Algorithmic Game Theory

Truthful Mechanisms
for Welfare Maximization

Vangelis Markakis
markakis@gmail.com

Designing welfare maximizing truthful auctions for single parameter environments

Single parameter auctions

- For the single-item case, we saw that the Vickrey auction is ideal
- We would like to achieve the same properties for any other type of auction
  - truthfulness and individual rationality [incentive guarantees]
  - welfare maximization [economic performance guarantees]
  - implementation in polynomial time [computational performance guarantees]
- Can we achieve all 3 properties for any single-parameter environment?

Knapsack auctions

- We will see an illustration for knapsack auctions
- k identical items for sale
- Each bidder i has a publicly known demand for w_i items
  - Inelastic demand
  - The mechanism should either give w_i items to the bidder or should not give him anything
- Each bidder i submits a bid b_i for his value per unit
- Real value per unit = v_i
- Assume the demands $(w_1, w_2, ..., w_n)$ are known to the mechanism
  - Say bidders have no incentive to lie about them
- Only private information to bidder i is v_i

Knapsack auctions

Alternative view of knapsack auctions

- The auctioneer has a resource of total capacity k (a knapsack)
- Each bidder requires size w_i , if he is served
- Each bidder has a value v_i w_i , if he is served
- The auctioneer needs to select a subset of bidders to serve so as not to exceed the capacity k

Feasible allocations:

- $(x_1, x_2, \ldots, x_n)$ with $x_i \in \{0, 1\}$, and $\sum_i w_i x_i <= k$
- Just like the feasible solutions of a knapsack problem

Knapsack auctions

Example

- Resource = the half-time break in the Champions League final
- Capacity k = total length of the break
- Each bidder corresponds to a company who wants to be advertised during the break
- The size w_i is the duration of the ad of bidder i
- The auctioneer needs to select a subset of bidders as winners and present their ads without exceeding the time capacity k

Knapsack auctions

- Let **b** = (b_1, b_2, ..., n_n) be the biding vector
- Need to decide the allocation and payment rule
- For the allocation rule:
  - Think of maximizing the social welfare
  - Then we have precisely the 0-1 Knapsack problem!

max $\sum_i b_i x_i$

s.t.

$\sum_i w_i x_i \le k$

$x_i \in \{0, 1\}$, for i =1,...,n

Knapsack auctions

Claim: The allocation rule that maximizes the social welfare is monotone

- Consider a winner and see what can happen if he increases his bid

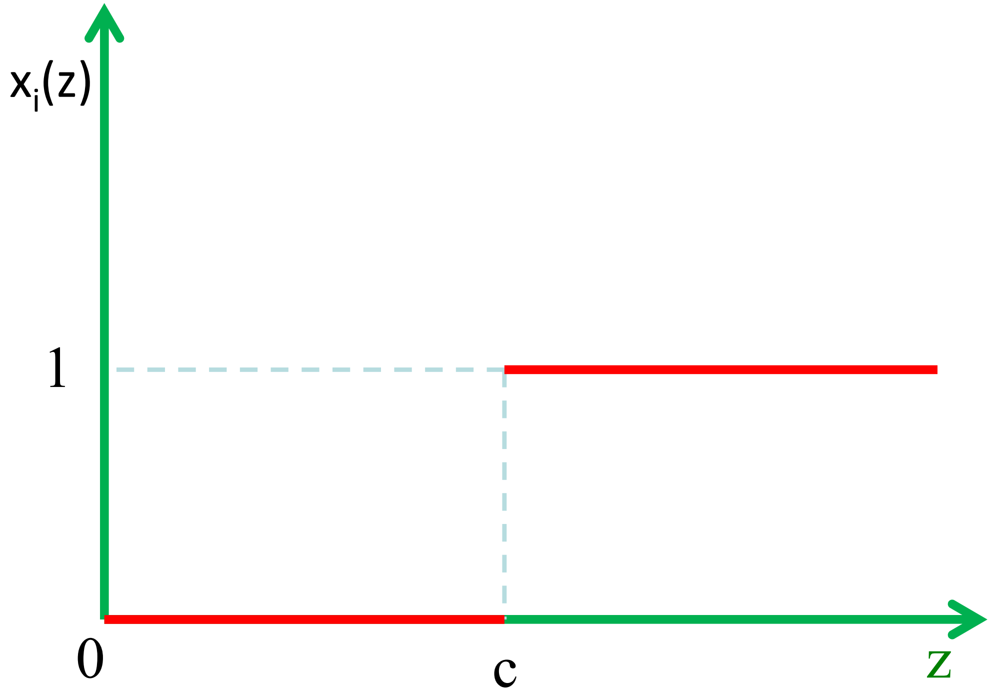
Hence, we can apply Myerson's lemma

How many jumps can we have for the allocation of a single player?

- At most one, a player can jump from being a loser ($x_i = 0$) to being a winner ($x_i = 1$)

Myerson's lemma and knapsack auctions

- The jump for a winner i happens at i's *critical bid*: the minimum he could bid and still be a winner, also known as *threshold bid*
- Generalization of the payment in Vickrey auction



Final mechanism:

- Solve the knapsack problem and find an optimal solution
- Give to each winner i, the requested number of items w_i
- Charge the winners their **critical bid**

Myerson's lemma and knapsack auctions

Does this mechanism achieve the desirable properties we wanted?

- truthfulness [YES]
- welfare maximization [YES]
- implementation in polynomial time [?]

