Games in Normal Form

Definition: A game in normal form consists of

- A set of players $N = \{1, 2, ..., n\}$
- For every player i, a set of available strategies S^i
- For every player i, a utility function $u_i: S^1 \times ... \times S^n \rightarrow \mathbb{R}$

- Strategy profile (configuration): any vector in the form $(s_1, ..., s_n)$, with $s_i \in S^i$
  - Every profile corresponds to an outcome of the game
  - The utility function describes the benefit/happiness that a player derives from the outcome of the game

2-player games in normal form

Consider a 2-player game with finite strategy sets

- $\mathbb{N} = \{1, 2\}$
- $S^1 = \{s_1, \ldots, s_n\}$
- $S^2 = \{t_1, \ldots, t_m\}$
- Utility functions:
  $u_1 : S^1 \times S^2 \rightarrow \mathbb{R}$, $u_2 : S^1 \times S^2 \rightarrow \mathbb{R}$

- Possible strategy profiles:

$(s_1, t_1), (s_1, t_2), (s_1, t_3), \ldots, (s_1, t_m)$

$(s_2, t_1), (s_2, t_2), (s_2, t_3), \ldots, (s_2, t_m)$

...

$(s_n, t_1), (s_n, t_2), (s_n, t_3), \ldots, (s_n, t_m)$

2-player games in normal form

The utility function of each player can be described by a matrix of size n x m

- We can think of player 1 as having to select a row
- And of player 2 as having to select a column

• A finite 2-player game in normal form is defined by a pair of n x m matrices (A, B), where:

- $A_{ij} = u_1(s_i, t_j)$, $B_{ij} = u_2(s_i, t_j)$
- Player 1 is referred to as the row player
- Player 2 is referred to as the column player

2-player games in normal form

Representation in matrix form:

For brevity, we will group together the values of the matrices A, B

$u_1(s_1, t_1), u_2(s_1, t_1)$	…, …	…, …	…, …	$u_1(s_1, t_m), u_2(s_1, t_m)$
$u_1(s_2, t_1), u_2(s_2, t_1)$	…, …	…, …	…, …	…, …
		$u_1(s_i, t_j), u_2(s_i, t_j)$	…, …	…, …
		…, …	…, …	…, …
…, …	…, …	…, …	…, …	$u_1(s_n, t_m), u_2(s_n, t_m)$

Dominant strategies

- Ideally, we would like a strategy that would provide the best possible outcome, regardless of what other players choose
- Definition: A strategy s_i of pl. 1 is *dominant* if

$$u_1(s_i, t_j) \geq u_1(s', t_j)$$

for every strategy $s' \in S^1$ and **every** strategy $t_j \in S^2$

- Similarly for pl. 2, a strategy t_j is dominant if

$$u_2(s_i, t_j) \geq u_2(s_i, t')$$

for every strategy $t' \in S^2$ and for **every** strategy $s_i \in S^1$

Dominant strategies

Even better:

•<u>Definition</u>: A strategy s_i of pl. 1 is ***strictly*** *dominant* if

$$u_1 (s_i, t_j) > u_1 (s', t_j)$$

for every strategy $s' \in S^1$ and every strategy $t_j \in S^2$

•Similarly for pl. 2

•In prisoner's dilemma, strategy D (confess) is strictly dominant

Observations:

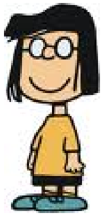
•There may be more than one dominant strategies for a player, but then they should yield the same utility under all profiles

•Every player can have at most one strictly dominant strategy

•A strictly dominant strategy is also dominant

Existence of dominant strategies

- Few games possess dominant strategies
- It may be too much to ask for
- E.g. in the BoS game, there is no dominant strategy:
  - Strategy B is not dominant for pl. 1:
    If pl. 2 chooses S, pl. 1 should choose S
  - Strategy S is also not dominant for pl. 1:
    If pl. 2 chooses B, pl. 1 should choose B



	B	S
B	(5, 1)	(0, 0)
S	(0, 0)	(1, 5)

- In all the examples we have seen so far, only prisoner’s dilemma possesses dominant strategies

Nash Equilibria

- Definition (Nash 1950): A strategy profile (s, t) is a Nash equilibrium, if no player has a unilateral incentive to deviate, given the other player's choice
- This means that the following conditions should be satisfied:
  1. $u_1(s, t) \geq u_1(s', t)$ for every strategy $s' \in S^1$
  2. $u_2(s, t) \geq u_2(s, t')$ for every strategy $t' \in S^2$

- One of the dominant concepts in game theory from 1950s till now
- Most other concepts in noncooperative game theory are variations/extensions/generalizations of Nash equilibria

