Locality and Winning Games

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Outline

1. Definitions and Notations
2. Locality of FO
3. Winning Games and Locality of FO Revisited

Gaifman graph

Gaifman graph

Given a $\sigma-$structure $\mathfrak{A}$, its *Gaifman graph* $G(\mathfrak{A})$ is defined as:

- $V(G(\mathfrak{A})) = A$ (the universe of $\mathfrak{A}$)
- $(x, y) \in E(G(\mathfrak{A}))$ iff
  - $x = y$
  - $\exists$ relation $R \in \sigma$, tuple $t \in R^{\mathfrak{A}}$ such that x, y appear in t

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1. If $\mathfrak{A}$ is an undirected graph then its Gaifman graph $G(\mathfrak{A})$ is simply $\mathfrak{A}$ with self loops

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Note: $\mathfrak{A}$ is always an undirected graph.

Distance in the Gaifman graph

Let $x, y \in V(G(\mathfrak{A}))$. We define the *distance* of x and y in the Gaifman graph as

$$d_{\mathfrak{A}}(x,y) = \begin{cases} \text{the length of the shortest path from } x \text{ to } y & , \exists \text{ path} \\ +\infty & , \nexists \text{ path} \end{cases}$$

The function defined is *non-negative*, *symmetric* and *subadditive*, satisfying all the properties of a *metric function*.

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Let $\overrightarrow{a} = (a_1, \ldots, a_n)$ and $\overrightarrow{b} = (b_1, \ldots, b_m)$ be tuples of elements of $V(G(\mathfrak{A}))$ and $c \in V(G(\mathfrak{A}))$. We define

$$d_{\mathfrak{A}}(\overrightarrow{a}, c) = \min_{1 \leq i \leq n} d_{\mathfrak{A}}(a_i, c) \text{ and } d_{\mathfrak{A}}(\overrightarrow{a}, \overrightarrow{b}) = \min_{1 \leq i \leq n} \min_{1 \leq j \leq m} d_{\mathfrak{A}}(a_i, b_j)$$

Balls and Neighborhoods

$r-$Ball

Let σ contain only relation symbols and let $\mathfrak{A}$ be a $\sigma-$structure and $\overrightarrow{a} = (a_1, \ldots, a_n) \in A^n$. We define the $r-$ball around $\overrightarrow{a}$ as

$$B_r^{\mathfrak{A}}(\overrightarrow{a}) = \{b \in A \mid d_{\mathfrak{A}}(\overrightarrow{a}, b) \leq r\}$$

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r−Neighborhood

The r−neighborhood of $\overrightarrow{a} = (a_1, \ldots, a_n) \in A^n$ is the σ_n−structure $N_r^{\mathfrak{A}}(\overrightarrow{a})$, where:

- the universe is $B_r^{\mathfrak{A}}(\overrightarrow{a})$
- each k−ary relation R is interpreted as $R^{\mathfrak{A}}$ restricted to $B_r^{\mathfrak{A}}(\overrightarrow{a})$; that is $R^{\mathfrak{A}} \cap (B_r^{\mathfrak{A}}(\overrightarrow{a}))^k$
- n additional constants are interpreted as $a_1, \ldots, a_n$

The $\leftrightarrows_d$ relation

Let $\mathfrak{A}, \mathfrak{B}$ be $\sigma-$structures, where σ only contains relation symbols.
Let $\overrightarrow{a} \in A^n$ and $\overrightarrow{b} \in B^n$. We write $(\mathfrak{A}, \overrightarrow{a}) \leftrightarrows_d (\mathfrak{B}, \overrightarrow{b})$
if there exists a bijection $f : A \rightarrow B$ such that for every $c \in A$

$$N_d^{\mathfrak{A}}(\overrightarrow{a}c) \cong N_d^{\mathfrak{B}}(\overrightarrow{b}f(c))$$

In the case of $n = 0$, we write $\mathfrak{A} \leftrightarrows_d \mathfrak{B}$
if there exists a bijection $f : A \rightarrow B$ such that for every $c \in A$

$$N_d^{\mathfrak{A}}(c) \cong N_d^{\mathfrak{B}}(f(c))$$

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The $\leftrightarrows_d$ relation says, in a sense, that locally two structures look the same, with respect to a certain bijection f; that is, f sends each element c into $f(c)$ that has the same neighborhood.