•Knapsack is an NP-complete problem

•The properties can be enforced only for special cases where Knapsack is easy

- If highest bid or highest demand is polynomial in n (by dynamic programming)
- If weights form a super-increasing sequence

Algorithmic Mechanism Design

- The requirement for low complexity usually comes in conflict with the other criteria
- Goal of algorithmic mechanism design: explore the trade-offs between the 3 main properties (or any other properties that we may require in a given setting)
  - Truthfulness
  - welfare maximization
  - implementation in polynomial time
- Approach: relax one of the criteria and see if we can achieve the others
- For Knapsack and in general whenever welfare maximization is NP-complete: resort to approximation algorithms

Knapsack auctions

Goal for Knapsack:

- Find an approximation algorithm for the social welfare
- Prove that it is monotone

Recall:

<u>Definition:</u> An algorithm *A*, for a maximization problem, achieves an approximation factor of γ ($\gamma \leq 1$), if for every instance I of the problem, the solution returned by *A* satisfies:

$$\text{SOL(I)} \geq \gamma \text{ OPT(I)}$$

Where OPT(I) is the value of the optimal solution for instance I

Knapsack auctions

- There are several heuristics and approximation algorithms for Knapsack, but not all of them are monotone
- A greedy ½-approximation:
  - For each bidder i, we care to evaluate the quantity b_i/w_i
  - Intuitively, we prefer bidders with small size/demand and large value
- Step 1: Sort and re-index the bidders so that

$$b_1/w_1 \geq b_2/w_2 \geq \ldots \geq b_n/w_n$$

- Step 2: Pick bidders in that order until the first time that adding someone exceeds the knapsack capacity
- Step 3: Return either the previous solution, or just the highest bidder if he achieves higher social welfare on his own

Knapsack auctions

- Why do we need the last step?
- Maybe there is a bidder with a very high value, but with a large demand as well
- The algorithm may not select this bidder in the first steps
- Step 3 ensures we do not miss out such highly-valued bidders

- Claim: This algorithm is monotone

- Theorem: Using Myerson's lemma, we can have a truthful polynomial time mechanism, that produces at least 50% of the optimal social welfare

Knapsack auctions

Going further

- Knapsack also admits an FPTAS (Fully Polynomial Time Approximation Scheme)
  - We can have a (1- ε)-approximation for any constant ε >0 [Ibarra, Kim '75]
  - But this is not a monotone algorithm
- [Briest, Krysta, Voecking '05]: A truthful FPTAS for Knapsack

- **Conclusion**: For a knapsack auction and any ε >0, we have a truthful mechanism that produces at least (1 – ε)-fraction of the optimal social welfare and runs in time polynomial in n and 1/ ε

General Approach

Suppose we have a single-parameter auction where the social welfare maximization problem is NP-hard

- Check if any of the known approximation algorithms for the problem is monotone (usually not)
- If not, then try to tweak it so as to make it monotone (sometimes feasible)
- Or design a new approximation algorithm that is monotone (hopefully without worsening the approximation guarantee)

Single-minded bidders

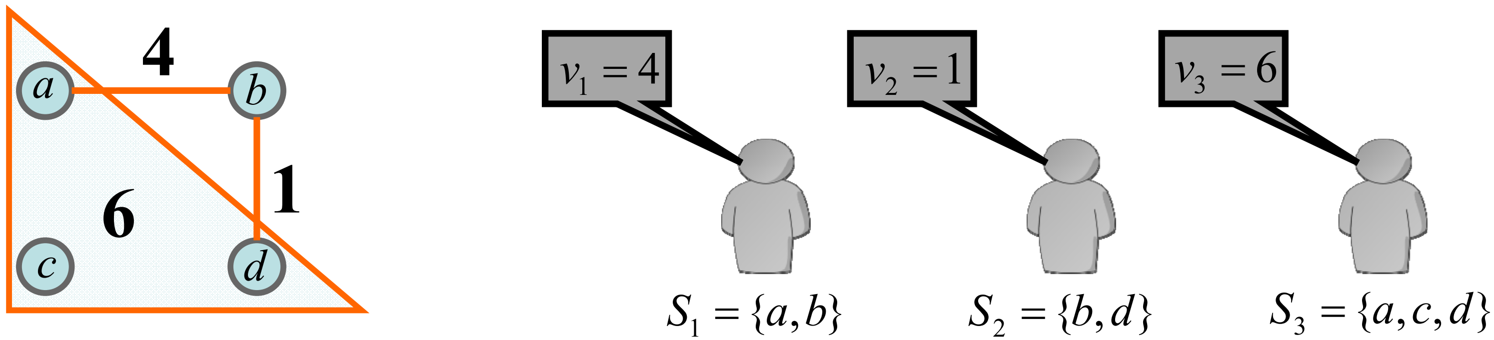
A single-paramerer auction with non-identical items

- The auctioneer has a set M of items for sale
- Each bidder i is interested in acquiring a specific subset of items, $S_i \subseteq M$ (known to the mechanism)
    - If the bidder does not obtain S_i (or a superset of it), his value is 0
- Each bidder submits a bid b_i for his value if he obtains the set

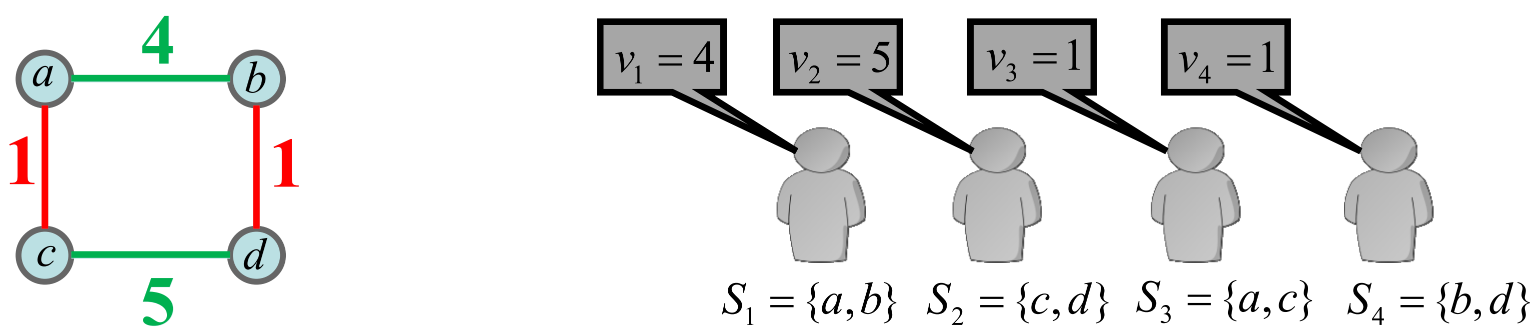
- Motivated by certain spectrum auctions
- Feasible allocations: the auctioneer needs to select winners who do not have overlapping sets

Single-minded bidders

Examples



- In the example above, the auctioneer can accept only 1 bidder as a winner
- In the example below, the auctioneer can accept up to 2 bidders as winners



Single-minded bidders

Social welfare maximization:

- Given the bids of the players, select a set of bidders with non-overlapping subsets, so as to maximize the sum of their bids
- It contains the SET PACKING problem, hence NP-hard
- Actually it gets even worse w.r.t. approximation

Theorem [Sandholm '99]: Under certain complexity theory assumptions, we cannot have an algorithm with approximation factor better than 1/sqrt(m)

Q: Can we have a 1/sqrt(m)-approximation?

