Algorithmic Game Theory

Introduction to Mechanism Design for Single Parameter Environments

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Mechanism Design

- What is mechanism design?
- It can be seen as reverse game theory
- Main goal: design the rules of a game so as to
  - avoid strategic behavior by the players
  - and more generally, enforce a certain behavior for the players or other desirable properties

- Applied to problems where a "social choice" needs to be made
  - i.e., an aggregation of individual preferences to a single joint decision
- strategic behavior = declaring false preferences in order to gain a higher utility

Examples

- Elections
  - Parliamentary elections, committee elections, council elections, etc
  - A set of voters
  - A set of candidates
  - Each voter expresses preferences according to the election rules
    - E.g., by specifying his single top choice, or by specifying his first few choices, or by submitting a full ranking of the candidates
  - Social choice: can be a single candidate (single-winner election) or a set of candidates (multi-winner election) or a ranking of the candidates

Examples

- Auctions
  - An auctioneer with some items for sale
  - A set of bidders express preferences (offers) over items
    - Or combinations of items
  - Preferences are submitted either through a valuation function, or according to some bidding language
  - Social choice: allocation of items to the bidders

Examples

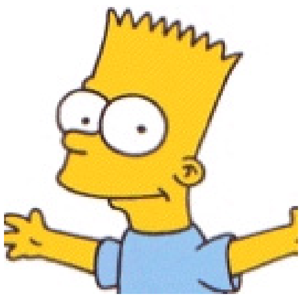
- Government policy making and referenda
  - A municipality is considering implementing a public project
  - Q1: Should we build a new road, a library or a tennis court?
  - Q2: If we build a library where shall we build it?
  - Citizens can express their preferences in an online survey or a referendum
  - Social choice: the decision of the municipality on what and where to implement

Specifying preferences

- In all the examples, the players need to submit their preferences in some form
- Representation of preferences can be done by
  - A valuation function (specifying a value for each possible outcome)
  - A ranking (an ordering on possible outcomes)
  - An approval set (which outcomes are approved)
- Possible conflict between increased expressiveness vs complexity of decision problem

Single-item Auctions

Auctions



Set of players
N = {1, 2, …, n}

1 indivisible good

Auctions

- A means of conducting transactions since antiquity
  - First references of auctions date back to ancient Athens and Babylon
- Modern applications:
  - Art works
  - Stamps
  - Flowers (Netherlands)
  - Spectrum licences
  - Other govermental licences
  - Pollution rights
  - Google ads
  - eBay
  - Bonds
  - ...

Auctions

- Earlier, the most popular types of auctions were
  - The English auction
    - The price keeps increasing in small increments
    - Gradually bidders drop out till there is only one winner left
  - The Dutch auction
    - The price starts at $+\infty$ (i.e., at some very high price) and keeps decreasing
    - Until there exists someone willing to offer the current price
  - There exist also many variants regarding their practical implementation
- These correspond to ascending or descending price trajectories

Sealed bid auctions

- Sealed bid: We think of every bidder submitting his bid in an envelope, without other players seeing it
  - It does not really have to be an envelope, bids can be submitted electronically
  - The main assumption is that it is submitted in a way that other bidders cannot see it
- After collecting the bids, the auctioneer needs to decide:
  - Who wins the item?
    - Easy! Should be the guy with the highest bid
      - Yes in most cases, but not always
  - How much should the winner pay?
    - Not so clear

Sealed bid auctions

Why do we view auctions as games?

- We assume every player has a valuation v_i for obtaining the good
- Available strategies: each bidder is asked to submit a bid b_i
  - $b_i \in [0, \infty)$
  - Infinite number of strategies
- The submitted bid b_i may differ from the real value v_i of bidder i

First price auction

Auction rules

- Let **b** = (b_1, b_2,..., b_n) the vector of all the offers
- Winner: The bidder with the highest offer
  - In case of ties: We assume the winner is the bidder with the lowest index (not important for the analysis)
  - E.g. if there is a tie among bidder 2 and bidder 4, the winner is bidder 2
- Winner's payment: the bid declared by the winner
- Utility function of bidder i,

$$u_i(\mathbf{b}) = \begin{cases} v_i - b_i\,, & \text{if i is the winner} \\ 0, & \text{otherwise} \end{cases}$$

Incentives in the first price auction

Analysis of first price auctions

- There are too many Nash equilibria
- Can we predict bidding behavior? Is some equilibrium more likely to occur?
- Hard to tell what exactly will happen in practice but we can still make some conclusions for first price auctions

Observation: Suppose that $v_1 \geq v_2 \geq v_3 \ldots \geq v_n$. Then the profile $(v_2, v_2, v_3, \ldots, v_n)$ is a Nash equilibrium

Corollary: The first price auction provides incentives to bidders to hide their true value

- This is highly undesirable when $v_1 - v_2$ is large

Auction mechanisms

We would like to explore alternative payment rules with better properties

Definition: For the single-item setting, an auction mechanism receives as input the bidding vector **b** = $(b_1, b_2, \ldots, b_n)$ and consists of

- an allocation algorithm (who wins the item)
- a payment algorithm (how much does the winner pay)

Most mechanisms satisfy individual rationality:

- Non-winners do not pay anything
- If the winner is bidder i, her payment will not exceed b_i (it is guaranteed that no-one will pay more than what she declared)

Auction mechanisms

Aligning Incentives

- Ideally, we would like mechanisms that do not provide incentives for strategic behavior
- How do we even define this mathematically?