Hanf-locality

An $m-$ary query Q on $\sigma-$structures is *Hanf-local* if there exists a number $d \geq 0$ such that for every $\mathfrak{A}, \mathfrak{B} \in \text{STRUCT}[\sigma]$, $\overrightarrow{a} \in A^m$, $\overrightarrow{b} \in B^m$

$$(\mathfrak{A}, \overrightarrow{a}) \leftrightarrows_d (\mathfrak{B}, \overrightarrow{b}) \text{ implies } (\overrightarrow{a} \in Q(\mathfrak{A}) \leftrightarrow \overrightarrow{b} \in Q(\mathfrak{B}))$$

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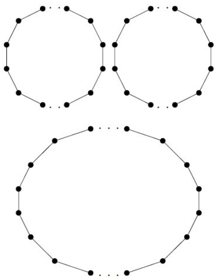
Using Hanf-locality for proving that a query Q is not definable in a logic L then amounts to showing:

- that every L-definable query is Hanf-local
- that Q is not Hanf-local

Example in Hanf-locality



Let's assume that the graph connectivity query Q is Hanf-local and $\text{hlr}(Q) = d$. Let $m > 2d+1$ and choose graphs G^1_m and G^2_m. We have $|V(G^1_m)| = |V(G^2_m)|$. Let $f : V(G^1_m) \to V(G^2_m)$ be a bijection. Since each cycle is of length $> 2d+1$, the $d-$neighborhood of any node a is a chain of length $2d$ with a in the middle. Hence, $G^1_m \leftrightarrows_d G^2_m$ which implies that G^1_m and G^2_m must agree on Q. But G^1_m is disconnected and G^2_m is connected. Thus, graph connectivity is not Hanf-local.

Gaifman-locality

An $m-$ary query Q, $m > 0$, on $\sigma-$structures, is *Gaifman-local* if there exists a number $d \geq 0$ such that for every $\mathfrak{A} \in \text{STRUCT}[\sigma]$ and $\overrightarrow{a_1}, \overrightarrow{a_2} \in A^m$

$$N_d^{\mathfrak{A}}(\overrightarrow{a_1}) \cong N_d^{\mathfrak{A}}(\overrightarrow{a_2}) \text{ implies } (\overrightarrow{a_1} \in Q(\mathfrak{A}) \leftrightarrow \overrightarrow{a_2} \in Q(\mathfrak{A}))$$

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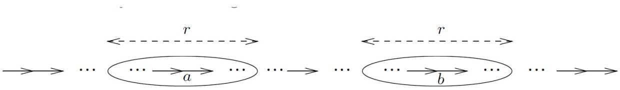
Using Gaifman-locality for proving that a non-Boolean query Q is not definable in a logic L then amounts to showing:

- that every L-definable query is Gaifman-local
- that Q is not Gaifman-local

Example in Gaifman-locality



Let's assume that the transitive closure query Q is Gaifman-local, and let $\text{lr}(Q)$.

If a and b are at a distance $> 2r + 1$ from each other and the start and the endpoints, then the $r-$neighborhoods of (a, b) and (b, a) are isomorphic, since each is a disjoint union of two chains of length $2r$.

Hence, this implies that (a, b) belongs to the output of Q iff (b, a) belongs to the output of Q, which contradicts the assumption that Q defines transitive closure.

Thus, transitive closure is not Gaifman-local.