Single-minded bidders

$s_i=|S_i|$

[Lehmann, O' Callaghan, Shoham '01]:

- Order the bidders in decreasing order of $b_i/\text{sqrt}(s_i)$
- Accept each bidder in this order unless overlapping with previously accepted bidders
- Payment i: largest bid b_j for set S_j with nonempty intersection with S_i .

- This algorithm achieves
  - 1/sqrt(m)-approximation, where m = |M|
  - 1/d-approximation, where $d = \max_i s_i$
  - Monotonicity and truthfulness.

Final conclusion: truthful polynomial time mechanism with the best possible approximation to the social welfare

Single-minded bidders

$s_i = |S_i|$

- Order the bidders in decreasing order of $b_i/\text{sqrt}(s_i)$
- Accept each bidder in this order unless overlapping with previously accepted bidders

- *A* algorithm's solution (set of indices accepted by Greedy)
- O optimal solution (set of indices accepted by OPT)

Wlog. assume that $O \cap A = \emptyset$.

Partition O into O_i, $i \in A$, s.t. $j \in O_i$ if $j \in O$ and $S_i \cap S_j \neq \emptyset$.

$$\sum_{j \in O_i} v_j \leq \frac{v_i}{\sqrt{s_i}} \sum_{j \in O_i} \sqrt{s_j} \qquad \text{Greedy property}$$

$$\leq \frac{v_i}{\sqrt{s_i}} \sqrt{\sum_{j \in O_i} s_j} \sqrt{|O_i|} \qquad \text{Cauchy-Schwarz ineq.}$$

$$\leq \frac{v_i}{\sqrt{s_i}} \sqrt{m} \sqrt{s_i} \qquad |O_i| \leq s_i \text{ and } \textstyle\sum_{j \in O_i} s_j \leq m$$

$$\leq v_i \sqrt{m}$$

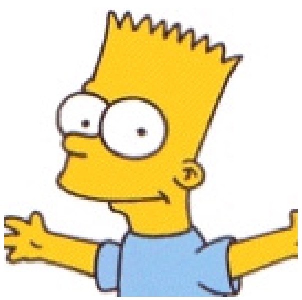
Quick Summary of Previous Lecture

- Single parameter bidders: private information of bidder i is single value v_i , expressed by bid b_i
- Myerson's Lemma: truthful mechanism iff monotone allocation, payments are uniquely determined (and virtually always easy to compute).
  - 2^{nd} price / Vickrey auction is the **only** truthful single-item auction.
  - Optimal is always monotone: if allocation problem is easy, we also get computational efficiency.
  - If allocation problem is hard, we seek monotone poly-time approximation algorithms.
  - (1-1/k)-approximation in time $O(n^{k+1})$ and FPTAS for Knapsack (with demand known).
  - Single-minded bidders / set packing: Greedy wrt $b_i/\text{sqrt}(s_i)$ is monotone and O(sqrt(m))-approximation (best possible approximation in polynomial time).

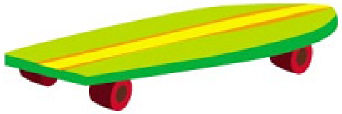
Multi-dimensional Bidders / Combinatorial Auctions

The model

Set of players
N = {1, 2, …, n}



Set of indivisible goods
M = {1, 2, …, m}



Combinatorial Auctions

- Any auction with multiple items for sale
- The players may be allowed to express interest / bids on various combinations of goods
- In practice very active field within the last 10-15 years
  - Spectrum licences
  - The FCC incentive auction:
    - https://www.fcc.gov/about-fcc/fcc-initiatives/incentive-auctions
  - Transportation routes
  - Logistics

Combinatorial auctions

- In practice, it seems economically more efficient and profitable to sell the items **together** than have a separate auction for each good
- Main challenges:
  - Algorithmic: How shall we design the allocation rule (especially if we have many overlaps in what the players want the most)?
  - Game-theoretic: Can we generalize Myerson's lemma to get truthful mechanisms?

Valuation functions

- So far we studied settings where a single parameter v_i determined all the information we needed for a player
- Most general scenario: consider that each player has a valuation function defined for every subset of the items
- $v_i : P(M) \rightarrow R$
  - where P(M) = powerset of M (all subsets of M)
  - For every $S \subseteq M$,
    - $v_i(S)$ = utility derived for player i if he acquires set S

      = maximum amount willing to pay for acquiring S
- We always assume **monotonicity** ("free-disposal"): for all $T \subseteq S$, $v_i(T) \leq v_i(S)$.

