdot \boldsymbol{x}'(z) \tag{3}$$

Myerson's lemma

- Allocation rule **x** is truthful (and thus, monotone) => find appropriate payments **p**

$$\boldsymbol{p}'(z) = z \cdot \boldsymbol{x}'(z) \tag{3}$$

We assume $\boldsymbol{p}(0) = 0$ (normalization) and solve (3):

$$\boldsymbol{p}_i(b_i, \boldsymbol{b}_{-i}) = \int_0^{b_i} z \cdot \boldsymbol{x}_i'(z, \boldsymbol{b}_{-i}) dz = b_i \cdot \boldsymbol{x}_i(b_i, \boldsymbol{b}_{-i}) - \int_0^{b_i} \boldsymbol{x}_i(z, \boldsymbol{b}_{-i}) dz$$

$$\boldsymbol{p}_i(b_i, \boldsymbol{b}_{-i}) = b_i \cdot \boldsymbol{x}_i(b_i, \boldsymbol{b}_{-i}) - \int_0^{b_i} \boldsymbol{x}_i(z, \boldsymbol{b}_{-i}) dz$$

$$i\text{'s utility:} \quad u_i(b_i, \boldsymbol{b}_{-i}) = (v_i - b_i) \cdot \boldsymbol{x}_i(b_i, \boldsymbol{b}_{-i}) + \int_0^{b_i} \boldsymbol{x}_i(z, \boldsymbol{b}_{-i}) dz$$

Myerson's lemma

- Any monotone allocation rule **x** is truthful with payments **p**

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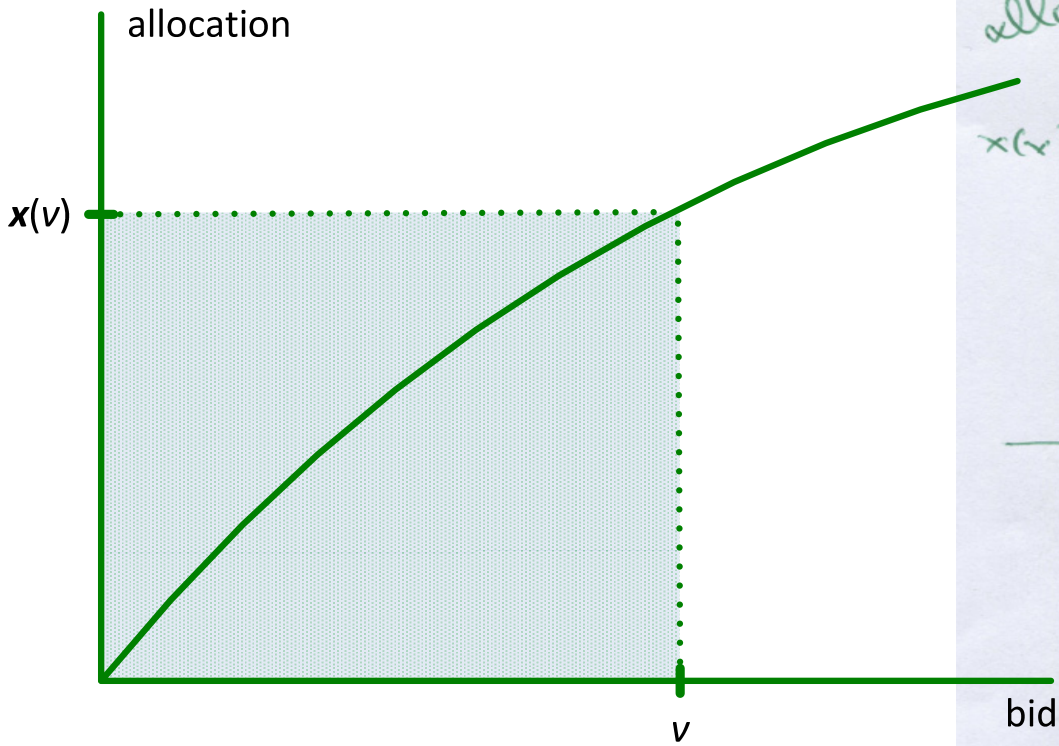