An attempt:

Definition: A mechanism is called truthful (or strategyproof, or incentive compatible) if for every bidder i, and for every profile $\mathbf{b}_{-i}$ of the other bidders, it is a **dominant strategy** for i to declare her real value v_i, i.e., it holds that

$$u_i(v_i, \mathbf{b}_{-i}) \geq u_i(b', \mathbf{b}_{-i}) \text{ for every } b' \neq v_i$$

Auction mechanisms

- In a truthful mechanism, every rational agent knows what to play, independently of what the other bidders are doing
- It is a win-win situation:
  - The auctioneer knows that players should not strategize
  - The bidders also know that they should not spend time on trying to find a different strategy
- Very powerful property for a mechanism
- Fact: The first-price mechanism is not truthful

Are there truthful mechanisms?

The 2nd price mechanism (Vickrey auction)

[Vickrey '61]

- **Allocation algorithm:** same as before, the bidder with the highest offer
  - In case of ties: we assume the winner is the bidder with the lowest index
- **Payment algorithm:** the winner pays the 2nd highest bid
- Hence, the auctioneer offers a discount to the winner

Observation: the payment does not depend on the winner's bid!

- The bid of each player determines if he wins or not, but not what he will pay

The 2^{nd} price mechanism (Vickrey auction)

[Vickrey '61] (Nobel prize in economics, 1996)

- **Theorem:** The 2^{nd} price auction is a truthful mechanism

Proof sketch:

- Fix a bidder i, and let $\mathbf{b}_{-i}$ be an arbitrary bidding profile for the rest of the players
- Let $b^* = \max_{j \neq i} b_j$
- Consider now all possible cases for the final utility of bidder i, if he plays v_i
  - $v_i < b^*$
  - $v_i > b^*$
  - $v_i = b^*$
  - In all these different cases, we can prove that bidder i does not become better off by deviating to another strategy

Optimization objectives

What do we want to optimize in an auction?

Usual objectives:

- Social welfare (the total welfare produced for the involved entities)
- Revenue (the payment received by the auctioneer)

We will focus first on social welfare

Optimization objectives

What do we want to optimize in an auction?

Definition: The utilitarian social welfare produced by a bidding vector **b** is SW(**b**) = $\sum_i u_i(\mathbf{b})$

- The summation includes the auctioneer's utility (= the auctioneer's payment)
- The auctioneer's payment cancels out with the winner's payment

➢ For the single-item setting, SW(**b**) = the value of the winner for the item

➢ An auction is welfare maximizing if it always produces an allocation with optimal social welfare when bidders are truthful

Vickrey auction: an ideal auction format

Summing up:

Theorem: The 2^{nd} price auction is

- truthful [incentive guarantees]
- welfare maximizing [economic performance guarantees]
- implementable in polynomial time [computational performance guarantees]

Even though the valuations are private information to the bidders, the Vickrey auction solves the welfare maximization problem as if the valuations were known

Generalizations to single-parameter environments

Single-parameter mechanisms

- In many cases, we do not have a single item to sell, but multiple items
- But still, the valuation of a bidder could be determined by a single number (e.g., value per unit)
- Note: the valuation function may depend on various other parameters, but we assume only a single parameter is private information to the bidder
  - The other parameters may be publicly known information
- We can treat all these settings in a unified manner
- Our focus: **Direct revelation** mechanisms
  - The mechanism asks each bidder to submit the parameter that completely determines her valuation function

Examples of single-parameter environments

- **Single-item auctions:**
  - One item for sale
  - each bidder is asked to submit his value for acquiring the item
- **k-item unit-demand auctions**
  - k identical items for sale
  - each bidder submits his value per unit and can win at most one unit
- **Knapsack auctions**
  - k identical items, each bidder has a value for obtaining a certain number of units
- **Single-minded auctions**
  - a set of (non-identical) items for sale
  - each bidder is interested in acquiring a specific subset of items (known to the mechanism)
  - Each bidder submits his value for the set she desires

Examples of single-parameter environments

- Sponsored search auctions
  - multiple advertising slots available, arranged from top to bottom
  - each bidder interested in acquiring as high a slot as possible
  - each bidder submits his value per click
- Public project mechanisms
  - deciding whether to build a public project (e.g., a park)
  - each bidder submits his value for having the project built

In all these settings, we can have multiple winners in the auction

Some Notation

- Suppose we have n players
- Let v_i be the parameter that is private information to player i
  - Usually v_i corresponds to value per unit, or value obtained at the desirable outcome, or maximum amount willing to pay (dependent on the context)

General form of direct-revelation mechanisms for single-parameter problems:

- Input: The bidding vector $\mathbf{b} = (b_1,\ldots, b_n)$ by the players
  - each b_i may differ from v_i
- Allocation rule: Choose an allocation $\mathbf{x}(\mathbf{b}) = (x_1(\mathbf{b}), x_2(\mathbf{b}),\ldots, x_n(\mathbf{b}))$
  - $x_i(\mathbf{b})$ = number of units received by pl. i or more generally the decision on what is allocated to i
- Payment rule: $\mathbf{p}(\mathbf{b}) = (p_1(\mathbf{b}), p_2(\mathbf{b}),\ldots, p_n(\mathbf{b}))$
  - $p_i(\mathbf{b})$ = payment for bidder i

Some Notation

- We will use (**x**, **p**) to refer to a mechanism with allocation function **x**, and payment function **p**
- Final utility of bidder i in a mechanism M = (**x**, **p**):
  - $u_i(\mathbf{b}) = v_i\, x_i(\mathbf{b}) - p_i(\mathbf{b})$
  - Quasi-linear form of utility functions
- For simplicity, we often write $(x_1, x_2, \ldots, x_n)$ instead of $(x_1(\mathbf{b}), x_2(\mathbf{b}), \ldots, x_n(\mathbf{b}))$
- We focus on mechanisms that satisfy Individual Rationality:
  - If a bidder i is a non-winner ($x_i(\mathbf{b}) = 0$), then $p_i(\mathbf{b}) = 0$
  - For winners, the payment rule satisfies $p_i(\mathbf{b}) \in [0, b_i\, x_i(\mathbf{b})]$ for every bidding vector **b** and every i
  - The auctioneer can never ask a bidder for a payment higher than her declared total value for what she won

