# Θεωρητική Πληροφορική Ι (ΣΗΜΜΥ)

# Υπολογιστική Πολυπλοκότητα

Εργαστήριο Λογικής και Επιστήμης Υπολογισμών
Εθνικό Μετσόβιο Πολυτεχνείο

# Πληροφορίες Μαθήματος

**Θεωρητική Πληροφορική Ι** (ΣΗΜΜΥ)
**Υπολογιστική Πολυπλοκότητα** (ΑΛΜΑ)

- Διδάσκοντες: Σ. Ζάχος, Ά. Παγουρτζής
- Βοηθοί Διδασκαλίας: Α. Αντωνόπουλος, Α. Χαλκή
- Επιμέλεια Διαφανειών: Α. Αντωνόπουλος
- Δευτέρα: 17:00 - 20:00 (1.1.31, Παλιά Κτίρια ΗΜΜΥ, ΕΜΠ)
  Πέμπτη: 15:00 - 17:00 (1.1.31, Παλιά Κτίρια ΗΜΜΥ, ΕΜΠ)
- Ώρες Γραφείου: Μετά από κάθε μάθημα
- Σελίδα: http://courses.corelab.ntua.gr/complexity
- Βαθμολόγηση:

| | |
|---|---|
| Διαγώνισμα: | 6 μονάδες |
| Ασκήσεις: | 2 μονάδες |
| Ομιλία: | 2 μονάδες |
| Quiz: | 1 μονάδα |

# Computational Complexity

## Graduate Course

Antonis Antonopoulos

Computation and Reasoning Laboratory
National Technical University of Athens


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# Bibliography


**Textbooks**

1. C. Papadimitriou, **Computational Complexity**, Addison Wesley, 1994
2. S. Arora, B. Barak, **Computational Complexity: A Modern Approach**, Cambridge University Press, 2009
3. O. Goldreich, **Computational Complexity: A Conceptual Perspective**, Cambridge University Press, 2008

**Lecture Notes**

1. L. Trevisan, **Lecture Notes in Computational Complexity**, 2002, UC Berkeley
2. J. Katz, **Notes on Complexity Theory**, 2011, University of Maryland
3. Jin-Yi Cai, **Lectures in Computational Complexity**, 2003, University of Wisconsin Madison

# Contents

- **Computational Complexity**: Quantifying the amount of computational resources required to solve a given task. *Classify* computational problems according to their inherent difficulty in complexity classes, and prove relations among them.

- **Structural Complexity**: "The study of the relations between various complexity classes and the global properties of individual classes. [...] The goal of structural complexity is a *thorough understanding of the relations between the various complexity classes and the internal structure of these complexity classes*." [J. Hartmanis]

## Decision Problems

- Have answers of the form "*yes*" or "*no*".
- Encoding: each instance $x$ of the problem is represented as a *string* of an alphabet $\Sigma$ ($|\Sigma| \geq 2$).
- Decision problems have the form "Is $x$ in $L$?", where $L$ is a *language*, $L \subseteq \Sigma^*$.

- So, for an encoding of the input, using the alphabet $\Sigma$, we associate the following language with the decision problem $\Pi$:

$L(\Pi) = \{x \in \Sigma^* \mid x \text{ is a representation of a "yes" instance of the problem } \Pi\}$

## Example

- Given a number $x$, is this number prime? ($x \stackrel{?}{\in}$ PRIMES)
- Given graph $G$ and a number $k$, is there a clique with $k$ (or more) nodes in $G$?

## Search Problems

- Have answers of the form of an **object**.
- **Relation** $R(x, y)$ connecting instances $x$ with answers (objects) $y$ we wish to find for $x$.
- Given instance $x$, find a $y$ such that $(x, y) \in R$.

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## Example

FACTORING: Given integer $N$, find its prime decomposition:

$$N = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$$

## Optimization Problems

- For each instance $x$ there is a **set of Feasible Solutions** $F(x)$.
- To each $s \in F(x)$ we map a positive integer $c(x)$, using **the objective function** $c(s)$.
- We search for the solution $s \in F(x)$ which minimizes (or maximizes) the objective function $c(s)$.

## Optimization Problems

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## Example

- The **Traveling Salesperson Problem** (TSP):
  Given a finite set $C = \{c_1, \ldots, c_n\}$ of cities and a distance $d(c_i, c_j) \in \mathbb{Z}^+, \forall (c_i, c_j) \in C^2$, we ask for a permutation $\pi$ of $C$, that minimizes this quantity:

$$\sum_{i=1}^{n-1} d(c_{\pi(i)}, c_{\pi(i+1)}) + d(c_{\pi(n)}, c_{\pi(1)})$$

# A Model Discussion

- There are many computational models (RAM, Turing Machines etc).
- The **Church-Turing Thesis** states that all computation models are equivalent. That is, every computation model can be simulated by a Turing Machine.

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- In Complexity Theory, we consider **efficiently computable** the problems which are solved (aka the languages that are decided) in **polynomial number of steps** (*Edmonds-Cobham Thesis*).

# A Model Discussion

- There are many computational models (RAM, Turing Machines etc).
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- In Complexity Theory, we consider **efficiently computable** the problems which are solved (aka the languages that are decided) in **polynomial number of steps** (*Edmonds-Cobham Thesis*).

**Efficiently Computable $\equiv$ Polynomial-Time Computable**

## Summary

- Computational Complexity classifies problems into classes, and studies the relations and the structure of these classes.
- We have decision problems with boolean answer, or function/optimization problems which output an object as an answer.
- Given some nice properties of polynomials, we identify polynomial-time algorithms as efficient algorithms.

# Contents

## Definition

A Turing Machine $M$ is a quintuple $M = (Q, \Sigma, \delta, q_0, F)$:

- $Q = \{q_0, q_1, q_2, q_3, \ldots, q_n, q_{\text{yes}}, q_{\text{no}}\}$ is a finite set of states.
- $\Sigma$ is the alphabet. The tape alphabet is $\Gamma = \Sigma \cup \{\sqcup\}$.
- $q_0 \in Q$ is the initial state.
- $F \subseteq Q$ is the set of final states.
- $\delta : (Q \setminus F) \times \Gamma \to Q \times \Gamma \times \{S, L, R\}$ is the transition function.

- A TM is a "programming language" with a single data structure (a tape), and a cursor, which moves left and right on the tape.

- Function $\delta$ is the **program** of the machine.

# Turing Machines and Languages

**Definition**

Let $L \subseteq \Sigma^*$ be a language and $M$ a TM such that, for every string $x \in \Sigma^*$:

- If $x \in L$, then $M(x) =$ "yes"
- If $x \notin L$, then $M(x) =$ "no"

Then we say that ***M* decides *L***.

- Alternatively, we say that $M(x) = L(x)$, where $L(x) = \chi_L(x)$ is the *characteristic function* of $L$ (if we consider 1 as "yes" and 0 as "no").
- If $L$ is decided by some TM $M$, then $L$ is called a **recursive language**.

### Definition

If for a language $L$ there is a TM $M$, which if $x \in L$ then $M(x) =$ "yes", and if $x \notin L$ then $M(x) \uparrow$, we call $L$ **recursively enumerable**.

*By $M(x) \uparrow$ we mean that $M$ does not halt on input $x$ (it runs forever).

### Theorem

*If $L$ is recursive, then it is recursively enumerable.*

**Proof**: *Exercise*

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**Proof**: *Exercise*

### Definition

If $f$ is a function, $f : \Sigma^* \to \Sigma^*$, we say that a TM $M$ computes $f$ if, for any string $x \in \Sigma^*$, $M(x) = f(x)$. If such $M$ exists, $f$ is called a **recursive function**.

- Turing Machines can be thought as algorithms for solving string related problems.

# Multitape Turing Machines

- We can extend the previous Turing Machine definition to obtain a Turing Machine with multiple tapes:

**Definition**

A k-tape Turing Machine $M$ is a quintuple $M = (Q, \Sigma, \delta, q_0, F)$:

- $Q = \{q_0, q_1, q_2, q_3, \ldots, q_n, q_{\text{halt}}, q_{\text{yes}}, q_{\text{no}}\}$ is a finite set of states.
- $\Sigma$ is the alphabet. The tape alphabet is $\Gamma = \Sigma \cup \{\sqcup\}$.
- $q_0 \in Q$ is the initial state.
- $F \subseteq Q$ is the set of final states.
- $\delta : (Q \setminus F) \times \Gamma^k \to Q \times (\Gamma \times \{S, L, R\})^k$ is the transition function.

# Bounds on Turing Machines

- We will characterize the "performance" of a Turing Machine by the amount of *time* and *space* required on instances of size $n$, when these amounts are expressed as a function of $n$.

**Definition**

Let $T : \mathbb{N} \to \mathbb{N}$. We say that machine $M$ operates within time $T(n)$ if, for any input string $x$, the time required by $M$ to reach a final state is at most $T(|x|)$. Function $T$ is a **time bound** for $M$.

**Definition**

Let $S : \mathbb{N} \to \mathbb{N}$. We say that machine $M$ operates within space $S(n)$ if, for any input string $x$, $M$ visits at most $S(|x|)$ locations on its work tapes (excluding the input tape) during its computation. Function $S$ is a **space bound** for $M$.

# Multitape Turing Machines

**Theorem**

*Given any k-tape Turing Machine $M$ operating within time $T(n)$, we can construct a TM $M'$ operating within time $\mathcal{O}\left(T^2(n)\right)$ such that, for any input $x \in \Sigma^*$, $M(x) = M'(x)$.*

**Proof**: See Th.2.1 (p.30) in [1].

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**Proof**: See Th.2.1 (p.30) in [1].