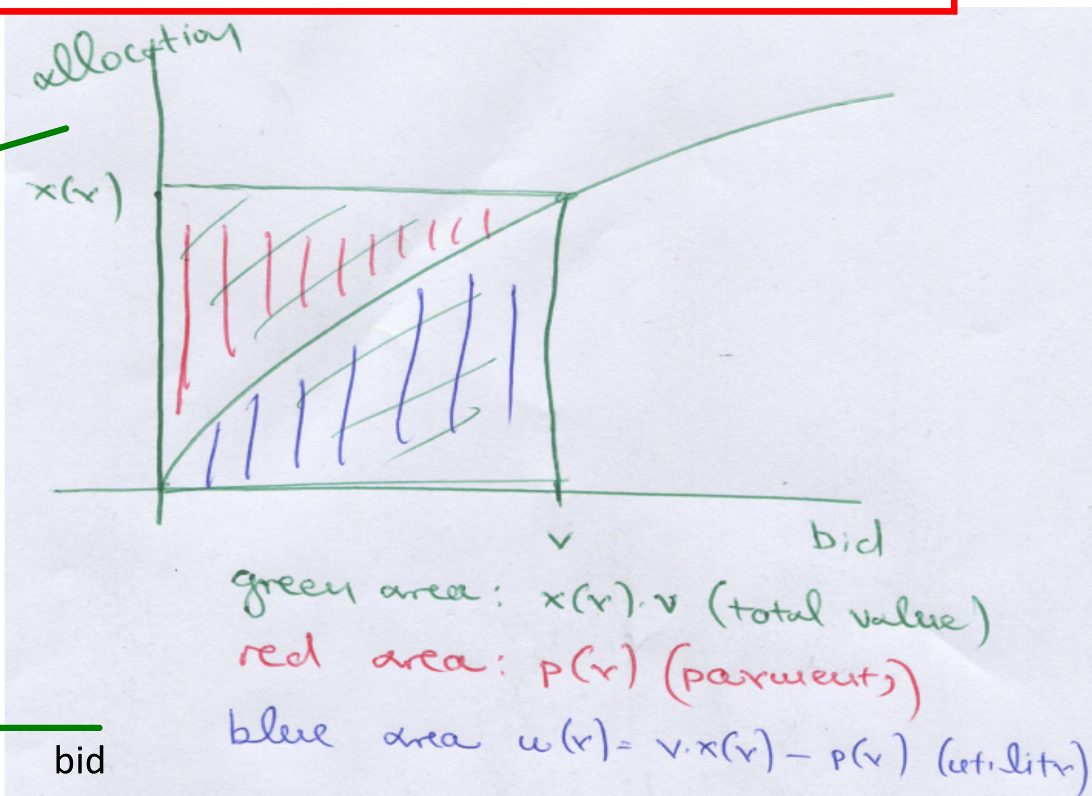

Myerson's lemma

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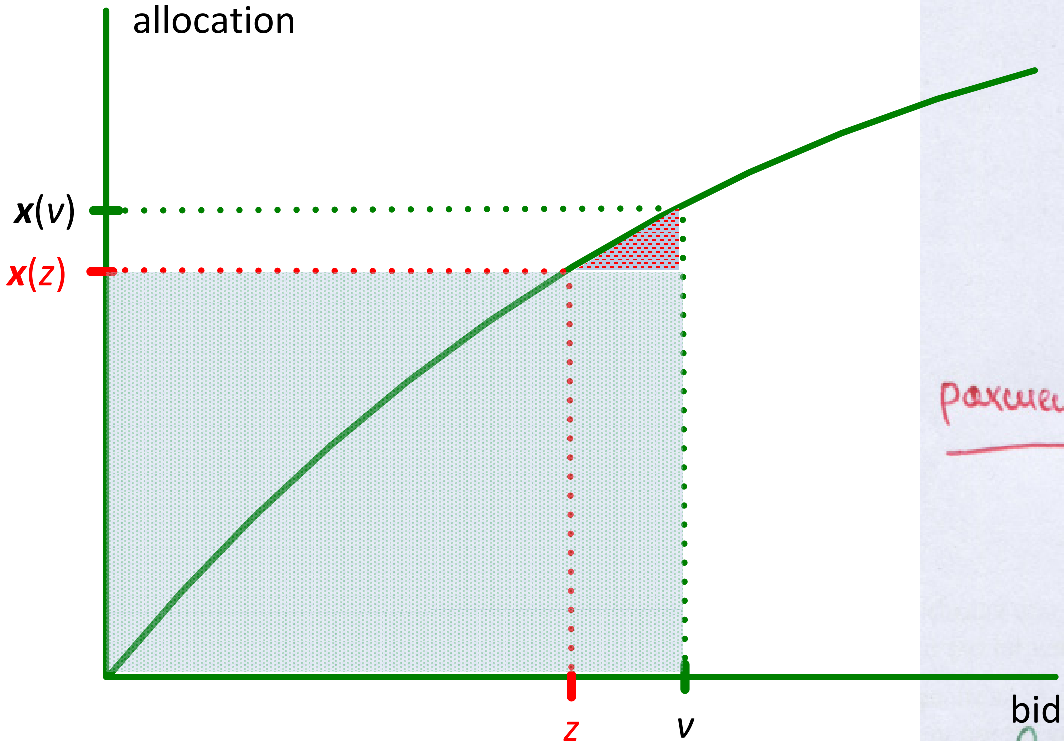