Examples of valuation functions

Additive valuation functions

- For every $S \subseteq M$, $v_i(S) = \sum_{j \in S} v_{ij}$
  - where v_{ij} = utility of acquiring item j
- Hence, the function can be completely determined by specifying the vector $(v_{i1}, v_{i2}, \ldots, v_{im})$
- m parameters for each bidder
- In such cases, the goods can be auctioned independently:
  - The value of an item is not affected by other items that a bidder may have already obtained

Examples of valuation functions

- In practice, the items may be interrelated with each other and additive valuations are not appropriate
- The value they add to a player may depend on the other items that the player has
- The items may exhibit
  - **Complementarity**: some items may be valuable only when they are sold together with other items (e.g. left and right shoe)
  - **Substitutability:** some items may be of similar type and should not be sold together to the same player (e.g. 2 cars with the same features)

Examples of valuation functions

Subadditive functions

- For any 2 disjoint subsets $S \subseteq M$, $T \subseteq M$,

$$v_i(S \cup T) \leq v_i(S) + v_i(T)$$

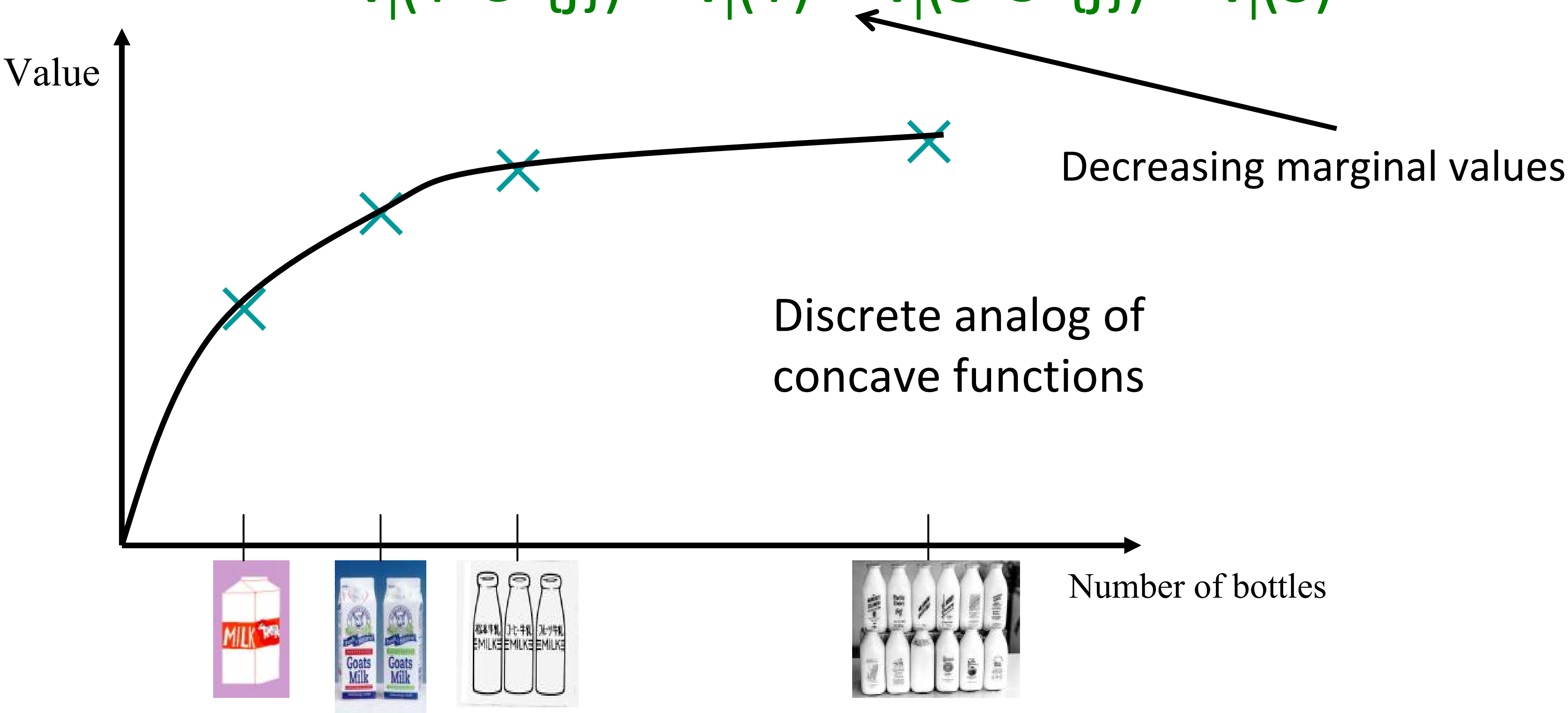
- In this case, we have substitutability among the goods
- They are also called complement-free functions (since we do not have complementarity)

Examples of valuation functions

Submodular functions

For any 2 subsets S, T, with $S \subseteq T \subseteq M$, and for every $j \notin T$

$$v_i(T \cup \{j\}) - v_i(T) \leq v_i(S \cup \{j\}) - v_i(S)$$



Examples of valuation functions

- Submodular functions form a special class of subadditive valuations
- Hence, they also do not exhibit complementarity
- They play a key role in micro-economic theory
- Expressing the fact that utility gets “saturated” as we keep allocating substitutes to the same player

Examples of valuation functions

Symmetric submodular

- Special case of submodular functions, where all goods are identical
  - Hence, the final utility depends only on how many items the player receives
- Applicable for multi-unit auctions
  - E.g., auctions for government bonds fall under this framework
- For k identical items, such functions can be represented by a vector of k marginal values
  - $(m_i(1), m_i(2), \ldots, m_i(k))$ with $m_i(j) \geq m_i(j+1)$
  - Where $m_i(j)$ = additional utility to the player for obtaining the j-th unit, if the player already has j-1 units

