

Algorithmic Game Theory

Mechanisms for Revenue Maximization

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Designing mechanisms for maximizing revenue for single parameter environments

Social Welfare vs Revenue

- Many reasons for focusing on **social welfare**:
 - In government auctions, revenue may not be the first priority
 - Also in competitive markets, greedily maximizing revenue may cause customers leave towards other sellers
- **Strong positive results** for social welfare maximization.
 - If we do not care about **computational efficiency**, we can **always** have a truthful mechanism that maximizes welfare (VCG)
 - We often have good approximations in polynomial time.
- Question: Similar results for **revenue maximization**?

Social Welfare vs Revenue

An illuminating example:

- Consider **1 item** and only **1 bidder** with private **value v**
- Only truthful mechanisms: set a price r **independent of the declared bid**
 - Any other pricing rule that depends on the bid is not **truthful**
 - These are called **posted price** mechanisms.
- If $v \geq r$, the bidder will buy the item, $sw = v$ and revenue = r
- If not, $sw = \text{revenue} = 0$

Social Welfare vs Revenue

- How do we **maximize social welfare** in this setting?
 - Easy, **just set $r = 0$**
 - All we care about for social welfare is that the bidder gets the item
 - We do **not need to know** the exact value of v
 - With more bidders, we also do not need to know the exact values to maximize welfare, **only who is the highest bidder**
 - Social welfare is quite special and relatively simple.

Social Welfare vs Revenue

- How do we **maximize revenue**?
 - **Optimal** revenue we can extract: **equal to v**
 - If we knew v , we would just set $r := v$
 - But v is private information!
 - Optimal revenue **really depends on** the exact form of the **valuation** function
 - E.g., if we just set $r = 100$, then the mechanism does well only for bidders with $v \geq 100$ (and not too large!).
 - For $v < 100$, it performs terribly

A Model for Revenue Maximization

Conclusions and modeling approach:

- Not easy to compare mechanisms
- We need to consider a **different model**
- Usual approach: Average case or **Bayesian analysis**

A Bayesian Model for Revenue Maximization

For **single-parameter** environments:

- Each bidder i has a value v_i which is private information
- For each bidder i , the **value v_i** is drawn from a **probability distribution F_i** on some interval $[0, v_{\max}]$, with $v_{\max} \neq +\infty$
 - $F_i(z) = \Pr[v_i \leq z]$
- The distributions F_1, F_2, \dots, F_n are all **independent**
- Mechanism **knows the distributions** (but not the values)
 - Typically derived from historical data
- Objective: design an auction to maximize **expected revenue**

Goal: Characterize truthful mechanisms that **maximize expected revenue.**

A Bayesian Model for Revenue Maximization

Back to **single item** and **single bidder**

- Value v of the bidder drawn from distribution F
- Suppose we **post a price r**
- Expected revenue = **$r \cdot \Pr[v \geq r] = r \cdot (1 - F(r))$**
- It reduces to optimizing posted price r
 - Optimal price r is called **monopoly price** of F .
- E.g., if F is uniform in $[0, 1]$, then $F(z) = z$
- Optimal mechanism: post **$r = 1/2$** with **expected revenue $1/4$**

A Bayesian Model for Revenue Maximization

Single-item auction with two bidders?

- This already gets more complex
- Can we start with something simple first?

2nd price auction with a **reserve price**

- Fix a reserve price r
- **Allocation rule:** If no bidder exceeds r , nobody gets the item. Otherwise, winner is the highest bidder
- **Payment rule:** $\max \{\text{reserve price}, 2^{\text{nd}} \text{ highest bid}\}$

A Bayesian Model for Revenue Maximization

Single-item auction with two bidders:

- Reserve prices are used in practice to boost revenue
- **Main advantage:** much better revenue for the cases where 2nd highest bid is low
- **Main disadvantage:** in some cases nobody wins (no revenue)
 - Hopefully latter happens with small probability
- Is the **optimal mechanism** very far from such a format?

Expected Revenue for Single-Parameter Bidders

- We focus on **single-parameter bidders**, monotone allocations and Myerson's truthful payments.
 - Due to truthfulness, **bids = true values**.

- Maximize $\mathbb{E}_{v_1 \sim F_1, \dots, v_n \sim F_n} \left[\sum_{i=1}^n p_i(\mathbf{v}) \right] = \sum_{i=1}^n \mathbb{E}_{\mathbf{v}_{-i}} \left[\mathbb{E}_{v_i} [p_i(v_i, \mathbf{v}_{-i})] \right]$

- Due to independence, we focus on **single bidder i**.