Examples of single-parameter environments

Describing the feasible allocations

- **Single-item auctions:**
  - $x_i \in \{0, 1\}$ for every i, and $\Sigma_i x_i = 1$
- **k-item unit-demand auctions**
  - k identical items for sale
  - $x_i \in \{0, 1\}$, $\Sigma_i x_i <= k$
- **Knapsack auctions**
  - k identical items for sale
  - For each bidder, demand of w_i units
  - $x_i \in \{0, 1\}$ for every i, $\Sigma_i w_i x_i <= k$
- **Public project mechanisms**
  - Deciding whether to build a public project (e.g., a park)
  - Only 2 feasible allocations: (0, 0, ..., 0) or (1, 1, ..., 1)

Allocation rules and truthful mechanisms

- Can we understand how to derive truthful mechanisms?
- Actually, we can rephrase this as:
  - Suppose we are given an allocation rule **x**
  - Can we tell if **x** can be combined with a pricing rule **p**, so that (**x**, **p**) is a truthful mechanism?
- This would allow us to focus only on designing the allocation algorithm appropriately
- Consider the single-item auction
  - Allocation rule 1: Give the item to the highest bidder
  - Allocation rule 2: Give the item to the 2nd highest bidder
- For rule 1, we have seen how to turn it into a truthful mechanism (Vickrey auction)
- For rule 2?
  - We have not seen how to do this, but we have also not proved that it cannot be done

Allocation rules and truthful mechanisms

- Consider a mechanism with allocation rule **x**
- Fix a player i, and fix a profile $\mathbf{b}_{-i}$ for the other players
- Allocation to player i at a profile $\mathbf{b} = (z, \mathbf{b}_{-i})$ is given by $x_i(\mathbf{b})$
- Keeping $\mathbf{b}_{-i}$ fixed, we can view the allocation to player i as a function of his bid
    - $x_i = x_i(z, \mathbf{b}_{-i})$, if bidder i bids z

- Definition: An allocation rule is monotone if for every bidder i, and every profile $\mathbf{b}_{-i}$, the allocation $x_i(z, \mathbf{b}_{-i})$ to i is non-decreasing in z
- I.e., bidding higher can only get you more stuff

Monotonicity of allocation rules

Examples

- Back to the single-item auction
- The allocation rule that gives the item to the highest bidder is monotone
    - If a bidder wins at profile **b**, she continues to be a winner if she raises her own bid (keeping $\mathbf{b}_{-i}$ fixed)
    - If she was not a winner at **b**, then by raising her bid, she will either remain a non-winner or she will become a winner
- The allocation rule that gives the item to the 2^{nd} highest bidder is not monotone
    - If I am a winner and raise my bid, I may become the highest bidder and will stop being a winner

Myerson's lemma

[Myerson '81]

- Theorem: For every single-parameter environment,
  - An allocation rule **x** can be turned into a truthful mechanism if and only if it is monotone
  - If **x** is monotone, then there is a unique payment rule **p**, so that (**x**, **p**) is a truthful mechanism
    - Subject to the constraint that if $b_i = 0$, then $p_i = 0$

- One of the classic results in mechanism design
- In fact, in many cases we can also compute the payments by a simple formula

Myerson's lemma

- Allocation rule **x** is truthful =>
  Allocation rule **x** is monotone: forall z, y, (**x**(z) – **x**(y))(z – y) ≥ 0

If z is the true value:

$$\boldsymbol{x}(z) \cdot z - \boldsymbol{p}(z) \geq \boldsymbol{x}(y) \cdot z - \boldsymbol{p}(y) \tag{1}$$

If y is the true value:

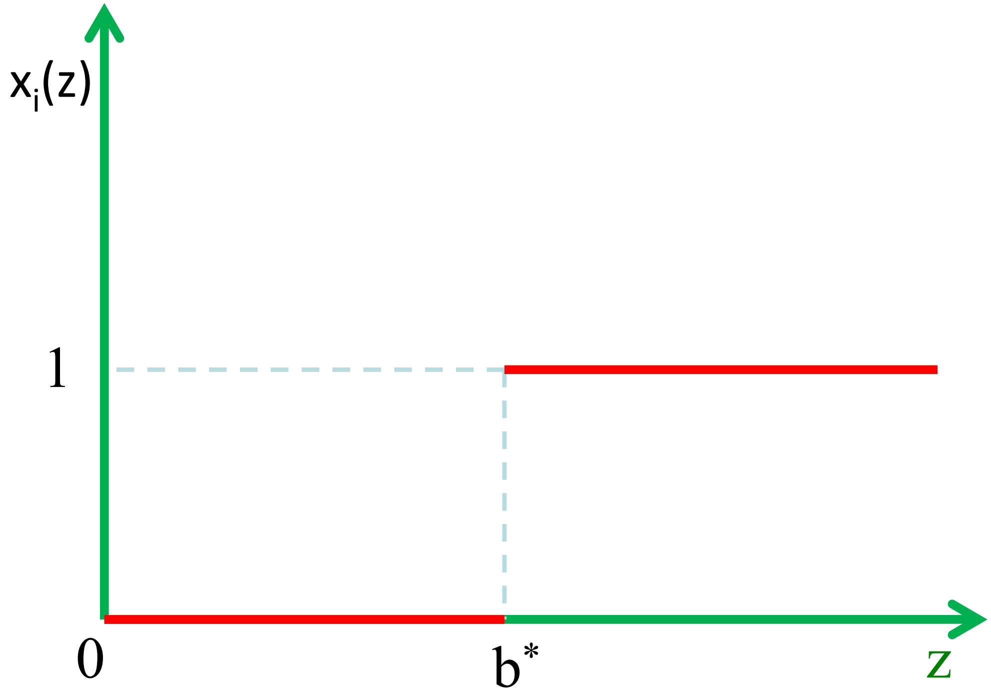
$$\boldsymbol{x}(y) \cdot y - \boldsymbol{p}(y) \geq \boldsymbol{x}(z) \cdot y - \boldsymbol{p}(z) \tag{2}$$