Hanf-locality vs Gaifman-locality

Most commonly Hanf-locality is used for Boolean queries. Then the definition says that for some $d \geq 0$, for every $\mathfrak{A}, \mathfrak{B} \in \mathsf{STRUCT}[\sigma]$, the condition $\mathfrak{A} \leftrightarrows_d \mathfrak{B}$ implies that $\mathfrak{A}$ and $\mathfrak{B}$ agree on Q.

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While Hanf-locality works well for Boolean queries, Gaifman-locality is often more helpful for non-Boolean queries.

The difference between Hanf-locality and Gaifman-locality is that the former relates two different structures, while the latter is talking about definability in one structure.

Hanf-locality of FO

Theorem 1

Every FO-definable query Q is Hanf-local.

If Q is defined by an FO$[k]$ formula then $\mathrm{hlr}(Q) \leq \dfrac{3^k - 1}{2}$.

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Theorem 1

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If Q is defined by an FO[k] formula then $\mathrm{hlr}(Q) \leq \frac{3^k - 1}{2}$.

We will us the following Lemma, without proof.

Lemma 1

If $(\mathfrak{A}, \overrightarrow{a}) \leftrightarrows_{3d+1} (\mathfrak{B}, \overrightarrow{b})$, then there exists a bijection $f : A \rightarrow B$ such that

$$(\mathfrak{A}, \overrightarrow{a}\, c) \leftrightarrows_{d} (\mathfrak{B}, \overrightarrow{b}\, f(c)), \text{ for all } c \in A$$

Hanf-locality of FO

Proof

- Base case: $k = 0$
  Let Q be a query defined by $\phi \in \mathrm{FO}[0]$ then $(\mathfrak{A}, \overrightarrow{a}) \leftrightarrows_0 (\mathfrak{B}, \overrightarrow{b})$ means that $(\overrightarrow{a}, \overrightarrow{b})$ defines a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, and thus $\overrightarrow{a}$ and $\overrightarrow{b}$ satisfy the same atomic formulas. Hence

$$\mathrm{hlr}(Q) = 0 \leq \frac{3^0 - 1}{2}$$

- Inductive hypothesis:

  Let Q be a query defined by $\phi \in \mathrm{FO}[k]$ then $\mathrm{hlr}(Q) \leq \dfrac{3^k - 1}{2}$

Hanf-locality of FO

- Inductive hypothesis:
  Let Q be a query defined by $\Phi \in \mathrm{FO}[k+1]$, then Φ is the Boolean combination of formulae of the form $\exists z \phi(\overrightarrow{x}, z)$, where $\mathrm{qr}(\phi) \leq k$. Then it suffices to show that for every query Q' defined by a formula of the form $\exists z \phi(\overrightarrow{x}, z)$ then $\mathrm{hlr}(Q')$ is bounded by the same number.
  Let $(\mathfrak{A}, \overrightarrow{a}) \leftrightarrows_{3\mathrm{hlr}(\phi)+1} (\mathfrak{B}, \overrightarrow{b})$ then by "Lemma 1" $\exists$ bijection $f : A \to B$ such that $(\mathfrak{A}, \overrightarrow{a} c) \leftrightarrows_{\mathrm{hlr}(\phi)} (\mathfrak{B}, \overrightarrow{b} f(c))$, for all $c \in A$.

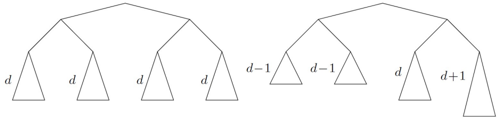
$$\mathfrak{A} \models \exists z \phi(\overrightarrow{a}, z) \iff \mathfrak{A} \models \phi(\overrightarrow{a}, c)$$
$$\iff \mathfrak{B} \models \phi(\overrightarrow{b}, f(c)) \iff \mathfrak{B} \models \exists z \phi(\overrightarrow{b}, z)$$

Hence $\mathrm{hlr}(Q) \leq 3 \cdot \mathrm{hlr}(Q') + 1 = 3 \cdot \dfrac{3^k - 1}{2} + 1 = \dfrac{3^{k+1} - 1}{2}$

Example

Let's assume that query Q tests for being a balanced binary tree and is defined by a formula in FO[k].