Examples of valuation functions

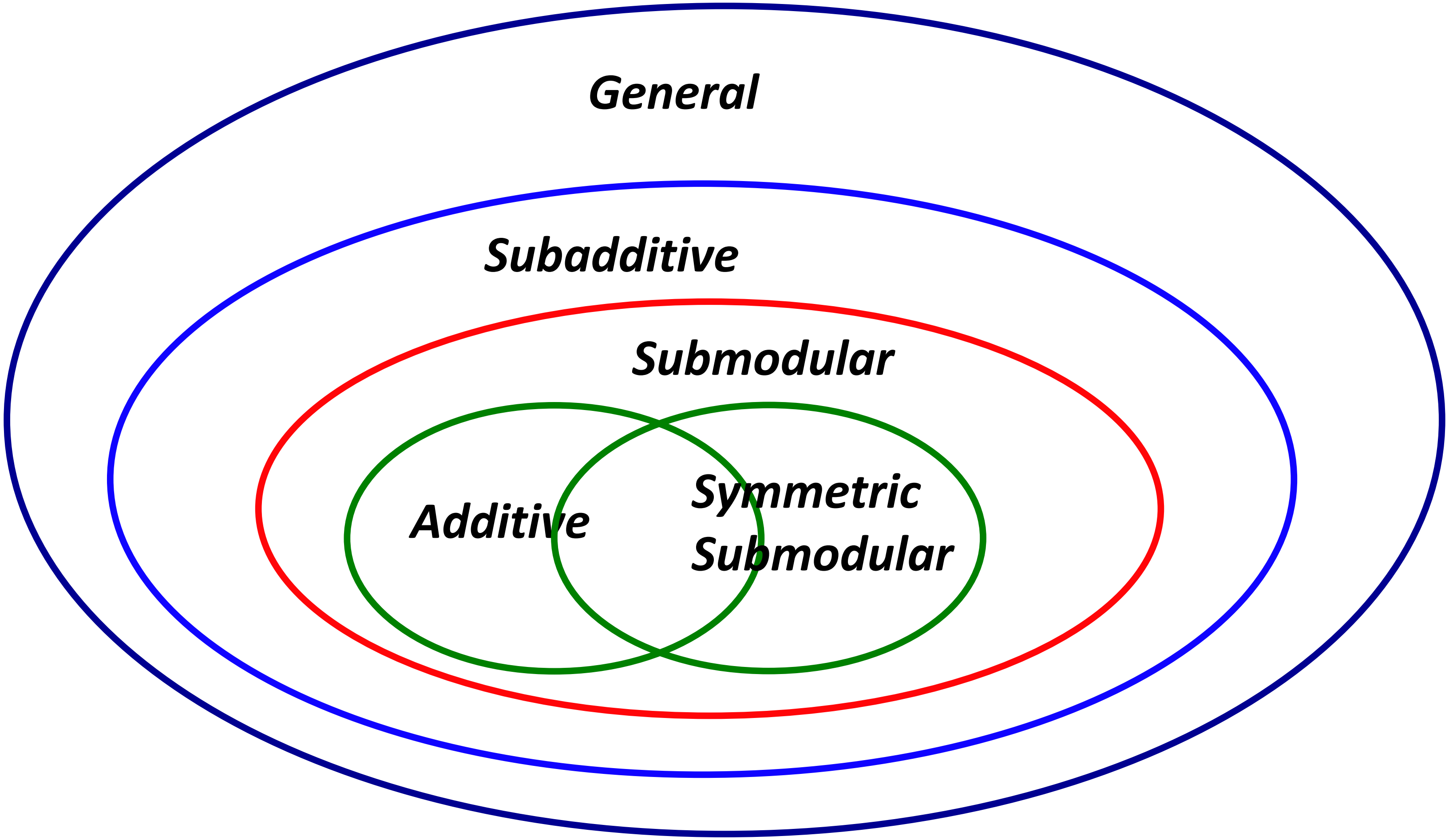
Superadditive functions

- For any 2 disjoint subsets $S \subseteq M$, $T \subseteq M$,

$$v_i(S \cup T) \geq v_i(S) + v_i(T)$$

- In this case, we have complementarity
- For example, the items may not have any value if they are sold on their own, but only when sold in bundles with other goods
  - Single-minded bidders fall under this class

Relations between different classes of valuation functions



Social Welfare Maximization

- Need to define social welfare in this more general setting
- Definition: Let **S** = (S_1, S_2, …, S_n) be an allocation of the items to the players, where S_i = subset assigned to player i. Then the social welfare derived from S is

$$SW(\mathbf{S}) = \Sigma_i \, v_i(S_i)$$

The SWM problem (Social Welfare Maximization):

Input: The valuation functions of the players (how?)

Output: Find an allocation $\mathbf{S}^*$ = (S_1 , S_2 , …, S_n) that produces the highest possible social welfare:

$$SW(\mathbf{S}^*) \geq SW(\mathbf{S}) \text{ for any other allocation } \mathbf{S}$$

Integer Programming Formulation

$$\max \sum_{i,S} x_{i,S} v_i(S)$$

$$\sum_{S} x_{i,S} \leq 1 \qquad \forall i \in [n]$$

$$\sum_{i,S:j \in S} x_{i,S} \leq 1 \qquad \forall j \in [m]$$

$$x_{i,S} \geq 0$$

$$\min \sum_{j \in [m]} p_j + \sum_{i \in [n]} u_i$$

$$u_i \geq v_i(S) - \sum_{j \in S} p_j \qquad \forall i, S$$

$$p_j \geq 0 \qquad \forall j \in [m]$$

$$u_i \geq 0 \qquad \forall i \in [n]$$

- p_j is the price of item *j* and
  u_i is the utility of bidder i

$$u_i = \max_{S} \{v_i(S) - p(S)\}$$

- Complementary slackness: in optimal solution (assuming integrality), each bidder gets a utility maximizing set and each item with positive price is allocated.
- Optimal solutions, if integral, correspond to equilibrium!

Walrasian (Competitive) Equilibrum

- **Competitive (Walrasian) equilibrium** is price vector **p** = (p_1, …, p_m) and allocation $\mathbf{S}^*$ = (S_1, …, S_m) such that
  - $v_i(S_i) - p(S_i) \geq v_i(S) - p(S)$, for any subset S of items.
  - Every item j with $p_j > 0$ is allocated.

- Example: two bidders Alice and Bob, two items x and y.
  - Alice has value 2 for x, y and x+y, 0 for empty set.
  - Bob has value 4 for x+y and 0 for anything else.
  - $p_x = p_y = 2$, Alice nothing, Bob x+y is equilibrium.
  - If Bob had value 3 for x+y and 0 for anything else, Walrasian equilibrium does **not** exist!