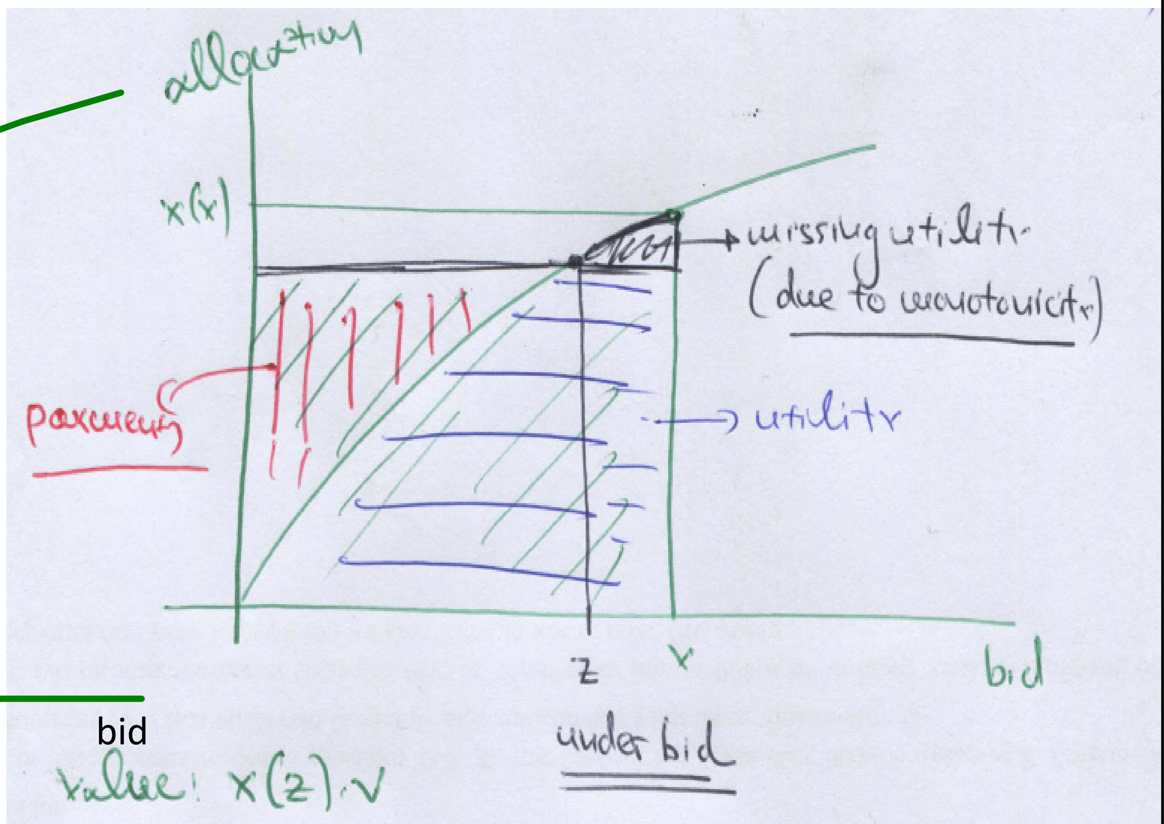

Myerson's lemma