This is a strong evidence of the robustness of our model:
*Adding a bounded number of strings does not increase their computational capabilities, and affects their efficiency only polynomially.*

# Linear Speedup

**Theorem**

*Let $M$ be a TM that decides $L \subseteq \Sigma^*$, that operates within time $T(n)$. Then, for every $\varepsilon > 0$, there is a TM $M'$ which decides the same language and operates within time $T'(n) = \varepsilon T(n) + n + 2$.*

**Proof**: See Th.2.2 (p.32) in [1].

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**Proof**: See Th.2.2 (p.32) in [1].

- If, for example, $T$ is linear, i.e. something like $cn$, then this theorem states that the constant $c$ can be made arbitrarily close to 1. So, it is fair to start using the $\mathcal{O}(\cdot)$ notation in our time bounds.
- A similar theorem holds for space:

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**Theorem**

*Let $M$ be a TM that decides $L \subseteq \Sigma^*$, that operates within space $S(n)$. Then, for every $\varepsilon > 0$, there is a TM $M'$ which decides the same language and operates within space $S'(n) = \varepsilon S(n) + 2$.*

# Nondeterministic Turing Machines

- We will now introduce an **unrealistic** model of computation:

**Definition**

A Turing Machine $M$ is a quintuple $M = (Q, \Sigma, \delta, q_0, F)$:

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# Nondeterministic Turing Machines

- In this model, an input is accepted if **there is** *some sequence* of nondeterministic choices that results in “yes”.
- An input is rejected if there is *no sequence* of choices that lead to acceptance.
- Observe the similarity with recursively enumerable languages.

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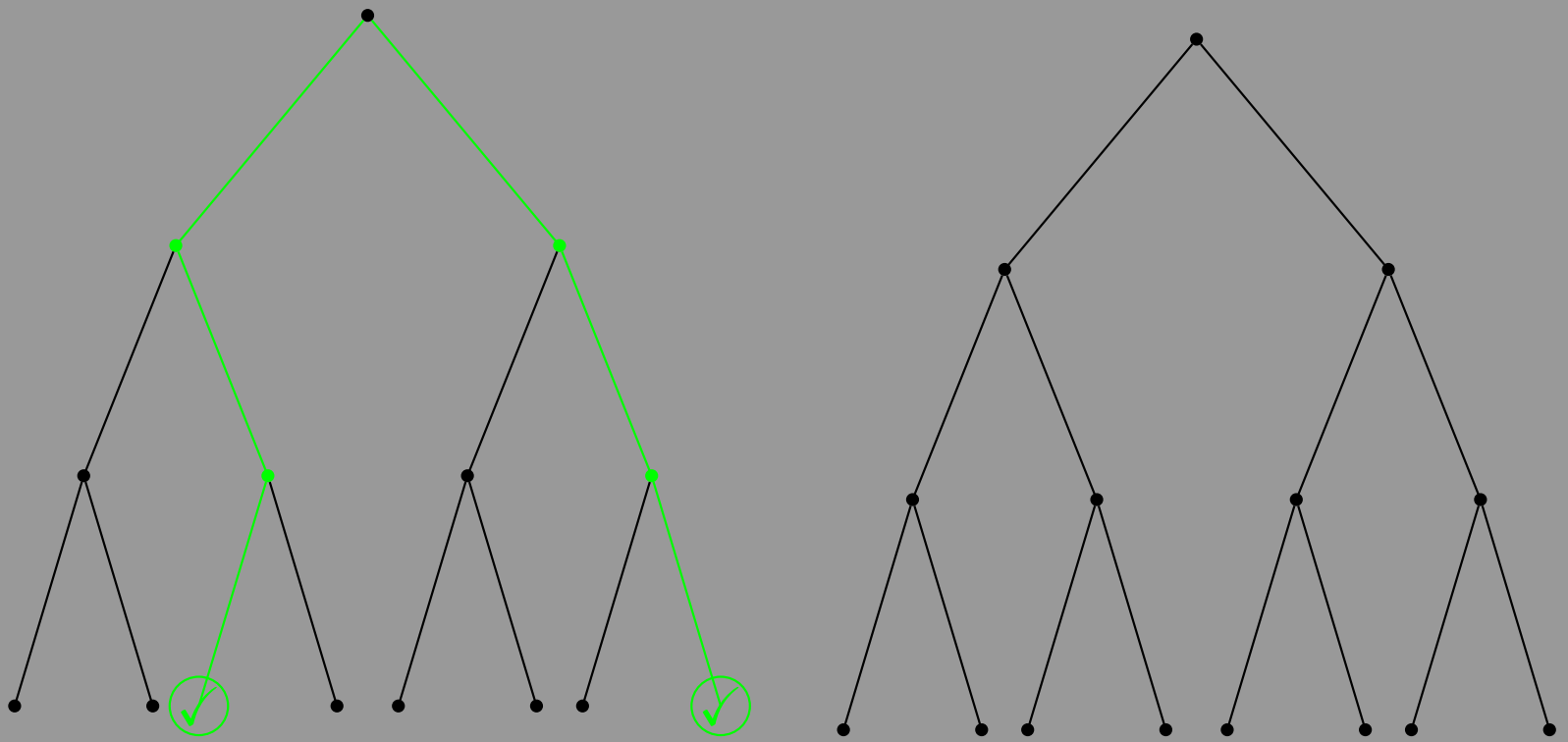
**Definition**

We say that $M$ operates within bound $T(n)$, if for every input $x \in \Sigma^*$ and every sequence of nondeterministic choices, $M$ reaches a final state within $T(|x|)$ steps.

- The above definition requires that $M$ does not have computation paths longer than $T(n)$, where $n = |x|$ the length of the input.
- The amount of time charged is the *depth* of the **computation tree**.

# Examples of Nondeterministic Computations

## Example



Accepting computation

Rejecting Computation

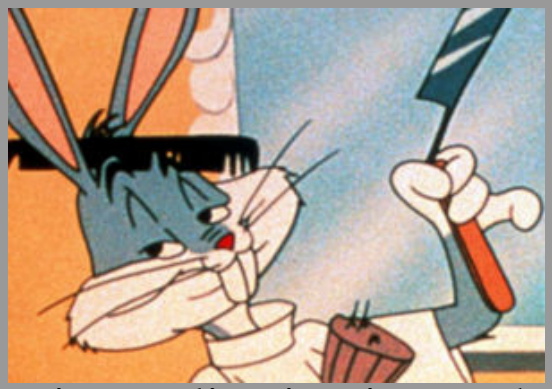
- Without loss of generality, the computation trees are binary, full and complete. (*why*?)

## Summary

- A recursive language is decided by a TM.
- A recursive enumerable language is accepted by a TM that halts only if $x \in L$.
- Multiple tape TMs can be simulated by a one-tape TM with quadratic overhead.
- Linear speedup justifies the $\mathcal{O}(\cdot)$ notation.
- Nondeterministic TMs move in "parallel universes", making different choices simultaneously.
- A Deterministic TM computation is a *path*.
- A Nondeterministic TM computation is a *tree*, i.e. exponentially many paths ran simultaneously.

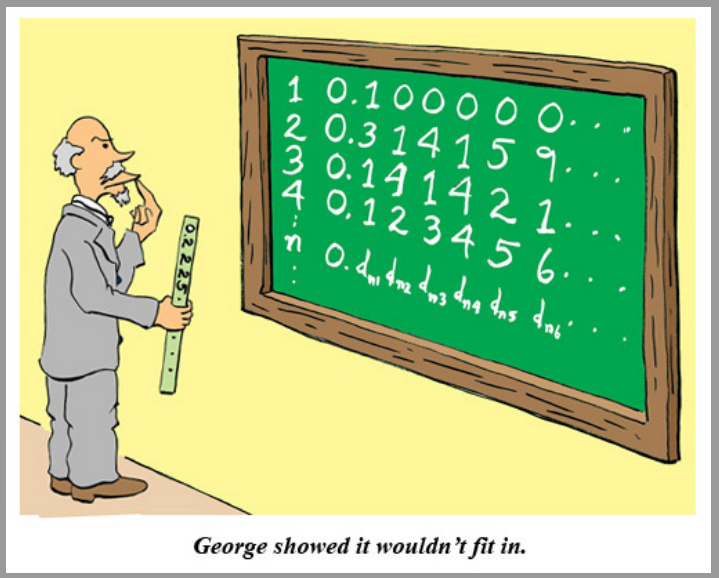
# Contents

# Diagonalization



*Suppose there is a town with just one barber, who is male. In this town, the barber shaves all those, and only those, men in town who do not shave themselves. Who shaves the barber?*

Diagonalization is a technique that was used in many different cases:




*George showed it wouldn't fit in.*

# Diagonalization

**Theorem**

*The functions from $\mathbb{N}$ to $\mathbb{N}$ are uncountable.*

**Proof**: Let, for the sake of contradiction that are countable: $\phi_1, \phi_2, \ldots$. Consider the following function: $f(x) = \phi_x(x) + 1$. This function must appear somewhere in this enumeration, so let $\phi_y = f(x)$. Then $\phi_y(x) = \phi_x(x) + 1$, and if we choose $y$ as an argument, then $\phi_y(y) = \phi_y(y) + 1$. □

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- Using the same argument:

**Theorem**

*The functions from $\{0,1\}^*$ to $\{0,1\}$ are uncountable.*

# Machines as strings

- It is obvious that we can represent a Turing Machine as a string: *just write down the description and encode it using an alphabet, e.g.* $\{0, 1\}$.
- We denote by $\llcorner M \lrcorner$ the TM $M$'s representation as a string.
- Also, if $x \in \Sigma^*$, we denote by $M_x$ the TM that $x$ represents.