Walrasian (Competitive) Equilibrum

- **Competitive (Walrasian) equilibrium** is price vector **p** = $(p_1, \ldots, p_m)$ and allocation $\mathbf{S}^*$ = $(S_1, \ldots, S_m)$ such that
  - $v_i(S_i) - p(S_i) \geq v_i(S) - p(S)$, for any subset S of items.
  - Every item j with $p_j > 0$ is allocated.
- **First Welfare Theorem:** (If exists,) Walrasian equilibrium maximizes social welfare, even among fractional solutions.

For any feasible (fractional) solution $x_{i,S}$, for any bidder i,

$$v_i(S_i) - \sum_{j \in S_i} p_j \geq \sum_S x_{i,S} \left(v_i(S) - \sum_{j \in S} p_j \right) \qquad (1)$$

by first condition and because $\sum_S x_{i,S} \leq 1$.

  - We sum up (1) and observe that sums of prices cancel out, because allocations must be disjoint.

Walrasian (Competitive) Equilibrum

- **Competitive (Walrasian) equilibrium** is price vector **p** = (p_1, …, p_m) and allocation $\mathbf{S}^*$ = (S_1, …, S_m) such that
  - $v_i(S_i) - p(S_i) \geq v_i(S) - p(S)$, for any subset S of items.
  - Every item j with $p_j > 0$ is allocated.
- **Second Welfare Theorem:** If LP admits integral optimal solution, then Walrasian equilibrium exists.
  - Follows from complementary slackness conditions.
- LP admits integral optimal solution for **gross substitutes**.
  - When price for item increases, the demand for other items does not decrease.
  - Walrasian equilibrium computed by natural tatonnement process. [Kelso-Crawford, '82] Special case of discrete convexity!!!
  - http://www.inbaltalgam.com/slides/GS%20Tutorial%20Part%20I.pdf and http://www.inbaltalgam.com/slides/GS%20Tutorial%20Part%20II.pdf

Walrasian Tatonnement

- Demand correspondence:

$$D(v,p) = \{S \subseteq U : v(S) - p(S) \geq v(T) - p(T),\ \forall T \subseteq U\}$$
$$D_i(p) = \{S \subseteq U : v_i(S) - p(S) \geq v_i(T) - p(T),\ \forall T \subseteq U\}$$

An item-price ascending auction for substitutes valuations:

Initialization:

For every item $j \in M$, set $p_j \leftarrow 0$.
For every bidder i let $S_i \leftarrow \emptyset$.

Repeat

For each i, let D_i be the demand of i at the following prices:
p_j for $j \in S_i$ and $p_j + \epsilon$ for $j \notin S_i$.
If for all i $S_i = D_i$, exit the loop;
Find a bidder i with $S_i \neq D_i$ and update:

- For every item $j \in D_i \setminus S_i$, set $p_j \leftarrow p_j + \epsilon$
- $S_i \leftarrow D_i$
- For every bidder $k \neq i$, $S_k \leftarrow S_k \setminus D_i$

Finally: Output the allocation $S_1, ..., S_n$.

Mechanisms for Combinatorial Auctions

How do the players describe their valuations to auctioneer?

- For a general function, the bidder would need to specify $v_i(S)$, for every $S \subseteq M$ (2^m numbers, prohibitive!)
- Three approaches:
  1. Some functions can be described with a small number of parameters
     - E.g. additive or symmetric submodular (m parameters)
  2. The auctioneer can ask the bidders during the auction for their values on certain subsets of items
     - **Value queries.**
     - No need to know the entire function.
  3. The auctioneer computes prices and let the bidders decide on their utility maximizing set.
     - **Demand queries** – NP-hard to compute, in general.
     - No information about valuation is given to auctioneer.

Mechanisms for Combinatorial Auctions

- Truthful mechanisms for combinatorial auctions?
- Can we generalize the 2nd price auction when we have multiple items?
- We need to generalize:
  - The allocation algorithm: with 1 item, the winner was the highest bidder
    - multiple winners (with non-overlapping sets of goods), but monotonicity still necessary!
  - The payment rule: with 1 item, we offered a «discount» to the winner
    - Adjust the discount to the more general setting (and we also need a separate discount for each winner)

Social welfare maximization

Example with additive valuations

- 3 players, 4 items
- The input can be determined by a 3 x 4 array

48	**41**	11	0
35	10	**50**	5
45	20	10	**25**

- Optimal allocation: $S^* = (S_1, S_2, S_3) = (\{1, 2\}, \{3\}, \{4\})$
- Optimal social welfare: 48 + 41 + 50 + 25 = 164

The VCG mechanism

- A generalization of the Vickrey auction
- Named after [Vickrey ’61, Clarke ’71, Groves ’73]

1. **S*** = $(S_1, S_2, \ldots, S_n)$ social welfare maximizing allocation.
2. Allocation rule: For i=1, ..., n, player i receives set S_i
3. Payment rule:
    - Payment of player i: $p_i = SW_{-i}^* - \Sigma_{j \neq i} v_j(S_j)$
      where SW_{-i}^* = optimal social welfare without player i
    - Every player pays the “externality” that his presence causes to the welfare of the others
    - Utility (value – payment) of player i: $u_i = SW^* - SW_{-i}^*$
    - Every player has utility equal to the increase in the social welfare due to his presence.

The VCG mechanism

In conclusion:

- Every player receives the items specified by the optimal allocation (w.r.t. the social welfare)
- His payment is determined by the declarations of the other players, just as in the Vickrey auction

Theorem: For any valuation functions, the VCG mechanism is truthful and maximizes the social welfare

Can we implement efficiently the VCG mechanism?

-Only when we can solve the SWM problem efficiently

The VCG Mechanism Truthfulness

Fix i and $\mathbf{b}_{-i}$. When the chosen outcome $\mathbf{x}(\mathbf{b})$ is ω^*, i's utility is

$$v_i(\omega^*) - p_i(\mathbf{b}) = \underbrace{\left[v_i(\omega^*) + \sum_{j \neq i} b_j(\omega^*)\right]}_{\text{(A)}} - \underbrace{\left[\max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega)\right]}_{\text{(B)}}.$$

- Part (B) is independent of i' bid b_i (truthfulness holds for any (B) that does not depend on b_i).
  - Part (B), a.k.a. Clarke pivot rule, ensures non-positive transfers (NPT) and individual rationality (IR).
- Bidding truthfully, i.e. $b_i = v_i$, allows the mechanism to maximize part (A), which is exactly what player i wants!
  - Players' incentives fully aligned with objective of mechanism!