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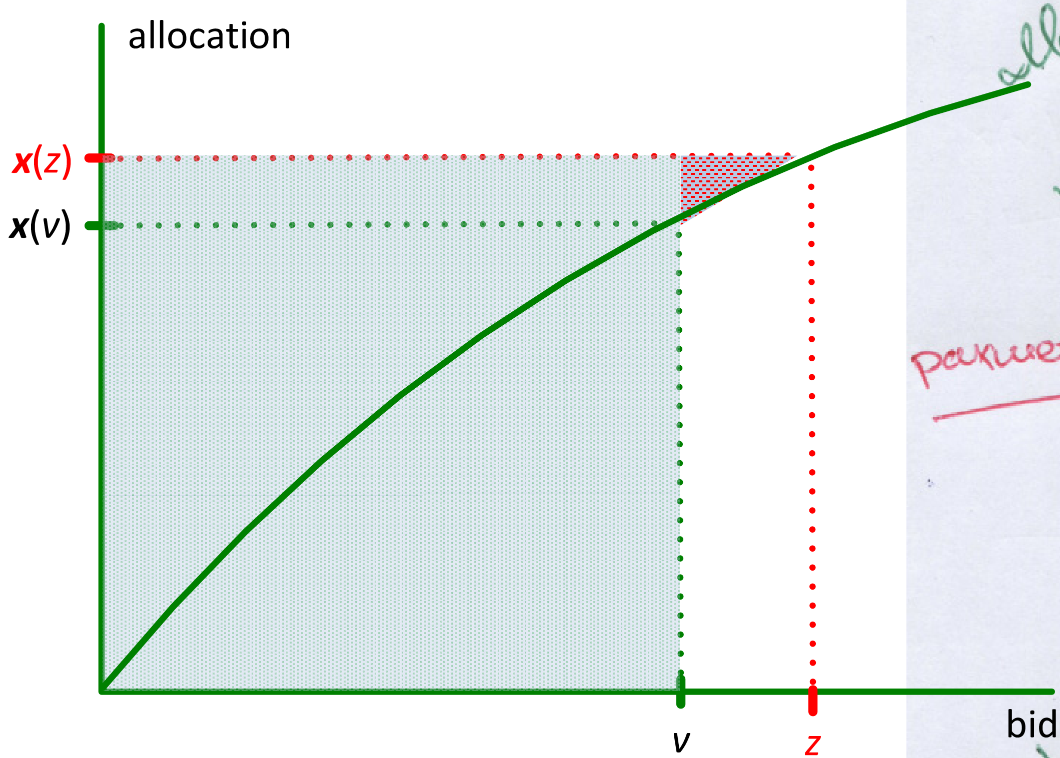