**Keep in mind that:**

- **Every string represents *some* TM.**
- **Every TM is represented by *infinitely many* strings.**

- There exists (*at least*) an uncomputable function from $\{0, 1\}^*$ to $\{0, 1\}$, since the set of all TMs is countable.

# The Universal Turing Machine

- So far, our computational models are specified to solve a single problem.
- Turing observed that there is a TM that can simulate any other TM $M$, given $M$'s description as input.

**Theorem**

*There exists a TM $\mathcal{U}$ such that for every $x, w \in \Sigma^*$, $\mathcal{U}(x, w) = M_w(x)$. Also, if $M_w$ halts within $T$ steps on input $x$, then $\mathcal{U}(x, w)$ halts within $CT \log T$ steps, where $C$ is a constant independent of $x$, and depending only on $M_w$'s alphabet size number of tapes and number of states.*

**Proof**: See section 3.1 in [1], and Th. 1.9 and section 1.7 in [2].

# The Halting Problem

- Consider the following problem: "*Given the description of a TM M, and a string x, will M halt on input x?* " This is called the HALTING PROBLEM.
- **We want to compute this problem ! ! !** (Given a computer program and an input, will this program enter an infinite loop?)
- In language form: $\mathrm{H} = \{\llcorner M \lrcorner; x \mid M(x) \downarrow\}$, where " $\downarrow$ " means that the machine halts, and " $\uparrow$ " that it runs forever.

**Theorem**

H *is recursively enumerable.*

**Proof**: See Th.3.1 (p.59) in [1]

- In fact, H is not just a recursively enumerable language:
  If we had an algorithm for deciding H, then we would be able to derive an algorithm for deciding any r.e. language (**RE**-complete).

# The Halting Problem

- But....

**Theorem**

H *is not recursive.*

**Proof**: See Th.3.1 (p.60) in [1]

- Suppose, for the sake of contradiction, that there is a TM $M_H$ that decides H.
- Consider the TM $D$:

$$D(\llcorner M \lrcorner) : \textbf{ if } M_H(\llcorner M \lrcorner; \llcorner M \lrcorner) = \text{“yes”} \textbf{ then } \uparrow \textbf{ else } \text{“yes”}$$

- What is $D(\llcorner D \lrcorner)$?

# The Halting Problem

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- What is $D(\llcorner D \lrcorner)$?
- If $D(\llcorner D \lrcorner) \uparrow$, then $M_H$ accepts the input, so $\llcorner D \lrcorner; \llcorner D \lrcorner \in$ H, so $D(D) \downarrow$.
- If $D(\llcorner D \lrcorner) \downarrow$, then $M_H$ rejects $\llcorner D \lrcorner; \llcorner D \lrcorner$, so $\llcorner D \lrcorner; \llcorner D \lrcorner \notin$ H, so $D(D) \uparrow$.

$\square$

- Recursive languages are a *proper* subset of recursive enumerable ones.
- Recall that the complement of a language $L$ is defined as:

$$\overline{L} = \{x \in \Sigma^* \mid x \notin L\} = \Sigma^* \setminus L$$

Theorem

1. *If $L$ is recursive, so is $\overline{L}$.*
2. *$L$ is recursive if and only if $L$ and $\overline{L}$ are recursively enumerable.*

**Proof**: Exercise

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- Recall that the complement of a language $L$ is defined as:

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**Proof**: Exercise

- Let $E(M) = \{x \mid (q_0, \triangleright, \varepsilon) \overset{M*}{\to} (q, y \sqcup x \sqcup, \varepsilon\}$
- $E(M)$ is the language *enumerated* by $M$.

Theorem

*$L$ is recursively enumerable iff there is a TM $M$ such that $L = E(M)$.*

# More Undecidability

- The HALTING PROBLEM, our first undecidable problem, was the first, but not the only undecidable problem. Its spawns a wide range of such problems, via *reductions*.
- To show that a problem $A$ is undecidable we establish that, if there is an algorithm for $A$, then there would be an algorithm for H, which is absurd.

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- To show that a problem $A$ is undecidable we establish that, if there is an algorithm for $A$, then there would be an algorithm for H, which is absurd.

**Theorem**

*The following languages are not recursive:*

1. $\{\llcorner M \lrcorner \mid M \text{ halts on all inputs}\}$
2. $\{\llcorner M \lrcorner; x \mid \text{There is a } y \text{ such that } M(x) = y\}$
3. $\{\llcorner M \lrcorner; x \mid \text{The computation of } M \text{ uses all states of } M\}$
4. $\{\llcorner M \lrcorner; x; y \mid M(x) = y\}$

# Rice's Theorem

- The previous problems lead us to a more general conclusion:

**Any non-trivial property of Turing Machines is undecidable**

- If a TM $M$ accepts a language $L$, we write $L = L(M)$.

Theorem (Rice's Theorem)

*Suppose that $\mathcal{C}$ is a proper, non-empty subset of the set of all recursively enumerable languages. Then, the following problem is undecidable:*

*Given a Turing Machine $M$, is $L(M) \in \mathcal{C}$?*

# Rice's Theorem

**Proof**:

See Th.3.2 (p.62) in [1]

- We can assume that $\emptyset \notin \mathcal{C}$ (*why?*).
- Since $\mathcal{C}$ is nonempty, $\exists\, L \in \mathcal{C}$, accepted by the TM $M_L$.
- Let $M_H$ the TM accepting the HALTING PROBLEM for an arbitrary input $x$. For each $x \in \Sigma^*$, we construct a TM $M$ as follows:

  $M(y)$ : **if** $M_H(x) =$ "yes" **then** $M_L(y)$ **else** $\uparrow$

- We claim that: $L(M) \in \mathcal{C}$ if and only if $x \in$ H.

# Rice's Theorem

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- We can assume that $\emptyset \notin \mathcal{C}$ (*why?*).
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  $M(y)$ : **if** $M_H(x) =$ "yes" **then** $M_L(y)$ **else** $\uparrow$

- We claim that: $L(M) \in \mathcal{C}$ if and only if $x \in$ H.
  **Proof of the claim**:
  - If $x \in$ H, then $M_H(x) =$ "yes", and so $M$ will accept $y$ or never halt, depending on whether $y \in L$. Then the language accepted by $M$ is exactly $L$, which is in $\mathcal{C}$.
  - If $M_H(x) \uparrow$, $M$ never halts, and thus $M$ accepts the language $\emptyset$, which is not in $\mathcal{C}$. □

## Summary

- TMs are encoded by strings.
- The Universal TM $\mathcal{U}(x, \llcorner M \lrcorner)$ can simulate any other TM $M$ along with an input $x$.
- The Halting Problem is recursively enumerable, but not recursive.
- Many other problems can be proved undecidable, by a *reduction* from the Halting Problem.
- Rice's theorem states that *any non-trivial property of TMs is an undecidable problem*.

# Contents

# Parameters used to define complexity classes:

- Model of Computation (Turing Machine, RAM, Circuits)
- Mode of Computation (Deterministic, Nondeterministic, Probabilistic)
- Complexity Measures (*Time, Space, Circuit Size-Depth*)
- Other Parameters (Randomization, Interaction)

# Our first complexity classes

**Definition**

Let $L \subseteq \Sigma^*$, and $T, S : \mathbb{N} \to \mathbb{N}$:

- We say that $L \in \mathbf{DTIME}[T(n)]$ if there exists a TM $M$ deciding $L$, which operates within the *time* bound $\mathcal{O}(T(n))$, where $n = |x|$.
- We say that $L \in \mathbf{DSPACE}[S(n)]$ if there exists a TM $M$ deciding $L$, which operates within *space* bound $\mathcal{O}(S(n))$, that is, for any input $x$, requires space at most $S(|x|)$.
- We say that $L \in \mathbf{NTIME}[T(n)]$ if there exists a *nondeterministic* TM $M$ deciding $L$, which operates within the time bound $\mathcal{O}(T(n))$.
- We say that $L \in \mathbf{NSPACE}[S(n)]$ if there exists a *nondeterministic* TM $M$ deciding $L$, which operates within space bound $\mathcal{O}(S(n))$.

# Our first complexity classes

- The above are **Complexity Classes**, in the sense that they are sets of languages.
- All these classes are parameterized by a function $T$ or $S$, so they are *families* of classes (for each function we obtain a complexity class).

# Our first complexity classes

- The above are **Complexity Classes**, in the sense that they are sets of languages.
- All these classes are parameterized by a function $T$ or $S$, so they are *families* of classes (for each function we obtain a complexity class).

**Definition (Complementary complexity class)**

For any complexity class $\mathcal{C}$, $co\mathcal{C}$ denotes the class: $\{\overline{L} \mid L \in \mathcal{C}\}$, where $\overline{L} = \Sigma^* \setminus L = \{x \in \Sigma^* \mid x \notin L\}$.

- We want to define "reasonable" complexity classes, in the sense that we want to "compute more problems", given more computational resources.

# Constructible Functions

- Can we use all computable functions to define Complexity Classes?

# Constructible Functions

- Can we use all computable functions to define Complexity Classes?

**Theorem (Gap Theorem)**

*For any computable functions r and a, there exists a computable function f such that* $f(n) \geq a(n)$, *and*

$$\mathbf{DTIME}[f(n)] = \mathbf{DTIME}[r(f(n))]$$

- That means, for $r(n) = 2^{2^{f(n)}}$, the incementation from $f(n)$ to $2^{2^{f(n)}}$ does not allow the computation of any new function!
- So, we must use some restricted families of functions:

# Constructible Functions

**Definition (Time-Constructible Function)**

A nondecreasing function $T : \mathbb{N} \to \mathbb{N}$ is **time constructible** if $T(n) \geq n$ and there is a TM $M$ that computes the function $x \mapsto \llcorner T(|x|) \lrcorner$ in time $T(n)$.