Summing up (1) and (2):

$$\boldsymbol{x}(z) \cdot z + \boldsymbol{x}(y) \cdot y \geq \boldsymbol{x}(y) \cdot z + \boldsymbol{x}(z) \cdot y \Leftrightarrow$$
$$(\boldsymbol{x}(z) - \boldsymbol{x}(y)) \cdot z \geq (\boldsymbol{x}(z) - \boldsymbol{x}(y)) \cdot y \Leftrightarrow$$
$$(\boldsymbol{x}(z) - \boldsymbol{x}(y)) \cdot (z - y) \geq 0$$

Myerson's lemma and payment formula

- For the payment rule, we need to look for each bidder at the allocation function $x_i(z, \mathbf{b}_{-i})$
- For the single-item truthful auction:
  - Fix $\mathbf{b}_{-i}$ and let $b^* = \max_{j \neq i} b_j$



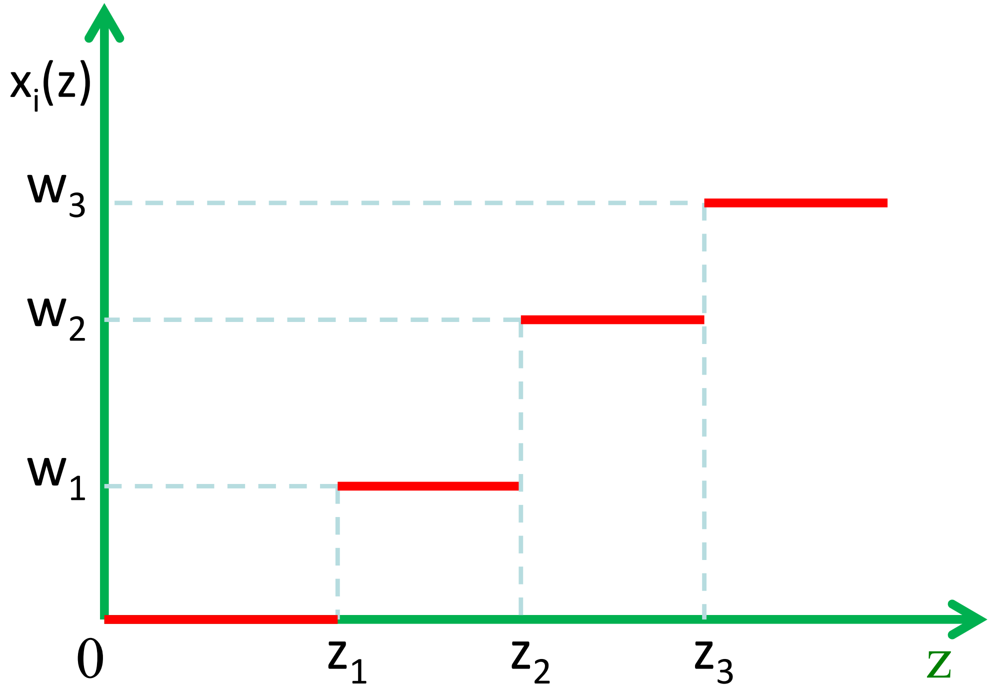
Facts:

- For any fixed $\mathbf{b}_{-i}$, the allocation function is piecewise linear with 1 jump
- The Vickrey payment is precisely the value at which the jump happens
- The jump changes the allocation from 0 to 1 unit

Myerson's lemma and payment formula

For most scenarios of interest

- The allocation is piecewise linear with multiple jumps
- The jump determines how many extra units the bidder wins



- Suppose bidder i bids b_i
- Look at the jumps of $x_i(z, b_{-i})$ in the interval $[0, b_i]$
- Suppose we have k jumps
- Jump at z_1: w_1
- Jump at z_2: $w_2 – w_1$
- Jump at z_3: $w_3 – w_2$
- …
- Jump at z_k: $w_k – w_{k-1}$