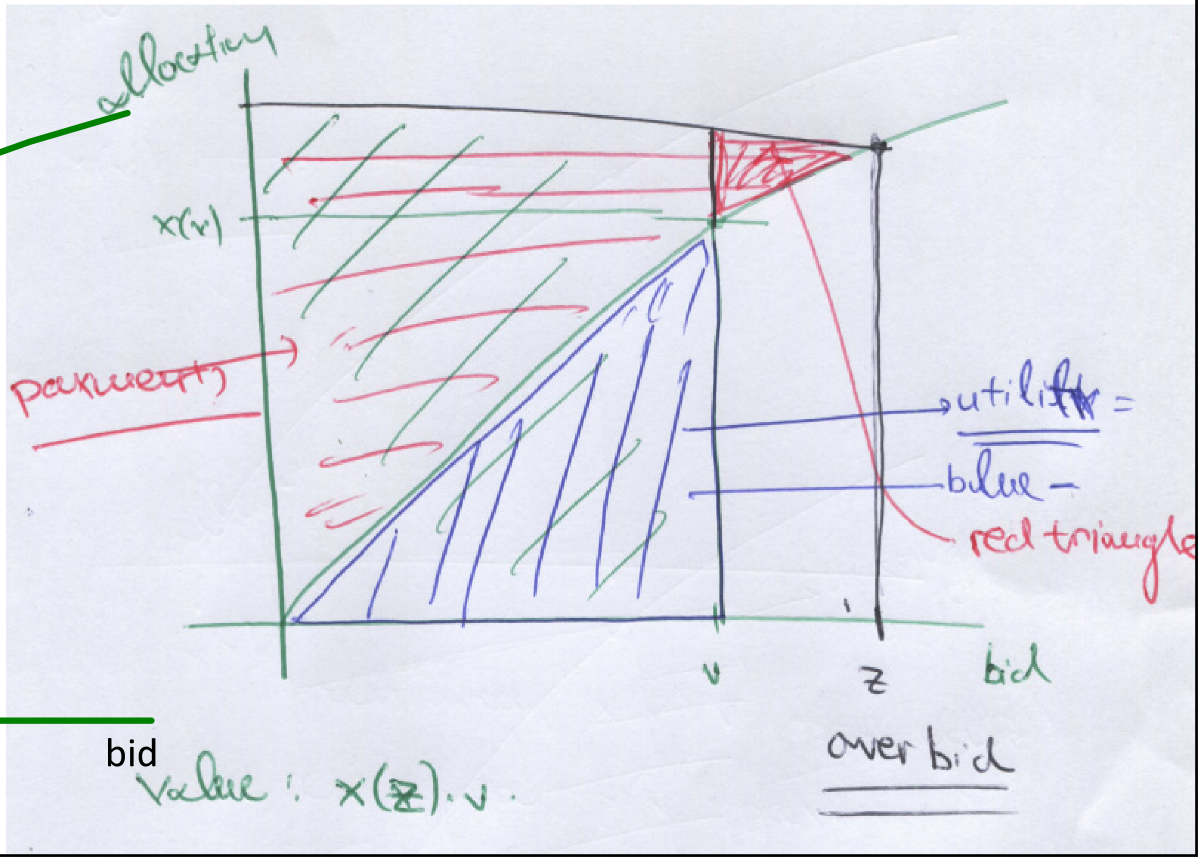

Applying Myerson's lemma

- Single-item auctions
- The allocation rule of giving the item to the highest bidder is monotone
- The payment rule of Myerson gives us precisely the Vickrey auction
  - Non-winners pay nothing: If a bidder i is not a winner, there is no jump within $[0, b_i]$ in the function $x_i(z, \mathbf{b}_{-i})$
  - The winner pays (2^{nd} highest bid) · [jump at 2^{nd} highest bid] = 2^{nd} highest bid

- Corollary: The Vickrey auction is the only truthful mechanism for single-item auctions, when the winner is the highest bidder