- We recall (dfn of expectation and Myerson's payments):

$$\mathbb{E}_{v_i} [p_i(v_i, \mathbf{v}_{-i})] = \int_0^{v_{\max}} p_i(v_i, \mathbf{v}_{-i}) f(v_i) dv_i \quad p_i(v_i, \mathbf{v}_{-i}) = \int_0^{v_i} z \cdot \mathbf{x}'_i(z, \mathbf{v}_{-i}) dz$$

- Therefore:

$$\mathbb{E}_{v_i} [p_i(v_i, \mathbf{v}_{-i})] = \int_0^{v_{\max}} p_i(v_i, \mathbf{v}_{-i}) f(v_i) dv_i = \int_0^{v_{\max}} \left[\int_0^{v_i} z \cdot \mathbf{x}'_i(z, \mathbf{v}_{-i}) dz \right] f(v_i) dv_i$$

Expected Revenue for Single-Parameter Bidders

- Reversing the order of integration:

$$\begin{aligned} \int_0^{v_{\max}} \left[\int_0^{v_i} z \cdot x'_i(z, \mathbf{v}_{-i}) dz \right] f_i(v_i) dv_i &= \int_0^{v_{\max}} \left[\int_z^{v_{\max}} f_i(v_i) dv_i \right] z \cdot x'_i(z, \mathbf{v}_{-i}) dz \\ &= \int_0^{v_{\max}} (1 - F_i(z)) \cdot z \cdot x'_i(z, \mathbf{v}_{-i}) dz. \end{aligned}$$

- Integration by parts and simplification:

$$\begin{aligned} &\int_0^{v_{\max}} \underbrace{(1 - F_i(z)) \cdot z}_f \cdot \underbrace{x'_i(z, \mathbf{v}_{-i})}_{g'} dz \\ &= \underbrace{(1 - F_i(z)) \cdot z \cdot x_i(z, \mathbf{v}_{-i}) \Big|_0^{v_{\max}}}_{=0-0} - \int_0^{v_{\max}} x_i(z, \mathbf{v}_{-i}) \cdot (1 - F_i(z) - z f_i(z)) dz \\ &= \int_0^{v_{\max}} \underbrace{\left(z - \frac{1 - F_i(z)}{f_i(z)} \right)}_{:=\varphi_i(z)} x_i(z, \mathbf{v}_{-i}) f_i(z) dz = \mathbb{E}_{v_i} \left[\varphi(v_i) \cdot \mathbf{x}_i(\mathbf{v}) \right] \end{aligned}$$

Virtual Valuations

We transform valuations to **virtual valuations**, that include information about valuation distribution.

Definition: For an agent i , with

- actual value v_i ,
- distribution F_i ,
- probability density function f_i ,

the virtual valuation at v_i is:

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \quad \leftarrow \text{“information rent” for agent } i$$

Optimal revenue extracted from i

Monopoly price of F where virtual valuation is 0:

$$\left(r(1 - F(r)) \right)' = 0 \Leftrightarrow r - \frac{1 - F(r)}{f(r)} = 0 \Leftrightarrow \varphi(r) = 0$$

Virtual Valuations

Example: **uniform** distribution on $[0, 1]$ for player i :

- distribution function: $F_i(z) = z$
- density function: $f_i(z) = 1$
- virtual valuation: $\phi_i(v_i) = v_i - (1-v_i)/1 = 2v_i - 1$

Observations:

- Virtual valuations can also take **negative values**, even though $v_i \geq 0$
- For any distribution, $\phi_i(v_i) \leq v_i$

Summary: $\mathbb{E}_{\mathbf{v} \sim F} [p_i(\mathbf{v})] = \mathbb{E}_{\mathbf{v} \sim F} [\varphi(v_i) \cdot \mathbf{x}_i(\mathbf{v})]$

Expected Revenue Equals Expected Virtual Welfare

Main result for revenue maximization:

Consider a **single-parameter** domain with valuation distributions F_1, F_2, \dots, F_n and let $F = F_1 \times F_2 \times \dots \times F_n$ be the product distribution.

For every truthful mechanism (\mathbf{x}, \mathbf{p})

$$E_{\mathbf{v} \sim F} [\sum_i p_i(\mathbf{v})] = E_{\mathbf{v} \sim F} [\sum_i \phi_i(v_i) \cdot x_i(\mathbf{v})]$$

Expected revenue

Expected virtual welfare

Surprisingly, finding the **revenue-optimal mechanism** reduces to **maximizing the expected virtual welfare**

Maximizing Virtual Welfare

- Although we care about payments, we reduced the problem to designing an appropriate allocation rule!
- How do we **maximize expected virtual welfare**?
 - Forget about the expectation and **maximize pointwise**.
 - For each profile $\mathbf{v} = (v_1, v_2, \dots, v_n)$, maximize $\sum_i \phi_i(v_i) \cdot x_i(\mathbf{v})$
 - This is simply a **welfare maximization problem**
 - With $\phi_i(v_i)$ playing the role of v_i
 - We apply Myerson's Lemma, but for the **virtual values**.
 - Allocation rule must be **monotone** (wrt bids / valuations v_i), as required for truthfulness.
- Whenever we can solve welfare maximization efficiently, we can also do it for the virtual welfare.