Implementing the VCG mechanism

How to compute the allocation and the payment rule of VCG:

- It suffices to solve n+1 instances of the SWM problem
- 1 instance with all players present to determine the allocation of the items
- n more instances with a different player absent each time (SWM with n-1 out of the initial n players)
- Final complexity: O(n) · (complexity of SWM)

Implementing the VCG mechanism

Additive valuations

- Input: n x m matrix
- Solving SWM: Easy, greedy algorithm
  - For every item j: give it to the player with the highest value
- Implementing the payment rule of VCG:
  - Easy, solve n more times SWM with 1 player absent each time

Implementing the VCG mechanism

Example with additive valuations

•3 players, 4 items

48	**41**	11	0
35	10	**50**	5
45	**20**	10	**25**

- Optimal allocation: $S^* = (S_1 , S_2 , S_3) = (\{1, 2\}, \{3\}, \{4\})$
- Optimal social welfare: 48 + 41 + 50 + 25 = 164

Implementing the VCG mechanism

Example with additive valuations

- 3 players, 4 items

48	**41**	11	0
35	10	**50**	5
45	**20**	10	**25**

Payments:

- $p_1 = SW_{-1}^* - \Sigma_{j \neq 1} v_j(S_j) = 140 – (50+25) = 65$
- $p_2 = SW_{-2}^* - \Sigma_{j \neq 2} v_j(S_j) = 125 – (89+25) = 11$
- Similarly, $p_3 = 5$

Implementing the VCG mechanism

Additive valuations

- What if we run m independent Vickrey auctions for every item separately?
- We get the same result!
- It is due to the fact that we have additive valuations (hence, the values of different items for a player are not correlated)

Corollary:

For additive valuations, the VCG mechanism is equivalent to executing an independent Vickrey auction for each item

Implementing the VCG mechanism

Submodular functions?

Good news

Theorem: The VCG mechanism can be implemented in polynomial time for **symmetric** submodular valuations

-Greedy (wrt. marginal values) allocation is optimal.

Bad news

- For general submodular valuations, SWM is NP-complete
  - Reduction from Knapsack
- The same also holds for subadditive valuations

Implementing the VCG mechanism

Submodular functions?

[Lehmann, Lehmann, Nisan ’01]: greedy, 1/2-approximation

- Fix an ordering of the goods, 1, 2, ..., m
- For j = 1, ..., m
  - ➢ Let $(S_1, S_2, \ldots, S_n)$ be the current allocation to the bidder
  - ➢ Allocate next good to the bidder with **currently highest marginal value** for this good
    - i.e., calculate $v_i(S_i \cup \{j\}) - v_i(S_i)$ for each player i
    - We measure how much extra welfare is derived by adding the good to the currently assigned bundle of a player

Implementing the VCG mechanism

Submodular functions?

- Further progress: (1 – 1/e ≈ 0.632)-approximation with value queries [Vondrak ’08]
- [Mirrokni, Schapira, Vondrak ’08]: Better approximation would require exponentially many value queries.
- Unfortunately these algorithms cannot be combined with the VCG payment formula to obtain a truthful mechanism
- Open problem to derive a truthful mechanism for submodular valuations with the best possible approximation to the social welfare

Truthful Mechanisms for Subadditive Valuations

Value Queries [Dobzinski, Nisan, Schapira 05]:

1. Query each bidder for values of all singleton sets and U.
2. Find best "matching" allocation where each bidder gets at most one good (maximum bipartite matching).
   - Complete bipartite graph with agents on the left, goods on the right, and weight $v_i(\{ j \})$ on each { agent i, good j} edge.
3. Return best of maximum "matching" and max$\{ v_i(U) \}$

- Algorithm finds optimal over **subset** of feasible allocations, that includes only "matchings" and "winner-takes-all".
  - **Maximal-in-Range** (MiR) mechanisms: optimize over a predetermined subset of feasible solutions (a.k.a. "range").
  - Allocation is optimal-in-range: truthfulness with VCG payments!
  - Range chosen to guarantee good approximation and polynomial-time optimization.

Truthful Mechanisms for Subadditive Valuations

Value Queries [Dobzinski, Nisan, Schapira 05]:

1. Query each bidder for values of all singleton sets and U.
2. Find best "matching" allocation where each bidder gets at most one good (maximum bipartite matching).
   - Complete bipartite graph with agents on the left, goods on the right, and weight $v_i(\{ j \})$ on each { agent i, good j} edge.
3. Return best of maximum "matching" and max{ $v_i(U)$ }

- Approximation ratio O(sqrt(m)) for subadditive valuations.
  - If most of OPT SW by "large sets" (cardinality $\geq$ sqrt(m), so at most sqrt(m) of them), max{ $v_i(U)$ } is sqrt(m)-approximation.
  - If most of OPT SW by "small sets" (cardinality < sqrt(m)) maximum "matching" is sqrt(m)-approximation (due to subadditivity and bound on cardinality).