Pictorially:

			t		
	(,)	(,)	(x_1,)	(,)	(,)
	(,)	(,)	(x_2,)	(,)	(,)
	(,)	(,)	(x_3,)	(,)	(,)
s	(, y_1)	(, y_2)	(x, y)	(, y_4)	(, y_5)
	(,)	(,)	(x_5,)	(,)	(,)

In order for (s, t) to be a Nash equilibrium:

- x must be greater than or equal to any x_i in column t
- y must be greater than or equal to any y_j in row s

Nash Equilibria

- We should think of Nash equilibria as “stable” profiles of a game
  - At an equilibrium, each player thinks that if the other player does not change her strategy, then he also does not want to change his own strategy
- Hence, no player would regret for his choice at an equilibrium profile (s, t)
  - If the profile (s, t) is realized, pl. 1 sees that he did the best possible, against strategy t of pl. 2,
  - Similarly, pl. 2 sees that she did the best possible against strategy s of pl. 1
- Attention: If both players decide to change simultaneously, then we may have profiles where they are both better off

Example 1: Prisoner's Dilemma

In small games, we can examine all possible profiles and check if they form an equilibrium

- (C, C): both players have an incentive to deviate to another strategy
- (C, D): pl. 1 has an incentive to deviate
- (D, C): Same for pl. 2
- (D, D): Nobody has an incentive to change

	D	C
D	5, 5	0, 15
C	15, 0	1, 1

Hence: The profile (D, D) is the unique Nash equilibrium of this game

– Recall that D is a dominant strategy for both players in this game

Corollary: If s is a dominant strategy of pl. 1, and t is a dominant strategy for pl. 2, then the profile (s, t) is a Nash equilibrium

Mixed strategies

- Definition: A mixed strategy of a player is a probability distribution on the set of his available choices
- If $S = (s_1, s_2, \dots, s_n)$ is the set of available strategies of a player, then a mixed strategy is a vector in the form $\mathbf{p} = (p_1, \dots, p_n)$, where

  $p_i \geq 0$ for $i=1, \dots, n$, and $p_1 + \dots + p_n = 1$
- p_j = probability for selecting the j-th strategy
- We can write it also as $p_j=p(s_j)$ = prob/ty of selecting s_j

Pure and Mixed strategies

- From now on, we refer to the available choices of a player as *pure strategies* to distinguish them from mixed strategies
- For 2 players with $S^1 = \{s_1, s_2, \ldots, s_n\}$ and $S^2 = \{t_1, t_2, \ldots, t_m\}$
- Pl. 1 has n pure strategies, Pl. 2 has m pure strategies
- Every pure strategy can also be represented as a mixed strategy that gives probability 1 to only a single choice
- E.g., the pure strategy s_1 can also be written as the mixed strategy (1, 0, 0, ..., 0)
- More generally: strategy s_i can be written in vector form as the mixed strategy $e^i = (0, 0, \ldots, 1, 0, \ldots, 0)$
  - 1 at position i, 0 everywhere else
  - Some times, it is convenient in the analysis to use the vector form for a pure strategy

Utility under Mixed Strategies

- Suppose that each player has chosen a mixed strategy in a game
- How does a player now evaluate the outcome of a game?
- We will assume that each player cares for his expected utility
  - Justified when games are played repeatedly
  - Not justified for more risk-averse or risk-seeking players

Expected utility (for 2 players)

- Consider a n x m game
- Pure strategies of pl. 1: $S^1 = \{s_1, s_2, \ldots, s_n\}$
- Pure strategies of pl. 2: $S^2 = \{t_1, t_2, \ldots, t_m\}$
- Let $\mathbf{p} = (p_1, \ldots, p_n)$ be a mixed strategy of pl. 1 and $\mathbf{q} = (q_1, \ldots, q_m)$ be a mixed strategy of pl. 2
- Expected utility of pl. 1:

$$u_1(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i \cdot q_j \cdot u_1(s_i, t_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} p(s_i) \cdot q(t_j) \cdot u_1(s_i, t_j)$$

- Similarly for pl. 2 (replace u_1 by u_2)

Nash equilibria with mixed strategies

- Definition: A profile of mixed strategies (**p**, **q**) is a Nash equilibrium if
  - $u_1(\mathbf{p}, \mathbf{q}) \geq u_1(\mathbf{p}', \mathbf{q})$ for any other mixed strategy **p**' of pl. 1
  - $u_2(\mathbf{p}, \mathbf{q}) \geq u_2(\mathbf{p}, \mathbf{q}')$ for any other mixed strategy **q**' of pl. 2