Then, "Theorem 1" yields $r = \text{hlr}(Q) \leq \dfrac{3^k - 1}{2}$.
Take d much larger that r and define trees T_1 and T_2 as shown.



Both T_1 and T_2 have $2^{d+3} - 1$ nodes and 2^{d+2} leaves and realize the same type of $r-$neighborhoods and hence $T_1 \leftrightarrows_r T_2$. But this contradicts the Hanf-locality of the balanced binary tree test, since T_1 is balanced, and T_2 is not.

Gaifman-locality of FO

Theorem 2

If Q is a Hanf-local non-Boolean query, then Q is Gaifman-local and

$$\mathrm{lr}(Q) \leq 3 \cdot \mathrm{hlr}(Q) + 1$$

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$$\mathrm{lr}(Q) \leq 3 \cdot \mathrm{hlr}(Q) + 1$$

We will use the following Lemma, without proof.

Lemma 2

If $\mathfrak{A} \leftrightarrows_d \mathfrak{B}$ and $N^{\mathfrak{A}}_{3d+1}(\overrightarrow{a}) \cong N^{\mathfrak{B}}_{3d+1}(\overrightarrow{b})$ then $(\mathfrak{A}, \overrightarrow{a}) \leftrightarrows_d (\mathfrak{B}, \overrightarrow{b})$

Gaifman-locality of FO

Proof

Let Q be a non-Boolean query on STRUCT$[\sigma]$ with $\text{hlr}(Q) = d$.
Let $\mathfrak{A}$ be a $\sigma-$structure and let $N^{\mathfrak{A}}_{3d+1}(\overrightarrow{a_1}) \cong N^{\mathfrak{A}}_{3d+1}(\overrightarrow{a_2})$.
Since $\mathfrak{A} \leftrightarrows_d \mathfrak{A}$ (identical function) and $N^{\mathfrak{A}}_{3d+1}(\overrightarrow{a_1}) \cong N^{\mathfrak{A}}_{3d+1}(\overrightarrow{a_2})$ by "Lemma 2" we have that $(\mathfrak{A}, \overrightarrow{a_1}) \leftrightarrows_d (\mathfrak{A}, \overrightarrow{a_2})$.
Since $\text{hlr}(Q) = d$

$$(\mathfrak{A}, \overrightarrow{a_1}) \leftrightarrows_d (\mathfrak{A}, \overrightarrow{a_2}) \text{ implies that } (\overrightarrow{a_1} \in Q(\mathfrak{A}) \iff \overrightarrow{a_2} \in Q(\mathfrak{A}))$$

Hence

$$N^{\mathfrak{A}}_{3d+1}(\overrightarrow{a_1}) \cong N^{\mathfrak{A}}_{3d+1}(\overrightarrow{a_2}) \text{ implies } (\overrightarrow{a_1} \in Q(\mathfrak{A}) \iff \overrightarrow{a_2} \in Q(\mathfrak{A}))$$

Thus

$$\text{lr}(Q) \leq 3 \cdot \text{hlr}(Q) + 1$$

Gaifman-locality of FO

By combining "Theorem 1" and "Theorem 2" we get

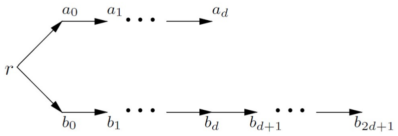
Corollary 1

Every FO-definable non-Boolean query Q is Gaifman-local.

If Q is defined by an FO[k] formula then $\mathrm{lr}(Q) \leq \dfrac{3^{k+1}-1}{2}$.