Truthful Mechanisms for Subadditive Valuations

Value Queries [Dobzinski, Nisan, Schapira ‘05]:

1. Query each bidder for values of all singleton sets and U.
2. Find best “matching” allocation where each bidder gets at most one good (maximum bipartite matching).
   - Complete bipartite graph with agents on the left, goods on the right, and weight $v_i(\{ j \})$ on each { agent i, good j} edge.
3. Return best of maximum “matching” and max{ $v_i(U)$ }

- **Theorem.** MiR algorithm above is truthful with VCG payments and achieves sqrt(m)-approximation for subadditive valuations.
- Maximal-in-Distributional Range gives sqrt(m)-approximation for CAs with **general** valuations [Lavi, Swamy ’05]
  https://www.cs.princeton.edu/~smattw/Teaching/521fa17lec19.pdf
  https://www.math.uwaterloo.ca/~cswamy/papers/mechdeslp-journ.pdf

Linear Programming Relaxation of Social Welfare Maximization

$$\max \sum_{i,S} x_{i,S} v_i(S)$$

$$\sum_{S} x_{i,S} \le 1 \qquad \forall i \in [n]$$

$$\sum_{i,S:j\in S} x_{i,S} \le 1 \qquad \forall j \in [m]$$

$$x_{i,S} \ge 0$$

$$\min \sum_{j\in[m]} p_j + \sum_{i\in[n]} u_i$$

$$u_i \ge v_i(S) - \sum_{j\in S} p_j \qquad \forall i, S$$

$$p_j \ge 0 \qquad \forall j \in [m]$$

$$u_i \ge 0 \qquad \forall i \in [n]$$

- p_j is the price of item *j* and
  u_i is the utility of bidder i

$$u_i = \max_S \{v_i(S) - p(S)\}$$

$$D_i(U_i, p) = \{S \subseteq U_i : v_i(S) - p(S) \ge v_i(T) - p(T),\ \forall T \subseteq U_i\}$$

Truthful Mechanisms for Submodular Valuations

Demand Queries [Krysta, Vocking, '12]:

Algorithm 1. Overselling MPU algorithm

For each good $e \in U$ do $p_e^1 := p_0$.
For each bidder $i = 1, 2, \ldots, n$ do
  Set $S_i := D_i(U_i, p^i)$, for a suitable $U_i \subseteq U$.
  Update for each good $e \in S_i$: $p_e^{i+1} := p_e^i \cdot 2$

- Binary search in optimal prices of goods!
- **Truthful** because prices p_i do not depend on bidder i and demand queries.
- If $p_0 = \max\{ v_i(U) \} / (4m)$, Alg1 allocates ≤ $\log_2(4m)+1$ copies of each good.
  - After allocating so many copies of good e, $p_e > \max\{ v_i(U) \}$ and no player can afford it anymore.

Truthful Mechanisms for Submodular Valuations

Lemma. p_e^* denotes final price of good e. Then,

$$\text{Alg} = \sum_{i=1}^{n} v_i(S_i) \geq \sum_{e \in U} p_e^* - m p_0$$

- Approximation ratio: compare social welfare of Alg and OPT
  - We get **Alg ≥ 3OPT/8** (but with logarithmic "overselling").

Truthful Mechanisms for Submodular Valuations

Algorithm 1. Overselling MPU algorithm

For each good $e \in U$ do $p_e^1 := p_0$.
For each bidder $i = 1, 2, \ldots, n$ do
  Set $S_i := D_i(U_i, p^i)$, for a suitable $U_i \subseteq U$.
  Update for each good $e \in S_i$: $p_e^{i+1} := p_e^i \cdot 2$

- "Overselling" is fixed with oblivious rounding and sets U_i
  - U_i is the set of available goods at step i.
  - **After** the demand query $D_i(U_i, p^i)$ is answered, S_i is allocated with probability $1/\log_2(4m)$
  - Approximation ratio increases by factor $O(\log_2(4m))$ for submodular valuations.
- Demand-query truthful approximations extends to budgeted bidders and liquid welfare: $\text{LiquidValuation}_i(S) = \min\{ v_i(S), B_i \}$

Negative Cycles, Monotonicity and Truthfulness

- Consider allocation (and truthfulness) from viewpoint of single bidder (as in Myerson's Lemma, but multi-dimensional)
  - Fix allocation rule **x**, other bids $\mathbf{b}_{-i}$ and payments **p**.
  - Consider allocation **x**(b), payments **p**(b) and utility v(**x**(b)) – **p**(b) of bidder i as functions of i's bid b and i's true valuation (a.k.a. type) v.
  - We want to characterize allocation rules **x** that allow for **truthful** payments **p** (similar to Myerson's Lemma).
  - Definition of truthfulness:

    $$v(\mathbf{x}(v)) - \mathbf{p}(v) \geq v(\mathbf{x}(b)) - \mathbf{p}(b), \text{ for all types } v, b$$
  - Focus on discrete domains (finite set of types), but everything generalizes to infinite (and continuous) domains.

Negative Cycles, Monotonicity and Truthfulness

- Let D set of all possible types.
- Correspondence graph G(D, E, w) is an edge-weighted complete directed graph on D.
  - Let b and b' be two types / vertices and o = **x**(b) and o' = **x**(b') corresponding outcomes.
  - w(b, b') = b(o) – b(o') (and w(b', b) = b'(o') – b'(o)).
  - When true type b, how much bidder prefers o (outcome if he is truthful) to o' (outcome if he misreports b')
  - Payments p function of outcomes (only)!
- Allocation **x is truthful** (without payments!) iff w(b, b') $\geq$ 0, for all edges (b, b').

Negative Cycles, Monotonicity and Truthfulness

- Correspondence graph G(D, E, w).
  - Let b, b' be types and o = **x**(b), o' = **x**(b') outcomes.
  - w(b, b') = b(o) – b(o') (and w(b', b) = b'(o') – b'(o)).
- Allocation **x** admits **truthful** payments **p** : Outcomes -> R_+, if all edges (b, b') become non-negative after we apply **p**:

  $$b(o) - p(o) \geq b(o') - p(o')$$
- Allocation **x** admits **truthful** payments **p** iff G(D, E, w) does not have negative cycles!
  - Truthful payments computed by Johnson's algorithm!
- If domain D is **convex**, allocation **x** admits **truthful** payments **p** iff G(D, E, w) does not have negative 2-cycles.
  - Weak monotonicity: b(o) – b'(o) ≥ b(o') – b'(o'), for all b, b' [Zaks, Yu '05]

Research questions on the implementation of truthful mechanisms

- Find special cases where SWM is solvable in polynomial time
- Design approximation algorithms for SWM for various types of valuation functions
- General problem with approximation algorithms: they cannot always be combined with some payment rule and get a truthful mechanism
- At the end, we need to understand how truthful mechanisms look like for multi-parameter environments, esp. when SWM is difficult