Myerson's lemma and payment formula

For most scenarios of interest

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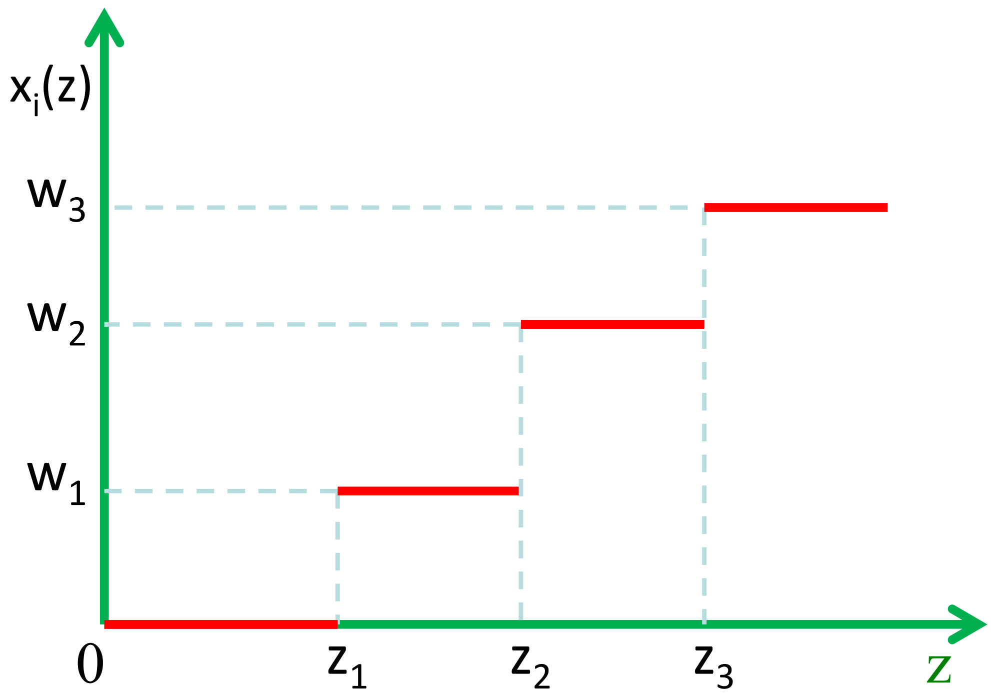

Payment formula

- For each bidder i at a profile b, find all the jump points within $[0, b_i]$
- $p_i(b) = \Sigma_j z_j \cdot$ [jump at z_j]
  $= \Sigma_j z_j \cdot [w_j - w_{j-1}]$
- The formula can also be generalized for monotone but not piecewise linear functions

Myerson's lemma

- Allocation rule **x** is truthful (and thus, monotone) => find appropriate payments **p**

If z is the true value:

$$\boldsymbol{x}(z) \cdot z - \boldsymbol{p}(z) \geq \boldsymbol{x}(y) \cdot z - \boldsymbol{p}(y) \tag{1}$$

If y is the true value:

$$\boldsymbol{x}(y) \cdot y - \boldsymbol{p}(y) \geq \boldsymbol{x}(z) \cdot y - \boldsymbol{p}(z) \tag{2}$$

Combining (1) and (2), we get:

$$z(\boldsymbol{x}(z) - \boldsymbol{x}(y)) \leq \boldsymbol{p}(y) - \boldsymbol{p}(z) \leq y(\boldsymbol{x}(z) - \boldsymbol{x}(y))$$

Assuming that y tends to z from above, in the limit, we get:

$$\boldsymbol{p}'(z) = z \cdot \boldsymbol{x}'(z) \tag{3}$$

Myerson's lemma

- Allocation rule **x** is truthful (and thus, monotone) => find appropriate payments **p**

$$\boldsymbol{p}'(z) = z \cdot \boldsymbol{x}'(z) \tag{3}$$

We assume $\boldsymbol{p}(0) = 0$ (normalization) and solve (3):

$$\boldsymbol{p}_i(b_i, \boldsymbol{b}_{-i}) = \int_0^{b_i} z \cdot \boldsymbol{x}_i'(z, \boldsymbol{b}_{-i}) dz = b_i \cdot \boldsymbol{x}_i(b_i, \boldsymbol{b}_{-i}) - \int_0^{b_i} \boldsymbol{x}_i(z, \boldsymbol{b}_{-i}) dz$$

$$\boldsymbol{p}_i(b_i, \boldsymbol{b}_{-i}) = b_i \cdot \boldsymbol{x}_i(b_i, \boldsymbol{b}_{-i}) - \int_0^{b_i} \boldsymbol{x}_i(z, \boldsymbol{b}_{-i}) dz$$

i's utility: $\quad u_i(b_i, \boldsymbol{b}_{-i}) = (v_i - b_i) \cdot \boldsymbol{x}_i(b_i, \boldsymbol{b}_{-i}) + \int_0^{b_i} \boldsymbol{x}_i(z, \boldsymbol{b}_{-i}) dz$

Myerson's lemma

- Any monotone allocation rule **x** is truthful with payments **p**

$$\boldsymbol{p}_i(b_i, \boldsymbol{b}_{-i}) = b_i \cdot \boldsymbol{x}_i(b_i, \boldsymbol{b}_{-i}) - \int_0^{b_i} \boldsymbol{x}_i(z, \boldsymbol{b}_{-i}) dz$$