**Definition (Space-Constructible Function)**

A nondecreasing function $S : \mathbb{N} \to \mathbb{N}$ is **space-constructible** if $S(n) > \log n$ and there is a TM $M$ that computes $S(|x|)$ using $S(|x|)$ space, given $x$ as input.

# Constructible Functions

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**Definition (Space-Constructible Function)**

A nondecreasing function $S : \mathbb{N} \to \mathbb{N}$ is **space-constructible** if $S(n) > \log n$ and there is a TM $M$ that computes $S(|x|)$ using $S(|x|)$ space, given $x$ as input.

- The restriction $T(n) \geq n$ is to allow the machine to read its input.
- The restriction $S(n) > \log n$ is to allow the machine to "remember" the index of the cell of the input tape that it is currently reading.

Complexity Classes

# Constructible Functions

- Also, if $f_1(n), f_2(n)$ are time/space-constructible functions, so are $f_1 + f_2, f_1 \cdot f_2$ and $f_1^{f_2}$.

Complexity Classes

# Constructible Functions

- Also, if $f_1(n), f_2(n)$ are time/space-constructible functions, so are $f_1 + f_2, f_1 \cdot f_2$ and $f_1^{f_2}$.
- If we use only *constructible* functions, we can prove **Hierarchy Theorems**, stating that with more resources we can compute more languages:

# Constructible Functions

- Also, if $f_1(n), f_2(n)$ are time/space-constructible functions, so are $f_1 + f_2, f_1 \cdot f_2$ and $f_1^{f_2}$.
- If we use only *constructible* functions, we can prove **Hierarchy Theorems**, stating that with more resources we can compute more languages:

**Theorem (Hierarchy Theorems)**

*Let $t_1$, $t_2$ be time-constructible functions, and $s_1$, $s_2$ be space-constructible functions. Then:*

1. *If* $t_1(n) \log t_1(n) = o(t_2(n))$, *then* $\mathbf{DTIME}(t_1) \subsetneq \mathbf{DTIME}(t_2)$.
2. *If* $t_1(n+1) = o(t_2(n))$, *then* $\mathbf{NTIME}(t_1) \subsetneq \mathbf{NTIME}(t_2)$.
3. *If* $s_1(n) = o(s_2(n))$, *then* $\mathbf{DSPACE}(s_1) \subsetneq \mathbf{DSPACE}(s_2)$.
4. *If* $s_1(n) = o(s_2(n))$, *then* $\mathbf{NSPACE}(s_1) \subsetneq \mathbf{NSPACE}(s_2)$.

Complexity Classes

# Simplified Case of Deterministic Time Hierarchy Theorem

**Theorem**

$\mathbf{DTIME}[n] \subsetneq \mathbf{DTIME}[n^{1.5}]$

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**Theorem**

$\mathbf{DTIME}[n] \subsetneq \mathbf{DTIME}[n^{1.5}]$

**Proof** (*Diagonalization*): See Th.3.1 (p.69) in [2]

Let $D$ be the following machine:

> On input $x$, run for $|x|^{1.4}$ steps $\mathcal{U}(M_x, x)$;
> If $\mathcal{U}(M_x, x) = b$, then return $1 - b$;
> Else return $0$;

- Clearly, $L = L(D) \in \mathbf{DTIME}[n^{1.5}]$
- We claim that $L \notin \mathbf{DTIME}[n]$:
  Let $L \in \mathbf{DTIME}[n] \Rightarrow \exists\, M :\ M(x) = D(x)\ \forall x \in \Sigma^*$, and $M$ works for $\mathcal{O}(|x|)$ steps.
  The time to simulate $M$ using $\mathcal{U}$ is $c|x| \log |x|$, for some $c$.

Complexity Classes

# Simplified Case of Deterministic Time Hierarchy Theorem

**Proof** (*cont'd*):

$\exists n_0 : \ n^{1.4} > cn \log n \ \forall n \geq n_0$

There exists a $x_M$, s.t. $x_M = \llcorner M \lrcorner$ and $|x_M| > n_0$ (*why?*) Then,

$D(x_M) = 1 - M(x_M)$ (while we have also that $D(x) = M(x), \ \forall x$)

Complexity Classes

# Simplified Case of Deterministic Time Hierarchy Theorem

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**Contradiction!!** $\square$

# Simplified Case of Deterministic Time Hierarchy Theorem

**Proof** (*cont'd*):
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$\mathbf{D(x_M) = 1 - M(x_M)}$ (while we have also that $D(x) = M(x), \ \forall x$)
**Contradiction!!** □

- So, we have the hierachy:

$$\mathbf{DTIME}[n] \subsetneq \mathbf{DTIME}[n^2] \subsetneq \mathbf{DTIME}[n^3] \subsetneq \cdots$$

- We will later see that the class containing the problems we can efficiently solve (recall the Edmonds-Cobham Thesis) is the class $\mathbf{P} = \bigcup_{c \in \mathbb{N}} \mathbf{DTIME}[n^c]$.

- Hierarchy Theorems tell us how classes of the same kind relate to each other, when we vary the complexity bound.
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**Theorem**

*Suppose that $T(n), S(n)$ are time-constructible and space-constructible functions, respectively. Then:*

1. $\mathbf{DTIME}[T(n)] \subseteq \mathbf{NTIME}[T(n)]$
2. $\mathbf{DSPACE}[S(n)] \subseteq \mathbf{NSPACE}[S(n)]$
3. $\mathbf{NTIME}[T(n)] \subseteq \mathbf{DSPACE}[T(n)]$
4. $\mathbf{NSPACE}[S(n)] \subseteq \mathbf{DTIME}[2^{\mathcal{O}(S(n))}]$

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**Corollary**

$$\mathbf{NTIME}[T(n)] \subseteq \bigcup_{c>1} \mathbf{DTIME}[c^{T(n)}]$$

**Proof**:

See Th.7.4 (p.147) in [1]

1. Trivial
2. Trivial
3. We can simulate the machine for each nondeterministic choice, using at most $T(n)$ steps in each simulation.
   There are *exponentially* many simulations, but we can simulate them one-by-one, *reusing the same space*.
4. Recall the notion of a configuration of a TM: For a $k$-tape machine, is a $2k-2$ tuple: $(q, i, w_2, u_2, \ldots, w_{k-1}, u_{k-1})$
   How many configurations are there?
   - $|Q|$ choices for the state
   - $n+1$ choices for $i$, and
   - Fewer than $|\Sigma|^{(2k-2)S(n)}$ for the remaining strings

   So, the total number of configurations on input size $n$ is at most $nc_1^{S(n)} = 2^{\mathcal{O}(S(n))}$.

**Proof** (*cont'd*):

Definition (Configuration Graph of a TM)

The configuration graph of $M$ on input $x$, denoted $G(M,x)$, has as **vertices** all the possible configurations, and there is an **edge** between two vertices $C$ and $C'$ if and only if $C'$ can be reached from $C$ in one step, according to $M$'s transition function.

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- So, we have reduced this simulation to REACHABILITY* problem (also known as S-T CONN), for which we know there is a poly-time ($\mathcal{O}\left(n^2\right)$) algorithm.
- So, the simulation takes $\left(2^{\mathcal{O}(S(n))}\right)^2 \sim 2^{\mathcal{O}(S(n))}$ steps. □

*REACHABILITY: Given a graph $G$ and two nodes $v_1, v_n \in V$, is there a path from $v_1$ to $v_n$?

# The essential Complexity Hierarchy

## Definition

$$\mathbf{L} = \mathbf{DSPACE}[\log n]$$

$$\mathbf{NL} = \mathbf{NSPACE}[\log n]$$

$$\mathbf{P} = \bigcup_{c \in \mathbb{N}} \mathbf{DTIME}[n^c]$$

$$\mathbf{NP} = \bigcup_{c \in \mathbb{N}} \mathbf{NTIME}[n^c]$$

$$\mathbf{PSPACE} = \bigcup_{c \in \mathbb{N}} \mathbf{DSPACE}[n^c]$$

$$\mathbf{NPSPACE} = \bigcup_{c \in \mathbb{N}} \mathbf{NSPACE}[n^c]$$

# The essential Complexity Hierarchy

Definition

$$\mathbf{EXP} = \bigcup_{c \in \mathbb{N}} \mathbf{DTIME}[2^{n^c}]$$

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$$\mathbf{EXPSPACE} = \bigcup_{c \in \mathbb{N}} \mathbf{DSPACE}[2^{n^c}]$$

$$\mathbf{NEXPSPACE} = \bigcup_{c \in \mathbb{N}} \mathbf{NSPACE}[2^{n^c}]$$

# The essential Complexity Hierarchy

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$$\mathbf{L} \subseteq \mathbf{NL} \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE} \subseteq \mathbf{NPSPACE} \subseteq \mathbf{EXP} \subseteq \mathbf{NEXP}$$

# Certificate Characterization of NP

**Definition**

Let $R \subseteq \Sigma^* \times \Sigma^*$ a binary relation on strings.

- $R$ is called **polynomially decidable** if there is a DTM deciding the language $\{x; y \mid (x, y) \in R\}$ in polynomial time.
- $R$ is called **polynomially balanced** if $(x, y) \in R$ implies $|y| \leq |x|^k$, for some $k \geq 1$.

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- $R$ is called **polynomially decidable** if there is a DTM deciding the language $\{x; y \mid (x, y) \in R\}$ in polynomial time.
- $R$ is called **polynomially balanced** if $(x, y) \in R$ implies $|y| \leq |x|^k$, for some $k \geq 1$.