Maximizing Virtual Welfare

For the **single-item** auction:

- Give the item to bidder with the highest virtual value.
- Actually, not always...
- Recall: a virtual value can take **negative values**
- Give it to bidder with the highest **positive** virtual value
- Sometimes, the item is not allocated to anyone.
- **Example:** Let F_i be the **uniform distribution** on $[0, 1]$
 - $\phi_i(v_i) = 2v_i - 1$
 - **Allocation rule:** give it to the highest bidder whose bid **exceeds $1/2$** (reserve price), if such bidder exists

Monotonicity of Virtual Welfare Maximization

- Is the **allocation rule** that maximizes the virtual welfare **monotone** (wrt. bids)?
 - If yes, then we are done by Myerson's lemma
 - Unfortunately this depends on the distributions

Definition: A distribution is called **regular** if the corresponding **virtual valuation** function is **non-decreasing**

- **Examples:** the uniform distribution and many other common distributions satisfy this
- **Non-regular distributions:** multi-modal distributions or with heavy tails

Monotonicity of Virtual Welfare Maximization

Observation: If we have regular distributions for all bidders, then the virtual welfare maximizing rule is monotone

Optimal mechanism for revenue maximization

Assumptions: **Independent** and **regular** distributions

- Collect the bids and transform each b_i into its corresponding **virtual bid** $\phi_i(b_i)$
- Choose an allocation (x_1, x_2, \dots, x_n) that maximizes the virtual welfare $\sum_i \phi_i(b_i) \cdot x_i$
- Charge each bidder according to Myerson's payment formula

Expected Revenue Maximization

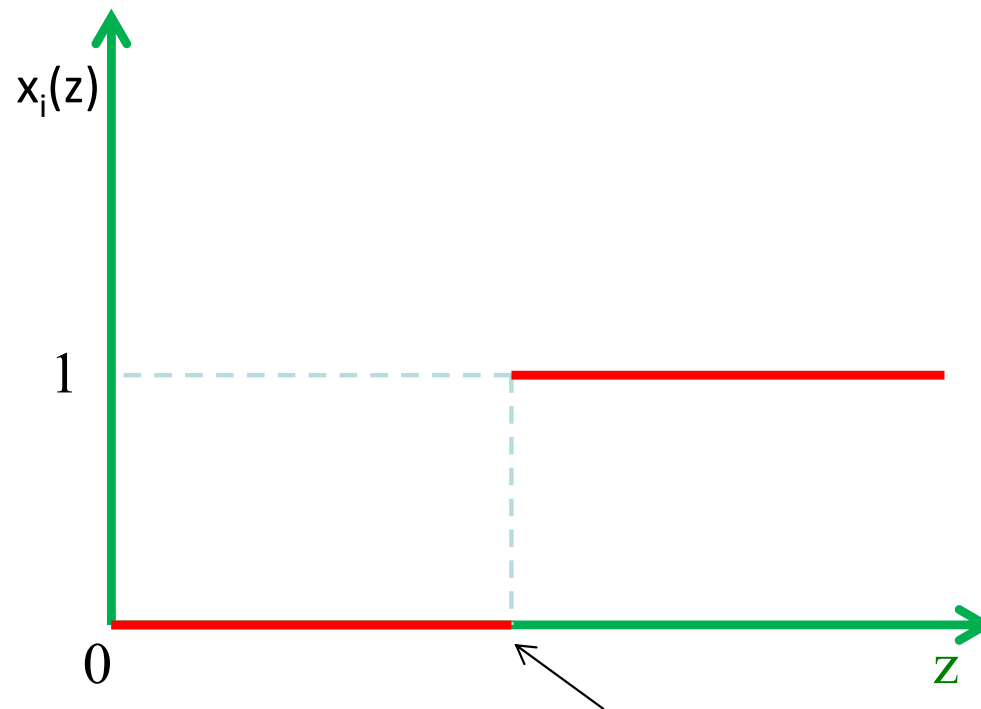
Let's apply this to **single-item** auctions with i.i.d bidders

Implementing the revenue-optimal mechanism

- Collect the bids and transform each b_i into its corresponding virtual bid $\phi_i(b_i)$
- **Allocation:** since the virtual valuation function is non-decreasing, for **i.i.d. bidders**, the **highest virtual value** corresponds to the **highest bidder**
 - Thus: we allocate the item to the highest bidder i , as long as $\phi_i(b_i) \geq 0$, otherwise, there is no winner
- **Payment:** need to find the threshold bid, where does the jump in the allocation occur?