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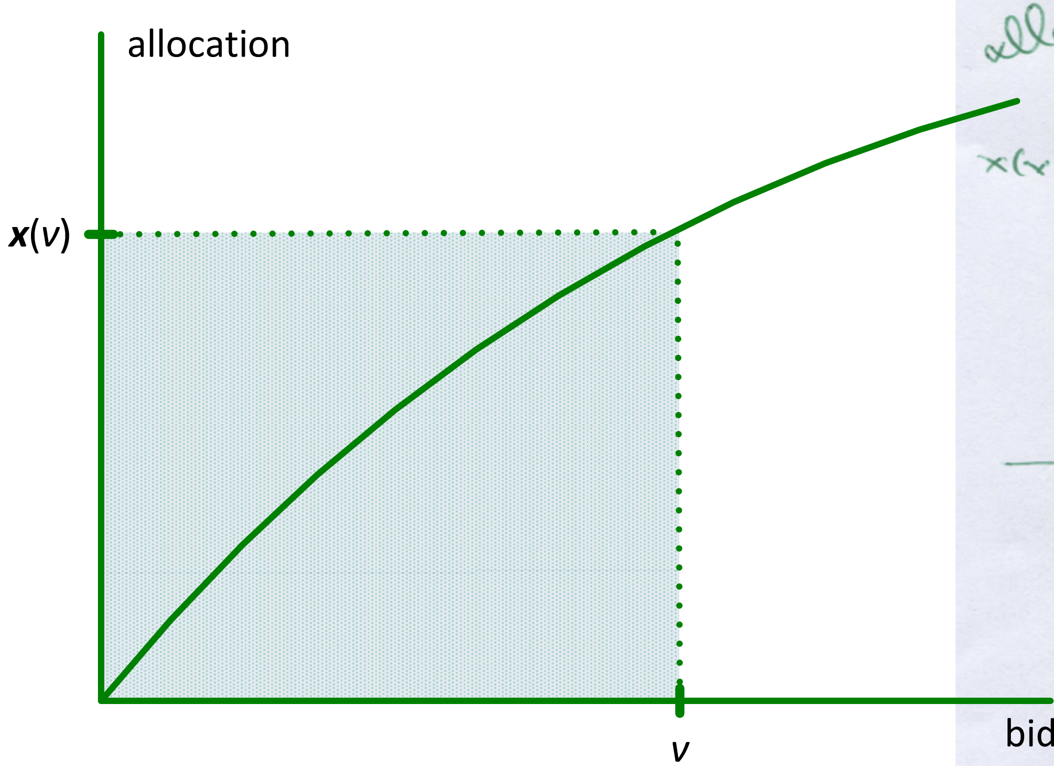