Example

Given a graph, two nodes *a* and *b* are in the same generation if there is a node *c* (common ancestor) such that the shortest paths from c to a and from c to b have the same length.
Let's assume that query Q tests if two nodes are in the same generation is FO−definable $\text{lr}(Q) = d$.



We have that $N_d^{\mathfrak{A}}(a_d, b_d) \cong N_d^{\mathfrak{A}}(a_d, b_{d+1})$. But this contradicts the Gaifman-locality of the same generation test, since a_d, b_d are in the same generation, and a_d, b_{d+1} are not.

Lower Bound

Suppose that σ is the vocabulary of undirected graphs: that is, $\sigma = \{E\}$ where E is binary. Define the following formulae:

- $d_0(x, y) = E(x, y)$
- $d_1(x, y) = \exists z(d_0(x, z) \wedge d_0(y, z))$
- $d_{k+1}(x, y) = \exists z(d_k(x, z) \wedge d_k(y, z))$

For an undirected graph, $d_k(x, y)$ holds iff there is a path of length 2^k between x and y; that is, if the distance between a and b is at most 2^k. Hence, $\mathrm{lr}(d_k) \geq 2^{k-1}$. However, $\mathrm{qr}(d_k) = k$, which shows that locality rank can be exponential in the quantifier rank.

Bijective Ehrenfeucht-Fraïssé game

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures in a relational vocabulary.
The $k-$round bijective Ehrenfeucht-Fraïssé game game is played by the same two players, the spoiler and the duplicator.

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- If $|A| \neq |B|$, then the duplicator loses before the game even starts.
- In the $i-$th round, the duplicator first selects a bijection $f_i : A \to B$. Then the spoiler plays either $a_i \in A$ or $b_i \in B$. The duplicator responds by either $f_i(ai)$ or $f_i^{-1}(b_i)$.

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The duplicator has a winning strategy after k rounds, if after k moves we have a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.
If the duplicator can win the $k-$round bijective game we write $\mathfrak{A} \equiv_k^{\text{bij}} \mathfrak{B}$ and clearly $\mathfrak{A} \equiv_k^{\text{bij}} \mathfrak{B}$ implies $\mathfrak{A} \equiv_k \mathfrak{B}$

Gaifman Theorem

Let σ be relational. Then every FO formula $\phi(\overrightarrow{x})$ over σ is equivalent to a Boolean combination of the following:

- local formula $\psi^{(r)}(\overrightarrow{x})$
- sentences of the form

$$\exists x_1, \ldots, x_n \left(\bigwedge_{i=1}^{s} a^{(r)}(x_i) \wedge \bigwedge_{1 \leq i < k \leq s} d^{>2r}(x_i, x_j) \right)$$

Furthermore,

- the transformation from ϕ to such a Boolean combination is effective
- if ϕ itself is a sentence, then only sentences of the above form appear in the Boolean combination
- if $\mathrm{qr}(\phi) = k$, and n is the length of $\overrightarrow{a}$, then the bounds on r and s are $r \leq 7^k, s \leq k + n$

Threshold Equivalence

Definition

Given two structures $\mathfrak{A}, \mathfrak{B}$ in a relational vocabulary, we write $\mathfrak{A} \leftrightarrows^{thr}_{d,m} \mathfrak{B}$ if for every isomorphism type τ of a $d-$neighborhood of a point either

- both $\mathfrak{A}$ and $\mathfrak{B}$ have the same number of points that $d-$realize τ, or
- both $\mathfrak{A}$ and $\mathfrak{B}$ have at least m points that $d-$realize τ

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Theorem

For each $k, l > 0$ there exist $d, m > 0$ such that for $\mathfrak{A}, \mathfrak{B} \in \mathrm{STRUCT}_l[\sigma]$,

$$\mathfrak{A} \leftrightarrows^{thr}_{d,m} \mathfrak{B} \text{ implies } \mathfrak{A} \equiv_k \mathfrak{B}$$

That's All Folks!

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