## Theorem

*Let $L \subseteq \Sigma^*$ be a language. $L \in$ **NP** if and only if there is a polynomially decidable and polynomially balanced relation R, such that:*

$$L = \{x \mid \exists y \, R(x, y)\}$$

- This $y$ is called **succinct certificate**, or **witness**.
- So, an **NP** **Search Problem** is the problem of *computing* witnesses.

**Proof**: See Pr.9.1 (p.181) in [1]

($\Leftarrow$) If such an $R$ exists, we can construct the following NTM deciding $L$:

"On input $x$, *guess* a $y$, such that $|y| \leq |x|^k$, and then test (in poly-time) if $(x, y) \in R$. If so, accept, else reject." Observe that an accepting computation exists if and only if $x \in L$.

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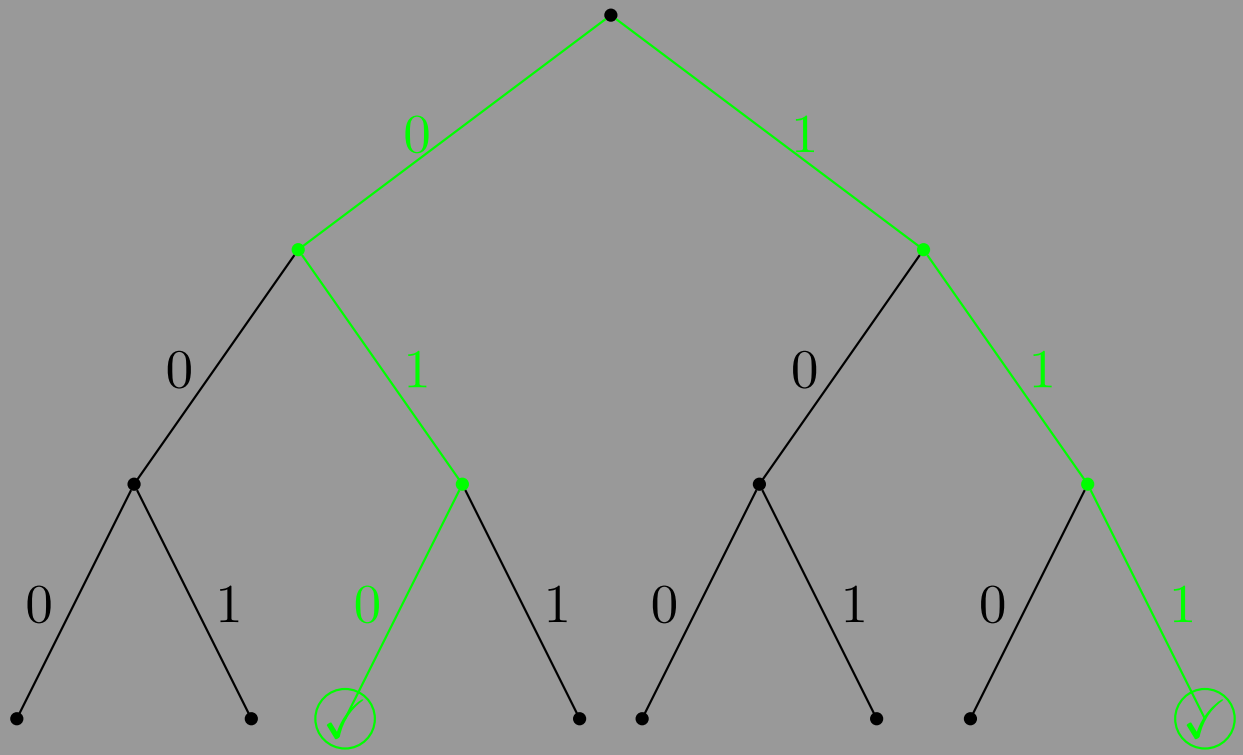
($\Rightarrow$) If $L \in \mathbf{NP}$, then $\exists$ an NTM $N$ that decides $L$ in time $|x|^k$, for some $k$. Define the following $R$:
"$(x, y) \in R$ if and only if $y$ is an **encoding** of an accepting computation of $N(x)$."
$R$ is polynomially balanced and decidable (*why?*), so, given by assumption that $N$ decides $L$, we have our conclusion. $\square$

# Certificate Characterization of NP

## Example (Encoding of a computation path)




- 010 and 111 encode accepting paths.

# Can creativity be automated?

As we saw:

- Class **P**: Efficient **Computation**
- Class **NP**: Efficient **Verification**
- So, if we can efficiently verify a mathematical proof, can we create it efficiently?

# Can creativity be automated?

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- Class **P**: Efficient **Computation**
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If $P = NP$...

- For every mathematical statement, and given a page limit, we would (quickly) generate a proof, if one exists.
- Given detailed constraints on an engineering task, we would (quickly) generate a design which meets the given criteria, if one exists.
- Given data on some phenomenon and modeling restrictions, we would (quickly) generate a theory to explain the data, if one exists.

See "A. Wigderson: Knowledge, Creativity and **P** versus **NP**"

# Complementary complexity classes

- Deterministic complexity classes are in general closed under complement ($co\mathbf{L} = \mathbf{L}$, $co\mathbf{P} = \mathbf{P}$, $co\mathbf{PSPACE} = \mathbf{PSPACE}$).
- Complementaries of non-deterministic complexity classes are very interesting:
- The class $co\mathbf{NP}$ contains all the languages that have **succinct disqualifications** (the analogue of *succinct certificate* for the class **NP**). The "no" instance of a problem in $co\mathbf{NP}$ has a short proof of its being a "no" instance.
- So:

$$\mathbf{P} \subseteq \mathbf{NP} \cap co\mathbf{NP}$$

- Note the *similarity* and the *difference* with $\mathbf{R} = \mathbf{RE} \cap co\mathbf{RE}$.

# Quantifier Characterization of Complexity Classes

## Definition

We denote as $\mathcal{C} = (Q_1/Q_2)$, where $Q_1, Q_2 \in \{\exists, \forall\}$, the class $\mathcal{C}$ of languages $L$ satisfying:

- $x \in L \Rightarrow Q_1 y \, R(x, y)$
- $x \notin L \Rightarrow Q_2 y \, \neg R(x, y)$

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- $x \in L \Rightarrow Q_1 y \, R(x, y)$
- $x \notin L \Rightarrow Q_2 y \, \neg R(x, y)$

- $\mathbf{P} = (\forall/\forall)$
- $\mathbf{NP} = (\exists/\forall)$
- $co\mathbf{NP} = (\forall/\exists)$

# Savitch's Theorem

- REACHABILITY $\in$ **NL.**

See Ex.2.10 (p.48) in [1]

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**Theorem (Savitch's Theorem)**

REACHABILITY $\in$ **DSPACE**$[\log^2 n]$

# Savitch's Theorem

- REACHABILITY $\in$ **NL**. See Ex.2.10 (p.48) in [1]

**Theorem (Savitch's Theorem)**

REACHABILITY $\in$ **DSPACE**$[\log^2 n]$

**Proof**: See Th.7.4 (p.149) in [1]

$REACH(x, y, i)$ : "*There is a path from x to y, of length* $\leq i$".

- We can solve REACHABILITY if we can compute $REACH(x, y, n)$, for any nodes $x, y \in V$, since any path in $G$ can be at most $n$ long.
- If $i = 1$, we can check whether $REACH(x, y, i)$.
- If $i > 1$, we use recursion:

**Proof** (*cont'd*):

```
def REACH(s,t,k)
    if k==1:
        if (s==t or (s,t) in edges): return true
    if k>1:
        for u in vertices:
            if (REACH(s,u, floor(k/2)) and
            (REACH(u,t,ceil(k/2))): return true
    return false
```

**Proof** (*cont'd*):

```
def REACH(s,t,k)
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    return false
```

- We generate all nodes $u$ one after the other, *reusing* space.
- The algorithm has recursion depth of $\lceil \log n \rceil$.
- For each recursion level, we have to store $s, t, k$ and $u$, that is, $\mathcal{O}(\log n)$ space.
- Thus, the total space used is $\mathcal{O}\left(\log^2 n\right)$. □

# Savitch's Theorem

## Corollary

**NSPACE**$[S(n)] \subseteq$ **DSPACE**$[S^2(n)]$, for any space-constructible function $S(n) \geq \log n$.

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**Proof**:

- Let $M$ be the nondeterministic TM to be simulated.
- We run the algorithm of Savitch's Theorem proof on the configuration graph of $M$ on input $x$.
- Since the configuration graph has $c^{S(n)}$ nodes, $\mathcal{O}\left(S^2(n)\right)$ space suffices. $\square$

# Savitch's Theorem

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**NSPACE**$[S(n)] \subseteq$ **DSPACE**$[S^2(n)]$, for any space-constructible function $S(n) \geq \log n$.

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**Corollary**

$$\textbf{PSPACE} = \textbf{NPSPACE}$$

# NL-Completeness

- In Complexity Theory, we "connect" problems in a complexity class with partial ordering relations, called **reductions**, which formalize the notion of "*a problem that is at least as hard as another*".
- A reduction must be computationally weaker than the class in which we use it.

# NL-Completeness

- In Complexity Theory, we "connect" problems in a complexity class with partial ordering relations, called **reductions**, which formalize the notion of "*a problem that is at least as hard as another*".
- A reduction must be computationally weaker than the class in which we use it.