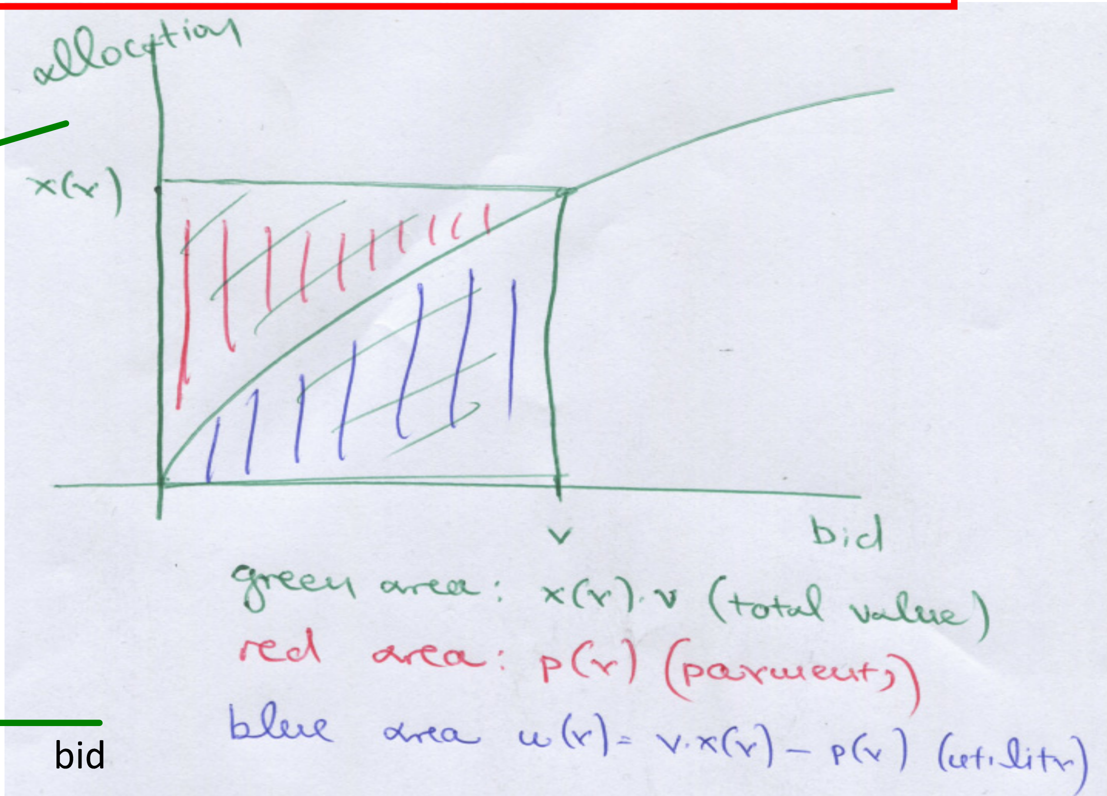

Myerson's lemma

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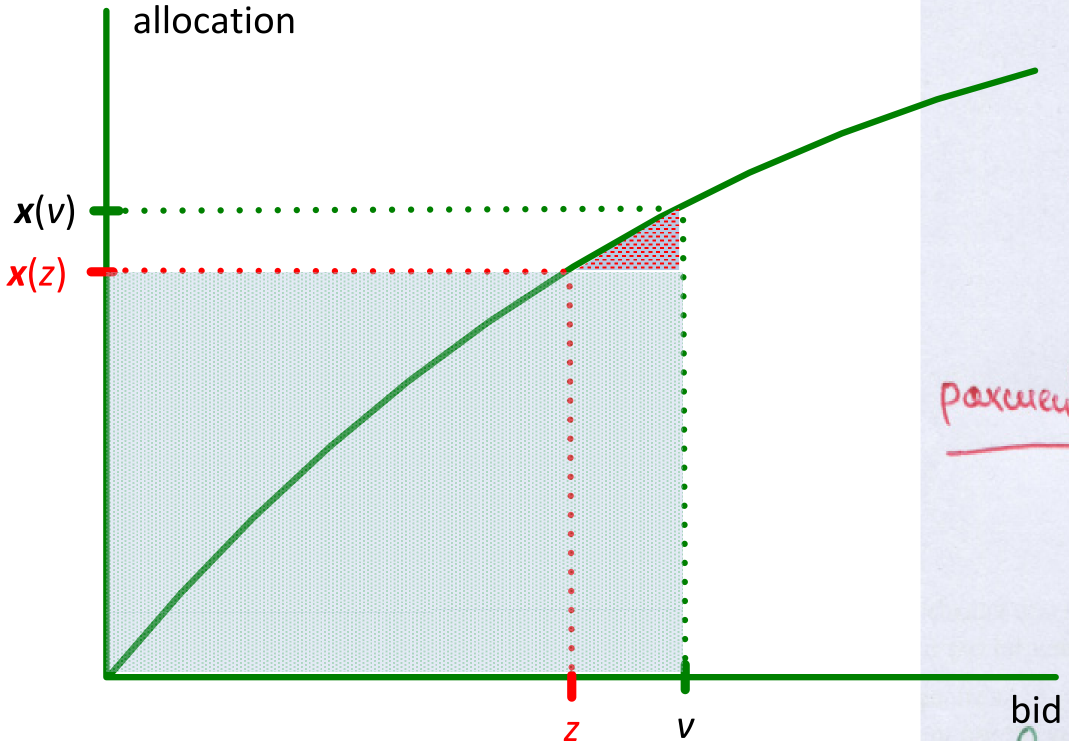