Expected Revenue Maximization

- Consider **i.i.d. bidders** with the uniform distribution on $[0, 1]$
- $\phi_i(z) = 2z - 1$ for every bidder i
- Let i be the winner, and fix a profile \mathbf{b}_{-i} for the other bidders
- The jump in the allocation can happen **either at the 2nd highest bid or at $1/2$**



2 cases to consider:

Case 1: at least one other bidder has a positive virtual bid

Case 2: no other bidder has a positive virtual bid

Payment = $\max\{2^{\text{nd}} \text{ highest bid}, \frac{1}{2}\}$

This is a **2nd price auction** with **reserve price = $1/2$**

Either 2nd highest bid
or $1/2$

Expected Revenue Maximization

More generally:

- Consider a single-item auction
- Suppose we have i.i.d. bidders with a regular distribution
- Let ϕ be the common virtual valuation function

Optimal mechanism: 2nd price auction with reserve = $\phi^{-1}(0)$

- i.e., the eBay format is optimal (with appropriate opening bid)
- Surprising that the optimal mechanism has such a simple format

Single-Item Auctions with Non I.I.D. Bidders

- Things become complicated when bidders are not i.i.d.
- For example, suppose bidders' valuations are **drawn independently** but from **different regular** distributions
- The revenue-optimal auction **does not resemble any format used in practice**
- It is also not easy to interpret as a **natural rule** to follow and does not have a practical appeal
- Current research: Identify simple auction rules for which we can prove they are near-optimal in terms of expected revenue
 - Based again on virtual valuations and on using **prophet inequalities** for estimating the derived revenue

Prophet Inequality and Simple Single-Item Auctions

- Let F_1, \dots, F_n be independent distributions, let X_1, \dots, X_n be realizations from F_1, \dots, F_n , and let $X^* = \max_i \{ X_i \}$.
 - Let $t : \text{Prob}[X^* \geq t] = 1/2$ (or simply $t = E[X^*]/2$)
 - Then, accepting an arbitrary $X_i \geq t$ (if any) guarantees an expected reward of $\geq E[X^*]/2$.
- Choose t s.t $\mathbb{P}_v \left[\max_i \varphi(v_i)^+ \geq t \right] = 1/2$ (or $t = \mathbb{E}_v \left[\max_i \varphi(v_i)^+ \right] / 2$)
 - Threshold t can be computed (or estimated), given F_1, \dots, F_n
- Give the item to arbitrary bidder i with $\phi_i(v_i) \geq t$, if any, at (i 's reserve) price r_i defined as $\phi_i(r_i) = t$.
 - If many candidate winners, any monotone selection works. E.g., highest bidder.
 - Also applies if bidders arrive online and offers are take-it-or-leave-it.

Prophet Inequality and Simple Single-Item Auctions

- Choose t s.t. $\mathbb{P}_v \left[\max_i \varphi(v_i)^+ \geq t \right] = 1/2$ (or $t = \mathbb{E}_v \left[\max_i \varphi(v_i)^+ \right] / 2$)
 - Threshold t can be computed (or estimated), given F_1, \dots, F_n
- Give the item to **arbitrary bidder i** with $\phi_i(v_i) \geq t$, if any, at (i's reserve) **price r_i** defined as $\phi_i(r_i) = t$.
 - If many candidate winners, choose **the highest bidder**.
- Prophet inequality implies **$\geq 50\%$ of optimal revenue!**
 - Simple, virtual valuations determine reserves, not the winner.
 - However, reserves are still player-dependent.
- **Open Problem:** how much of optimal revenue we can recover with **anonymous reserve prices**, if bidders are independent but **not identically** distributed.

Prior-Independent Auctions

- Design auctions that extract significant fraction of optimal revenue **without** resorting to **knowledge** of valuation distributions F_1, \dots, F_n
 - Distributions are used in the analysis of the auction, not in its design.
- Expected revenue of **Vickrey auction** with **$n+1$ i.i.d. bidders** from any regular distribution $F \geq$ expected revenue of optimal auction (Vickrey auction with **optimal reserve price** derived with **knowledge of F**) with **n i.i.d. bidders** from F .

Theorem 4.1 (Bulow-Klemperer Theorem [1]) *Let F be a regular distribution and n a positive integer. Then:*

$$\mathbf{E}_{v_1, \dots, v_{n+1} \sim F}[\text{Rev}(VA) (n + 1 \text{ bidders})] \geq \mathbf{E}_{v_1, \dots, v_n \sim F}[\text{Rev}(OPT_F) (n \text{ bidders})], \quad (6)$$

where VA and OPT_F denote the Vickrey auction and the optimal auction for F , respectively.³

Multi-Parameter Revenue Maximization

- A **much harder** problem!
- Recall Myerson's lemma does not hold any more for more complex valuations
- Not easy to characterize truthful mechanisms when the valuation functions depend on multiple private parameters of the bidders
- Very active research field even for auctions with a small number of items