## Definition

A language $L_1$ is **logspace reducible** to a language $L_2$, denoted $L_1 \leq_m^{\ell} L_2$, if there is a function $f : \Sigma^* \to \Sigma^*$, computable by a DTM in $\mathcal{O}(\log n)$ space, such that for all $x \in \Sigma^*$:

$$x \in L_1 \Leftrightarrow f(x) \in L_2$$

We say that a language $A$ is **NL**-complete if it is in **NL** and for every $B \in \mathbf{NL}$, $B \leq_m^{\ell} A$.

# NL-Completeness

**Theorem**

*REACHABILITY is* **NL***-complete.*

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*REACHABILITY is* **NL***-complete.*

**Proof**:

See Th.4.18 (p.89) in [2]

- We 've argued why REACHABILITY $\in$ **NL**.
- Let $L \in$ **NL**, that is, it is decided by a $\mathcal{O}(\log n)$ NTM $N$.
- Given input $x$, we can construct the *configuration graph* of $N(x)$.
- We can assume that this graph has a *single* accepting node.
- We can construct this in logspace: Given configurations $C, C'$ we can in space $\mathcal{O}(|C| + |C'|) = \mathcal{O}(\log |x|)$ check the graph's adjacency matrix if they are connected by an edge.
- It is clear that $x \in L$ if and only if the produced instance of REACHABILITY has a "yes" answer. $\square$

# Certificate Definition of NL

- We want to give a characterization of **NL**, similar to the one we gave for **NP**.
- A certificate may be polynomially long, so a logspace machine may not have the space to store it.
- So, we will assume that the certificate is provided to the machine on a separate tape that is **read once**.

# Certificate Definition of NL

**Definition**

A language $L$ is in **NL** if there exists a deterministic TM $M$ with an additional special read-once input tape, such that for every $x \in \Sigma^*$:

$$x \in L \Leftrightarrow \exists y, |y| \in poly(|x|), M(x, y) = 1$$

where by $M(x, y)$ we denote the output of $M$ where $x$ is placed on its input tape, and $y$ is placed on its special read-once tape, and $M$ uses at most $\mathcal{O}(\log |x|)$ space on its read-write tapes for every input $x$.

- What if remove the read-once restriction and allow the TM's head to move back and forth on the certificate, and read each bit multiple times?

# Immerman-Szelepscényi

**Theorem (The Immerman-Szelepscényi Theorem)**

$\overline{\text{REACHABILITY}} \in \mathbf{NL}$

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**Theorem (The Immerman-Szelepscényi Theorem)**

$\overline{\text{REACHABILITY}} \in \mathbf{NL}$

**Proof**: See Th.4.20 (p.91) in [2]

- It suffices to show a $\mathcal{O}(\log n)$ verification algorithm $A$ such that:
  $\forall\ (G, s, t)$, $\exists$ a polynomial certificate $u$ such that:
  $A((G, s, t), u) =$ "yes" iff $t$ is not reachable from $s$.
- $A$ has read-once access to $u$.
- $G$'s vertices are identified by numbers in $\{1, \ldots, n\} = [n]$
- $C_i$: "*The set of vertices reachable from s in* $\leq i$ *steps.*"
- Membership in $C_i$ is easily certified:
- $\forall i \in [n]$: $v_0, \ldots, v_k$ along the path from $s$ to $v$, $k \leq i$.
- The certificate is at most polynomial in $n$.

# The Immerman-Szelepscényi Theorem

**Proof** (*cont'd*):

- We can check the certificate using read-once access:
  1. $v_0 = s$
  2. for $j > 0$, $(v_{j-1}, v_j) \in E(G)$
  3. $v_k = v$
  4. Path ends within at most $i$ steps
- We now construct two types of certificates:
  1. A certificate that a vertex $v \notin C_i$, given $|C_i|$.
  2. A certificate that $|C_i| = c$, for some $c$, given $|C_{i-1}|$.
- Since $C_0 = \{s\}$, we can provide the 2nd certificate to convince the verifier for the sizes of $C_1, \ldots, C_n$
- $C_n$ is the set of vertices *reachable* from $s$.

# The Immerman-Szelepscényi Theorem

**Proof** (*cont'd*):

- Since the verifier has been convinced of $|C_n|$, we can use the 1st type of certificate to convince the verifier that $t \notin C_n$.

# The Immerman-Szelepscényi Theorem

**Proof** (*cont'd*):

- Since the verifier has been convinced of $|C_n|$, we can use the 1st type of certificate to convince the verifier that $t \notin C_n$.

- **Certifying that $v \notin C_i$, given $|C_i|$**
  The certificate is the list of certificates that $u \in C_i$, for every $u \in C_i$.
  The verifier will check:
  1. Each certificate is valid
  2. Vertex $u$, given a certificate for $u$, is larger than the previous.
  3. No certificate is provided for $v$.
  4. The total number of certificates is exactly $|C_i|$.

# The Immerman-Szelepscényi Theorem

**Proof** (*cont'd*):

**Certifying that $v \notin C_i$, given $|C_{i-1}|$**

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The verifier will check:

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**Proof** (*cont'd*):

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**Certifying that $|C_i| = c$, given $|C_{i-1}|$**

The certificate will consist of $n$ certificates, for vertices 1 to $n$, in ascending order.

The verifier will check all certificates, and count the vertices that have been certified to be in $C_i$. If $|C_i| = c$, it accepts. □

# The Immerman-Szelepscényi Theorem

**Corollary**

For every space constructible $S(n) > \log n$:

$$\mathbf{NSPACE}[S(n)] = co\mathbf{NSPACE}[S(n)]$$

**Proof**:

- Let $L \in \mathbf{NSPACE}[S(n)]$. We will show that $\exists$ $S(n)$ space-bounded NTM $\overline{M}$ deciding $\overline{L}$:
- $\overline{M}$ on input $x$ uses the above certification procedure on the *configuration graph* of $M$. $\square$

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**Corollary**

$$\mathbf{NL} = co\mathbf{NL}$$

# What about Undirected Reachability?

- UNDIRECTED REACHABILITY captures the phenomenon of configuration graphs with both directions.
- H. Lewis and C. Papadimitriou defined the class **SL** (**S**ymmetric **L**ogspace) as the class of languages decided by a **Symmetric Turing Machine** using logarithmic space.
- Obviously,

$$\boxed{\mathbf{L} \subseteq \mathbf{SL} \subseteq \mathbf{NL}}$$

- As in the case of **NL**, UNDIRECTED REACHABILITY is **SL**-complete.
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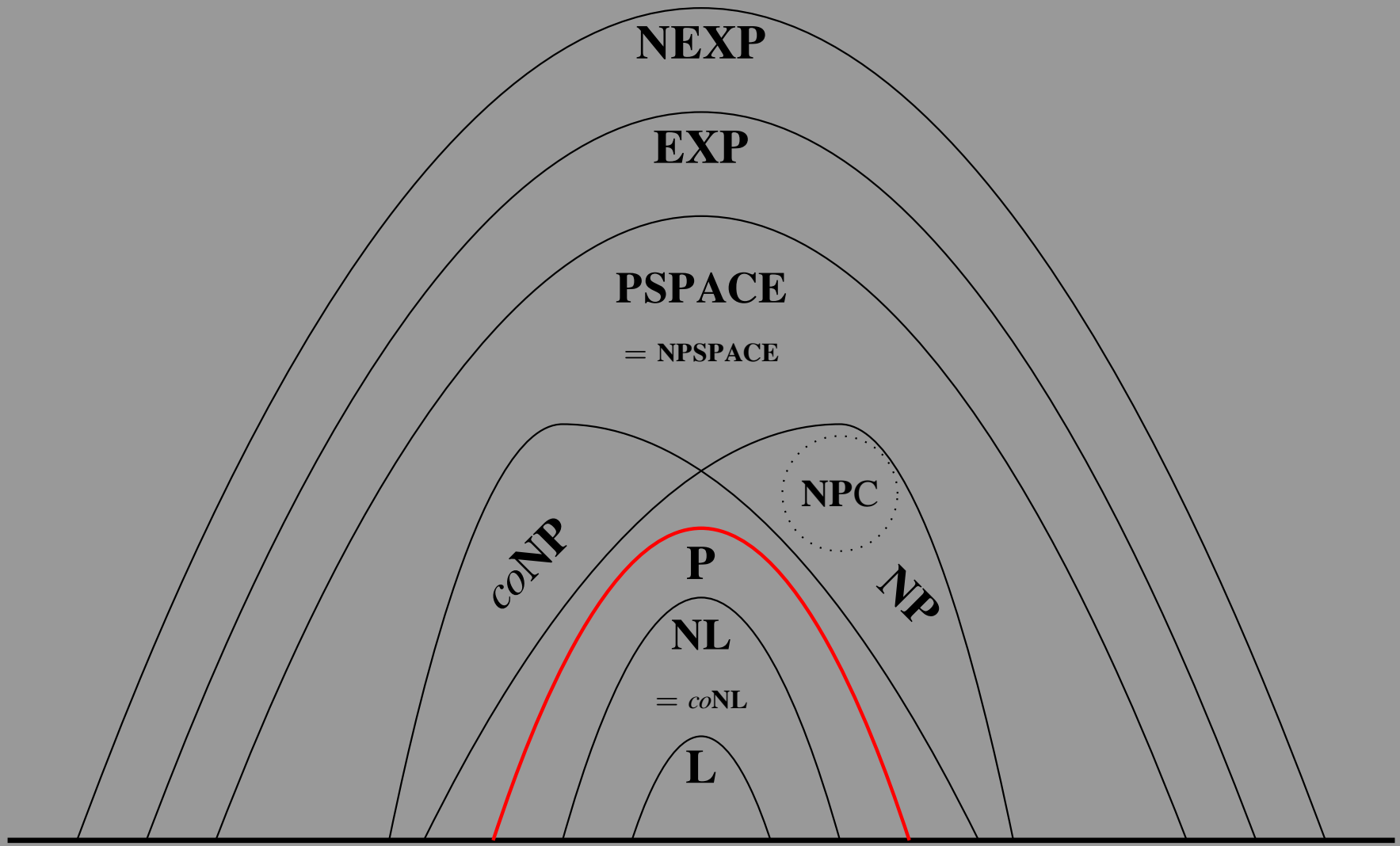
Theorem (Reingold, 2004)

$$\mathbf{L} = \mathbf{SL}$$

# Our Complexity Hierarchy Landscape



# Karp Reductions

**Definition**

A language $L_1$ is **Karp reducible** to a language $L_2$, denoted by $L_1 \leq_m^p L_2$, if there is a function $f : \Sigma^* \to \Sigma^*$, computable by a polynomial-time DTM, such that for all $x \in \Sigma^*$:

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$$x \in L_1 \Leftrightarrow f(x) \in L_2$$

**Definition**

Let $\mathcal{C}$ be a complexity class.