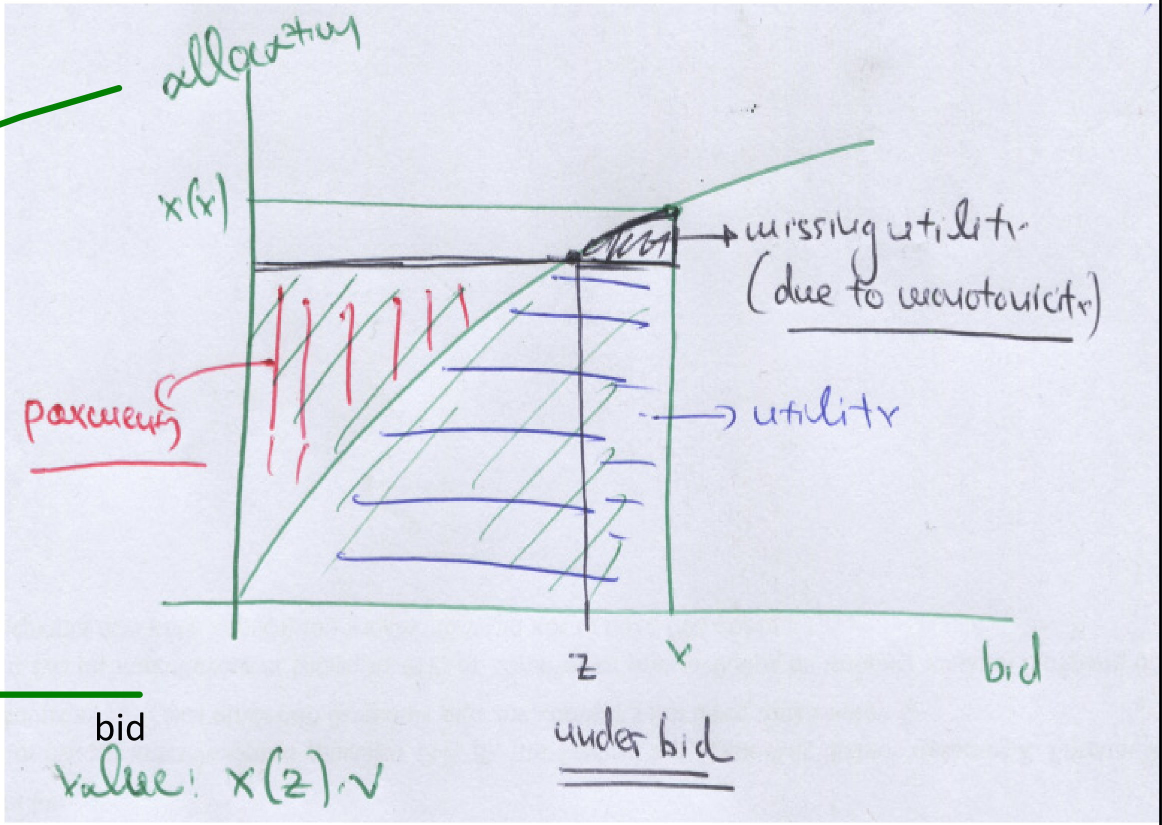

Myerson's lemma

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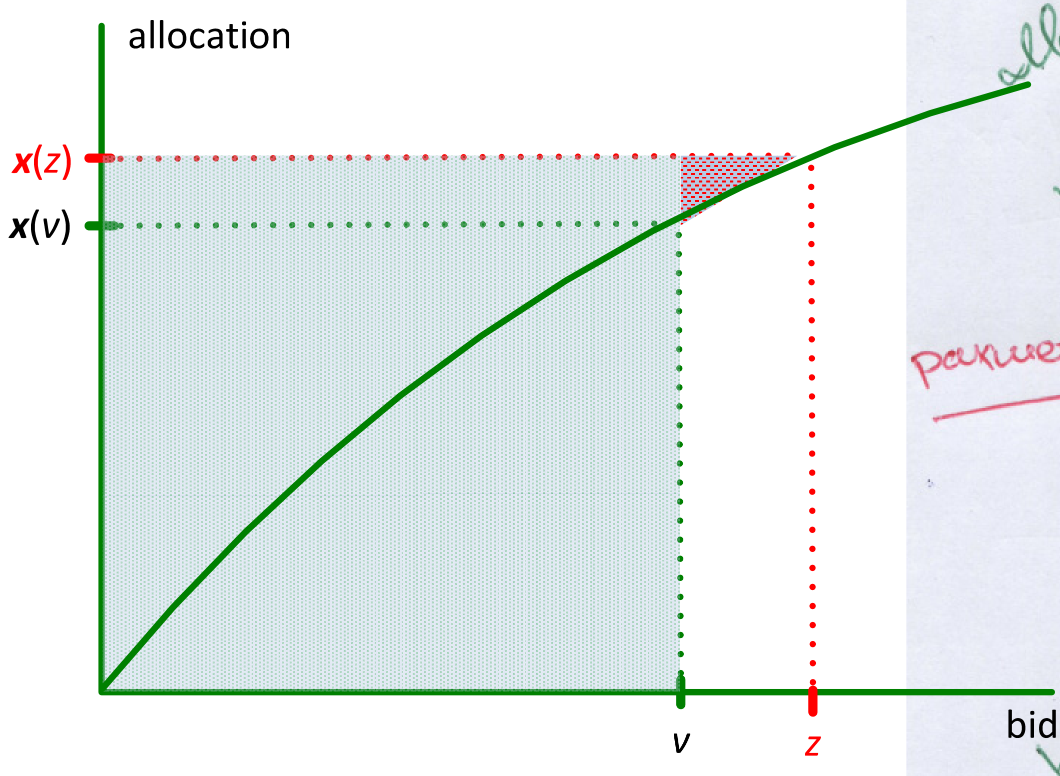

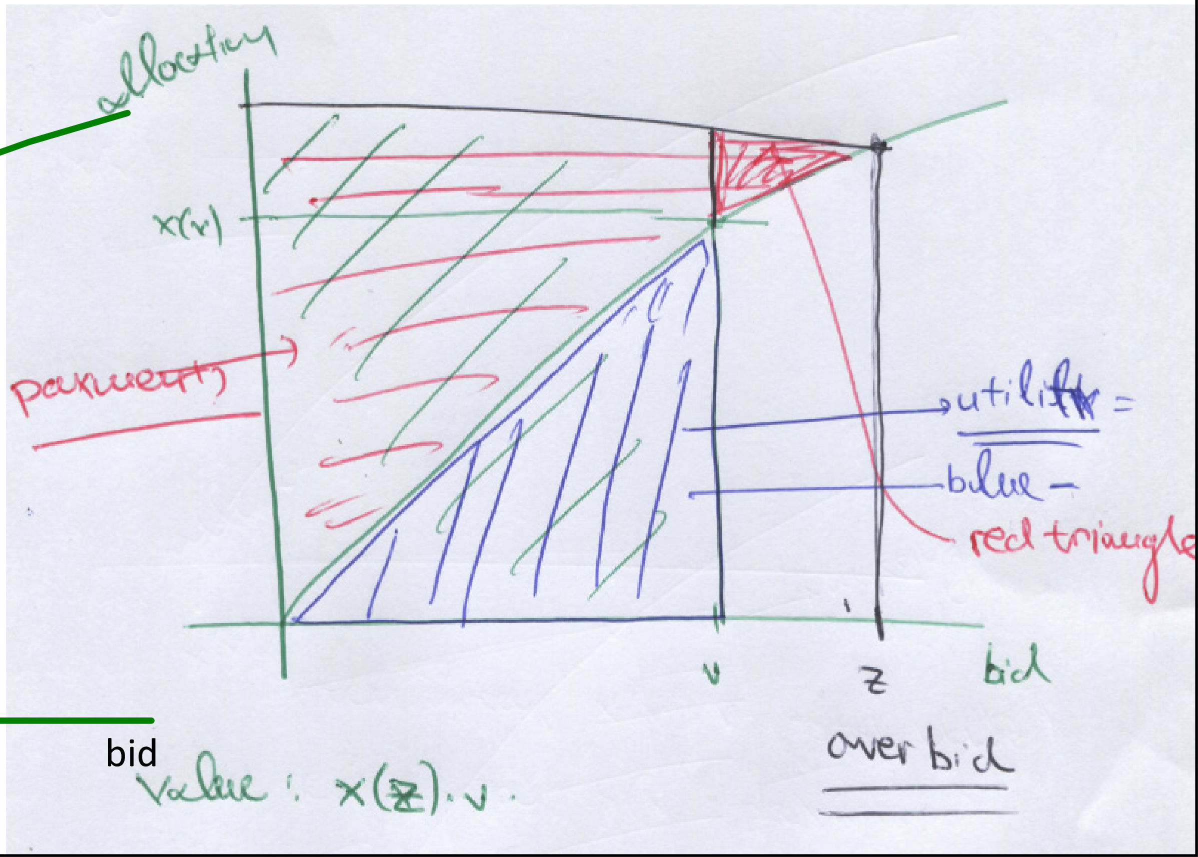

Applying Myerson's lemma

- Single-item auctions
- The allocation rule of giving the item to the highest bidder is monotone
- The payment rule of Myerson gives us precisely the Vickrey auction
  - Non-winners pay nothing: If a bidder i is not a winner, there is no jump within $[0, b_i]$ in the function $x_i(z, \mathbf{b}_{-i})$
  - The winner pays (2nd highest bid) · [jump at 2nd highest bid] = 2nd highest bid

- Corollary: The Vickrey auction is the only truthful mechanism for single-item auctions, when the winner is the highest bidder