- Again, we just demand that no player has a unilateral incentive to deviate to another strategy

- How do we verify that a profile is a Nash equilibrium?
  - There is an infinite number of mixed strategies!
  - Infeasible to check all these deviations

Nash equilibria with mixed strategies

- Corollary: It suffices to check only deviations to pure strategies
  - Because each mixed strategy is a convex combination of pure strategies

- Equivalent definition: A profile of mixed strategies ($\mathbf{p}$, $\mathbf{q}$) is a Nash equilibrium if
  - $u_1(\mathbf{p}, \mathbf{q}) \geq u_1(e^i, \mathbf{q})$ for every pure strategy e^i of pl. 1
  - $u_2(\mathbf{p}, \mathbf{q}) \geq u_2(\mathbf{p}, e^j)$ for every pure strategy e^j of pl. 2

- Hence, we only need to check n+m inequalities as in the case of pure equilibria

2 Player Zero-Sum Game

Column player

Row player

0	-1	1
1	0	-1
-1	1	0

Strategy:
A probability distribution

Row player tries to maximize the payoff, column player tries to minimize

2 Player Zero-Sum Game

Strategy:
A probability distribution

Column player

Row player

	A(i,j)	

You have to decide your strategy first.

Is it fair??

Von Neumann Minimax Theorem

$$\max_{y \in \Delta^m} \min_{x \in \Delta^n} yAx = \min_{x \in \Delta^n} \max_{y \in \Delta^m} yAx$$

Strategy set

Which player decides first doesn't matter!

e.g. paper, scissor, rock.

Key Observation

$$\max_{y \in \Delta^m} \min_{x \in \Delta^n} yAx$$

If the row player fixes his strategy,

then we can assume that x chooses a **pure** strategy

$$\min_{x \in \Delta^n} yAx$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_i \geq 0$$

Vertex solution
is of the form
(0,0,….,1,…0),
i.e. a pure strategy

Key Observation

$$\max_{y \in \Delta^m} \min_{x \in \Delta^n} yAx = \max_{y \in \Delta^m} \min_i (yA)_i$$

similarly

$$\min_{x \in \Delta^n} \max_{y \in \Delta^m} yAx = \min_{x \in \Delta^n} \max_j (Ax)_j$$

Primal Dual Programs

$$\max_{y \in \Delta^m} \min_i (yA)_i \qquad\qquad \min_{x \in \Delta^n} \max_j (Ax)_j$$

$$\max \quad t \qquad\qquad \min \quad w$$

$x_j \rightarrow$ $$\sum_{i=1}^{m} y_i a_{ij} \geq t \qquad\qquad \sum_{j=1}^{n} a_{ij} x_j \leq w$$

$w \rightarrow$ $$\sum_{i=1}^{m} y_i = 1 \qquad \text{duality} \qquad \sum_{j=1}^{n} x_j = 1$$

$$y_i \geq 0 \qquad\qquad x_j \geq 0$$

Existence of Nash equilibria

- Theorem [Nash 1951]: Every finite game possesses at least one equilibrium when we allow mixed strategies
  - Finite game: finite number of players and finite number of pure strategies per player
- Corollary: if a game does not possess an equilibrium with pure strategies, then it definitely has one with mixed strategies
- One of the most important results in game theory
- Nash's theorem resolves the issue of non-existence
  - By allowing a richer strategy space, existence is guaranteed, no matter how big or complex the game might be

Examples

- In Prisoner's dilemma or BoS, there exist equilibria with pure strategies
  - For such games, Nash's theorem does not add any more information. However, in addition to pure equilibria, we may also have some mixed equilibria
- Matching-Pennies: For this game, Nash's theorem guarantees that there exists an equilibrium with mixed strategies
  - In fact, it is the profile we saw: ((1/2, 1/2), (1/2, 1/2))
- Rock-Paper-Scissors?
  - Again the uniform distribution: ((1/3, 1/3, 1/3), (1/3, 1/3, 1/3))

Nash Equilibria: Computation

- Nash's theorem only guarantees the existence of Nash equilibria
  - Proof reduces to using Brouwer's fixed point theorem
- Brouwer's theorem: Let $f:D \rightarrow D$, be a continuous function, and suppose D is convex and compact. Then there exists x such that $f(x) = x$
  - Many other versions of fixed point theorems also available

Nash equilibria: Computation

- So far, we are not aware of efficient algorithms for finding fixed points [Hirsch, Papadimitriou, Vavasis ’91]
  - There exist exponential time algorithms for finding approximate fixed points
- Can we design polynomial time algorithms for 2-player games?
  - After all, it seems to be only a special case of the general problem of finding fixed points