- We say that a language $A$ is **$\mathcal{C}$-hard** (or $\leq_m^p$-hard for $\mathcal{C}$) if for every $B \in \mathcal{C}$, $B \leq_m^p A$.
- We say that a language $A$ is **$\mathcal{C}$-complete**, if it is $\mathcal{C}$-hard, and also $A \in \mathcal{C}$.

# Karp reductions vs logspace redutions

**Theorem**

*A logspace reduction is a polynomial-time reduction.*

**Proof**:

See Th.8.1 (p.160) in [1]

- Let $M$ the logspace reduction TM.
- $M$ has $2^{\mathcal{O}(\log n)}$ possible configurations.
- The machine is deterministic, so *no configuration can be repeated* in the computation.
- So, the computation takes $\mathcal{O}\left(n^k\right)$ time, for some $k$. □

# Circuits and CVP

## Definition (Boolean circuits)

For every $n \in \mathbb{N}$ an $n$-input, single output Boolean Circuit $C$ is a directed acyclic graph with $n$ sources and *one* sink.

- All nonsource vertices are called *gates* and are labeled with one of $\wedge$ (and), $\vee$ (or) or $\neg$ (not).
- The vertices labeled with $\wedge$ and $\vee$ have *fan-in* (i.e. number or incoming edges) 2.
- The vertices labeled with $\neg$ have *fan-in* 1.
- For every vertex $v$ of $C$, we assign a value as follows: for some input $x \in \{0,1\}^n$, if $v$ is the $i$-th input vertex then $val(v) = x_i$, and otherwise $val(v)$ is defined recursively by applying $v$'s logical operation on the values of the vertices connected to $v$.
- The *output* $C(x)$ is the value of the output vertex.

# Circuits and CVP

**Definition (CVP)**

Circuit Value Problem (CVP): Given a circuit $C$ and an assignment $x$ to its variables, determine whether $C(x) = 1$.

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**Example**

REACHABILITY $\leq_m^\ell$ CVP: Graph $G \rightarrow$ circuit $R(G)$:

- The gates are of the form:
  - $g_{i,j,k}$, $1 \leq i, j \leq n$, $0 \leq k \leq n$.
  - $h_{i,j,k}$, $1 \leq i, j, k \leq n$
- $g_{i,j,k}$ is **true** iff there is a path from $i$ to $j$ without intermediate nodes bigger than $k$.
- $h_{i,j,k}$ is **true** iff there is a path from $i$ to $j$ without intermediate nodes bigger than $k$, and $k$ is used.

# Circuits and CVP

## Example

- Input gates: $g_{i,j,0}$ is **true** iff ($i = j$ or $(i,j) \in E(G)$).
- For $k = 1, \ldots, n$: $h_{i,j,k} = (g_{i,k,k-1} \wedge g_{k,j,k-1})$
- For $k = 1, \ldots, n$: $g_{i,j,k} = (g_{i,j,k-1} \vee h_{i,j,k})$
- The output gate $g_{1,n,n}$ is **true** iff there is a path from 1 to $n$ using no intermediate paths above $n$ (sic).
- We also can compute the reduction in logspace: go over all possible $i, j, k$'s and output the appropriate edges and sorts for the variables $(1, \ldots, 2n^3 + n^2)$.

# Composing Reductions

**Theorem**

*If* $L_1 \leq_m^\ell L_2$ *and* $L_2 \leq_m^\ell L_3$, *then* $L_1 \leq_m^\ell L_3$.

**Proof**: See Prop.8.2 (p.164) in [1]

- Let $R, R'$ be the aforementioned reductions.
- We have to prove that $R'(R(x))$ is a logspace reduction.
- But $R(x)$ may by longer than $\log |x|$...

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- We have to prove that $R'(R(x))$ is a logspace reduction.
- But $R(x)$ may by longer than $\log |x|$...
- We simulate $M_{R'}$ by remembering the head position $i$ of the input string of $M_{R'}$, i.e. the output string of $M_R$.
- If the head moves to the right, we increment $i$ and simulate $M_R$ long enough to take the $i^{th}$ bit of the output.
- If the head stays in the same position, we just remember the $i^{th}$ bit.
- If the head moves to the left, we decrement $i$ and **start $M_R$ from the beginning**, until we reach the desired bit. □

# Closure under reductions

- Complete problems are the **maximal elements** of the reductions partial ordering.
- Complete problems capture the essence and difficulty of a complexity class.

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- If an **NP**-complete language is in **P**, then **P** $=$ **NP**.
- If $L$ is **NP**-complete, then $\bar{L}$ is *co***NP**-complete.
- If a *co***NP**-complete problem is in **NP**, then **NP** $=$ *co***NP**.

# P-Completeness

**Theorem**

*If two classes $\mathcal{C}$ and $\mathcal{C}'$ are both closed under reductions and there is an $L \subseteq \Sigma^*$ complete for both $\mathcal{C}$ and $\mathcal{C}'$, then $\mathcal{C} = \mathcal{C}'$.*

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*If two classes $\mathcal{C}$ and $\mathcal{C}'$ are both closed under reductions and there is an $L \subseteq \Sigma^*$ complete for both $\mathcal{C}$ and $\mathcal{C}'$, then $\mathcal{C} = \mathcal{C}'$.*

- Consider the **Computation Table** $T$ of a poly-time TM $M(x)$:
  $T_{ij}$ represents the contents of tape position $j$ at step $i$.
- But how to remember the head position and state?
  *At the $i^{th}$ step: if the state is $q$ and the head is in position $j$, then $T_{ij} \in \Sigma \times Q$.*
- We say that the table is **accepting** if $T_{|x|^k-1,j} \in (\Sigma \times \{q_{yes}\})$, for some $j$.
- Observe that $T_{ij}$ depends only on the contents of the **same** of **adjacent** positions at time $i-1$.

# P-Completeness

**Theorem**

*CVP is* **P***-complete.*

# P-Completeness

**Theorem**

*CVP is* **P***-complete.*

**Proof**: See Th. 8.1 (p.168) in [1]

- We have to show that for any $L \in \mathbf{P}$ there is a reduction $R$ from $L$ to CVP.
- $R(x)$ must be a variable-free circuit such that $x \in L \Leftrightarrow R(x) = 1$.
- $T_{ij}$ depends only on $T_{i-1,j-1}$, $T_{i-1,j}$, $T_{i-1,j+1}$.
- Let $\Gamma = \Sigma \cup (\Sigma \times Q)$.
- Encode $s \in \Gamma$ as $(s_1, \ldots, s_m)$, where $m = \lceil \log |\Gamma| \rceil$.
- Then the computation table can be seen as a table of binary entries $S_{ij\ell}$, $1 \leq \ell \leq m$.
- $S_{ij\ell}$ depends only on the $3m$ entries $S_{i-1,j-1,\ell'}$, $S_{i-1,j,\ell'}$, $S_{i-1,j+1,\ell'}$, where $1 \leq \ell' \leq m$.

# P-Completeness

**Proof** (*cont'd*):

- So, there are $m$ Boolean Functions $f_1, \ldots, f_m : \{0,1\}^{3m} \to \{0,1\}$ s.t.:
$$S_{ij\ell} = f_\ell(\overrightarrow{S}_{i-1,j-1}, \overrightarrow{S}_{i-1,j}, \overrightarrow{S}_{i-1,j+1})$$
- Thus, there exists a Boolean Circuit $C$ with $3m$ inputs and $m$ outputs computing $T_{ij}$.
- $C$ depends only on $M$, and has constant size.
- $R(x)$ will be $(|x|^k - 1) \times (|x|^k - 2)$ copies of $C$.
- The input gates are fixed.
- $R(x)$'s output gate will be the first bit of $C_{|x|^k-1,1}$.
- The circuit $C$ is fixed, so we can generate indexed copies of $C$, using $\mathcal{O}(\log |x|)$ space for indexing. □

# CIRCUIT SAT & SAT

**Definition (CIRCUIT SAT)**

Given Boolen Circuit $C$, is there a truth assignment $x$ appropriate to $C$, such that $C(x) = 1$?

**Definition (SAT)**

Given a Boolean Expression $\phi$ in CNF, is it satisfiable?

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**Example**

CIRCUIT SAT $\leq_m^\ell$ SAT:

- Given $C \rightarrow$ Boolean Formula $R(C)$, s.t. $C(x) = 1 \Leftrightarrow R(C)(x) = T$.
- Variables of $C \rightarrow$ variables of $R(C)$.
- Gate $g$ of $C \rightarrow$ variable $g$ of $R(C)$.

# CIRCUIT SAT & SAT

## Example

- Gate $g$ of $C \rightarrow$ clauses in $R(C)$:
  - $g$ **variable** gate: add $(\neg g \vee x) \wedge (g \vee \neg x)$ $\equiv g \Leftrightarrow x$
  - $g$ **TRUE** gate: add $(g)$
  - $g$ **FALSE** gate: add $(\neg g)$
  - $g$ **NOT** gate & $pred(g) = h$: add $(\neg g \vee \neg h) \wedge (g \vee h)$ $\equiv g \Leftrightarrow \neg h$
  - $g$ **OR** gate & $pred(g) = \{h, h'\}$: add $(\neg h \vee g) \wedge (\neg h' \vee g) \wedge (h \vee h' \vee \neg g)$ $\equiv g \Leftrightarrow (h \vee h')$
  - $g$ **AND** gate & $pred(g) = \{h, h'\}$: add $(\neg g \vee h) \wedge (\neg g \vee h') \wedge (\neg h \vee \neg h' \vee g)$ $\equiv g \Leftrightarrow (h \wedge h')$
  - $g$ **output** gate: add $(g)$
- $R(C)$ is satisfiable if and only if $C$ is.
- The construction can be done within $\log |x|$ space. $\square$

# Bounded Halting Problem

- We can define the time-bounded analogue of HP:

**Definition (Bounded Halting Problem (BHP))**

Given the code $\llcorner M \lrcorner$ of an NTM $M$, and input $x$ and a string $0^t$, decide if $M$ accepts $x$ in $t$ steps.

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**Theorem**

*BHP is* **NP***-complete.*

**Proof**:

- BHP $\in$ **NP**.
- Let $A \in$ **NP**. Then, $\exists$ NTM $M$ deciding $A$ in time $p(|x|)$, for some $p \in poly(|x|)$.
- The reduction is the function $R(x) = \langle \llcorner M \lrcorner, x, 0^{p(|x|)} \rangle$. $\square$

# Cook's Theorem

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*SAT is* **NP***-complete.*

**Proof**:

See Th.8.2 (p.171) in [1]

- SAT $\in$ **NP**.
- Let $L \in$ **NP**. We will show that $L \leq_m^\ell$ CIRCUIT SAT $\leq_m^\ell$ SAT.
- Since $L \in$ **NP**, there exists an NPTM $M$ deciding $L$ in $n^k$ steps.
- Let $(c_1, \ldots, c_{n^k}) \in \{0,1\}^{n^k}$ a certificate for $M$ (recall the binary encoding of the computation tree).

# Cook's Theorem

**Proof** (*cont'd*):

- If we fix a certificate, then the computation is *deterministic* (the language's Verifier $V(x, y)$ is a DPTM).
- So, we can define the **computation table** $T(M, x, \overrightarrow{c})$.
- As before, all non-top row and non-extreme column cells $T_{ij}$ will depend *only* on $T_{i-1,j-1}, T_{i-1,j}, T_{i-1,j+1}$ and the nondeterministic choice $c_{i-1}$.
- We now fixed a circuit $C$ with $3m + 1$ input gates.
- Thus, we can construct in $\log |x|$ space a circuit $R(x)$ with variable gates $c_1, \ldots c_{n^k}$ corresponding to the **nondeterministic choices** of the machine.
- $R(x)$ is satisfiable if and only if $x \in L$. □

# NP-completeness: Web of Reductions

- Many **NP**-complete problems stem from Cook's Theorem via reductions:
  - 3SAT, MAX2SAT, NAESAT
  - IS, CLIQUE, VERTEX COVER, MAX CUT
  - $\text{TSP}_{(\text{D})}$, 3COL
  - SET COVER, PARTITION, KNAPSACK, BIN PACKING
  - INTEGER PROGRAMMING (IP): Given $m$ inequalities oven $n$ variables $u_i \in \{0, 1\}$, is there an assignment satisfying all the inequalities?
- Always remember that these are **decision versions** of the corresponding **optimization problems**.

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- Always remember that these are **decision versions** of the corresponding **optimization problems**.
- But 2SAT, 2COL $\in$ **P**.

# NP-completeness: Web of Reductions

## Example

SAT $\leq_m^\ell$ IP:

- Every clause can be expressed as an inequality, eg:

$$(x_1 \vee \bar{x}_2 \vee \bar{x}_3) \longrightarrow x_1 + (1 - x_2) + (1 - x_3) \geq 1$$

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- This method is generalized by the notion of *Constraint Satisfaction Problems*.
- A **Constraint Satisfaction Problem** (CSP) generalizes SAT by allowing clauses of arbitrary form (instead of ORs of literals).

3SAT is the subcase of *qCSP*, where arity $q = 3$ and the constraints are ORs of the involved literals.

# Quantified Boolean Formulas

**Definition (Quantified Boolean Formula)**

A **Quantified Boolean Formula** $F$ is a formula of the form:

$$F = \exists x_1 \forall x_2 \exists x_3 \cdots Q_n x_n \ \phi(x_1, \ldots, x_n)$$

where $\phi$ is *plain* (quantifier-free) boolean formula.

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**Example**

$$F = \exists x_1 \forall x_2 \exists x_3 \left[(x_1 \vee \neg x_2) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3)\right]$$

The above is a True QBF ($(1, 0, 0)$ and $(1, 1, 1)$ satisfy it).

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**Theorem**

TQBF *is* **PSPACE**-*complete*.

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TQBF *is* **PSPACE**-*complete.*

**Proof**:

See Th. 19.1 (p.456) in [1] – Th.4.13 (p.84) in [2]

- TQBF $\in$ **PSPACE**:
  - Let $\phi$ be a QBF, with $n$ variables and length $m$.
  - Recursive algorithm $A(\phi)$:
  - If $n = 0$, then there are only constants, hence $\mathcal{O}(m)$ time/space.
  - If $n > 0$:
    $A(\phi) = A\left(\phi|_{x_1=0}\right) \vee A\left(\phi|_{x_1=1}\right)$, if $Q_1 = \exists$, and
    $A(\phi) = A\left(\phi|_{x_1=0}\right) \wedge A\left(\phi|_{x_1=1}\right)$, if $Q_1 = \forall$.
  - Both recursive computations can be run on *the same space*.
  - So $space_{n,m} = space_{n-1,m} + \mathcal{O}(m) \Rightarrow space_{n,m} = \mathcal{O}(n \cdot m)$.

# Quantified Boolean Formulas

**Proof** (*cont'd*):

- Now, let $M$ a TM with space bound $p(n)$.
- We can create the configuration graph of $M(x)$, having size $2^{\mathcal{O}(p(n))}$.
- $M$ accepts $x$ iff there is a path of length at most $2^{\mathcal{O}(p(n))}$ from the initial to the accepting configuration.
- Using Savitch's Theorem idea, for two configurations $C$ and $C'$ we have:
  $REACH(C, C', 2^i) \Leftrightarrow$
  $\Leftrightarrow \exists C'' \left[REACH(C, C'', 2^{i-1}) \wedge REACH(C'', C', 2^{i-1})\right]$

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- $M$ accepts $x$ iff there is a path of length at most $2^{\mathcal{O}(p(n))}$ from the initial to the accepting configuration.
- Using Savitch's Theorem idea, for two configurations $C$ and $C'$ we have:
  $REACH(C, C', 2^i) \Leftrightarrow$
  $\Leftrightarrow \exists C'' \left[REACH(C, C'', 2^{i-1}) \wedge REACH(C'', C', 2^{i-1})\right]$
- But, this is a bad idea: Doubles the size each time.
- Instead, we use additional variables:
  $\exists C'' \forall D_1 \forall D_2 \left[(D_1 = C \wedge D_2 = C'') \vee (D_1 = C'' \wedge D_2 = C')\right] \Rightarrow$
  $REACH(D_1, D_2, 2^{i-1})$

# Quantified Boolean Formulas

**Proof** (*cont'd*):

- The base case of the recursion is $C_1 \rightarrow C_2$, and can be encoded as a quantifier-free formula.
- The size of the formula in the $i^{th}$ step is $space_i \leq space_{i-1} + \mathcal{O}\left(p(n)\right) \Rightarrow \mathcal{O}\left(p^2(n)\right)$. □

# *Logical Characterizations

- **Descriptive complexity** is a branch of computational complexity theory and of finite model theory that characterizes complexity classes by the *type of logic* needed to express the languages in them.

# *Logical Characterizations

- **Descriptive complexity** is a branch of computational complexity theory and of finite model theory that characterizes complexity classes by the *type of logic* needed to express the languages in them.

**Theorem (Fagin's Theorem)**

*The set of all properties expressible in Existential Second-Order Logic is precisely* **NP**.

**Theorem**

*The class of all properties expressible in Horn Existential Second-Order Logic with Successor is precisely* **P**.

- HORNSAT is **P**-complete.

## Summary 1/2

- We define complexity classes using a computation model/mode and complexity measures.
- Time/Space constructible functions are used as complexity measures.
- Classes of the same kind form *proper hierarchies*.
- **NP** is the class of *easily verifiable* problems: given a *certificate*, one can efficiently verify that it is correct.
- Savitch's Theorem implies that **PSPACE** = **NPSPACE**.

## Summary 2/2

- Reductions relate problems with respect to hardness.
- Complete problems reflect the difficulty of the class.
- REACHABILITY is **NL**-complete.
- Immerman-Szelepscényi's Theorem implies that $\mathbf{NL} = co\mathbf{NL}$.
- Circuit Value Problem (CVP) is **P**-complete under logspace reductions.
- CIRCUIT SAT and SAT are **NP**-complete.
- True Quantified Boolean Formula (TQBF) is **PSPACE**-complete.