Algorithms for Data Science: Lecture 4

Vasileios Nakos

National Technical University of Athens

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Streaming algorithms

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$\star$ Compress data "on-the-fly": store a small piece of information sufficient to answer approximate answer for the data set.

Applications

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- Very large Databases, i.e. estimating the size of a join.
- Manipulation of astronomical (satellite imagery), financial data.
- GPS or seismometer readings to detect geological anomalies, sensor networks etc
- Training Machine Learning models when the training data set is huge.

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Output: Number of *distinct* elements in x

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Implementations used by Google (Sawzall, Dremel, PowerDrill), Yahoo, Twitter, Facebook Presto, etc.

Google
Big Query

F_2 moment estimation

Input: Vector $x \in \mathbb{R}^n$, updates (i, Δ) causing $x_i \leftarrow x_i + \Delta$.

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Output: An approximate answer to $\sum_{i=1}^{n} x_i^2$.

- Anomaly detection in traffic monitoring.
- Detect DoS attacks.
- Database optimization engine to estimate *self join size*.
- Subroutine in many other streaming algorithms.

Architecture of a streaming algorithm

Implements two routines, **Update()** and **Query()** using space S usually *sublinear* in the size of the input, i.e. does **not** store the whole input.

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- Distinct elements: $O(\frac{\log\log n}{\epsilon^2} + \log n)$ bits of space to estimate number of distinct elements up to $(1+\epsilon)$
- F_2 estimation: $O(\frac{\log n}{\epsilon^2})$ bits of space to estimate the F_2 moment up to $1+\epsilon$, i.e. find V such that

$$(1-\epsilon)\sum_{i=1}^{n} x_i^2 \leq V \leq (1+\epsilon)\sum_{i=1}^{n} x_i^2 .$$

Streaming algorithms are almost always *randomized* and *approximate*.

Let X be a discrete random variable that takes values on $\{\ldots, -1, 0, 1 \ldots\}$. Then

- (expectation) $\mathbb{E}(X) := \sum_{i=-\infty}^{\infty} i \cdot Pr[X = i]$
- (variance) $\mathrm{Var}(X) := \mathbb{E}(X^2) - \mathbb{E}(X)^2$

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- (Markov's inequality) For a variable X that takes only positive values we have $Pr[X \geq x] \leq \frac{\mathbb{E}(X)}{x}$.

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- (Markov's inequality) For a variable X that takes only positive values we have $Pr[X \geq x] \leq \frac{\mathbb{E}(X)}{x}$.
- (Chebyshev's inequality) $Pr[|X - \mathbb{E}(X)| \geq \lambda] \leq \frac{\mathrm{Var}(X)}{\lambda^2}$.

Chebyshev's inequality is very useful in the design of randomized algorithms, showing that an estimator concentrates around its expected value.

Question: How tight is Chebyshev's inequality?

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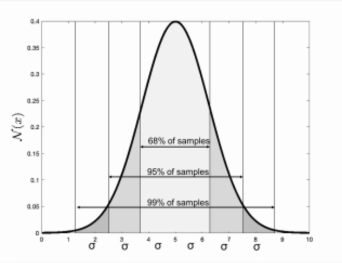

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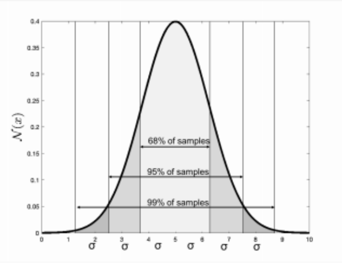

Follow the "68-95-99 rule" for Gaussian bell-curve $\mathcal{N}(0, \sigma^2)$.

- Chebyshev's inequality versus true value.
- $Pr[|X - \mathbb{E}(X)| \geq 1\sigma] \leq 100\%$ vs $Pr[|X - \mathbb{E}(X)| \geq 1\sigma] \approx 32\%$
- $Pr[|X - \mathbb{E}(X)| \geq 2\sigma] \leq 25\%$ vs $Pr[|X - \mathbb{E}(X)| \geq 1\sigma] \approx 5\%$
- $Pr[|X - \mathbb{E}(X)| \geq 3\sigma] \leq 11\%$ vs $Pr[|X - \mathbb{E}(X)| \geq 1\sigma] \approx 1\%$
- $Pr[|X - \mathbb{E}(X)| \geq 4\sigma] \leq 6\%$ vs $Pr[|X - \mathbb{E}(X)| \geq 1\sigma] \approx 0.01\%$

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Back to Distinct Elements

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Output: Number of *distinct* elements D in x. Example:
$0, 1, 0, 0, 0, 1, 2 \rightarrow 3$

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The (idealized) Flajolet-Martin algorithm:

- Choose a random hash function $h : [n] \rightarrow [0, 1]$
- $S = \infty$
- For every element e set $S \leftarrow \min\{S, h(e)\}$.
- Output $\frac{1}{S} - 1$.

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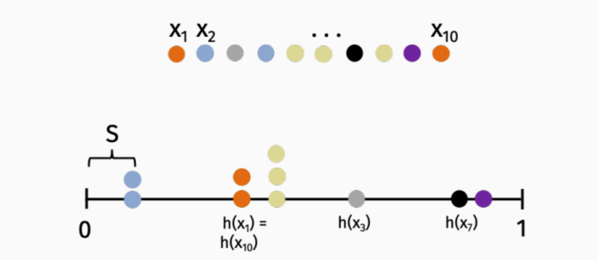
We must store S and a description of $h \ldots$

So what is V?

S is the minimum hash value ever seen so far.



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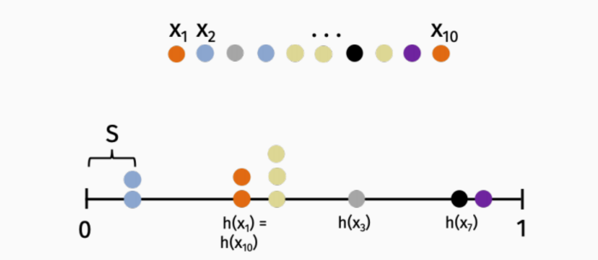

If an element comes many times, it will be always mapped to the same position. We return $\frac{1}{S} - 1$.

Lemma

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$$\mathbb{E}(S) = \int_0^1 Pr[S \geq \lambda] d\lambda =$$

$$\int_0^1 (1 - \lambda)^D d\lambda = \frac{1}{D+1}$$

We know that $\mathbb{E}(S) = \frac{1}{D+1}$. **Estimate** $\tilde{D} = \frac{1}{S} - 1$**.**

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However,

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Thus, precise estimation of D reduces to showing that S concentrates around its value $\rightarrow$ Chebyshev's inequality!

Let's compute the variance.

Lemma

$\mathrm{Var}(S) = \mathbb{E}(S^2) - \mathbb{E}(S)^2 = \frac{2}{(D+1)(D+2)} - \frac{1}{(D+1)^2} \leq \frac{1}{(D+1)^2}$.

Similarly as before,

$$\mathbb{E}(S^2) = \int_0^1 Pr[S^2 \geq \lambda] d\lambda =$$
$$\int_0^1 Pr[S \geq \sqrt{\lambda}] d\lambda =$$
$$\int_0^1 (1 - \sqrt{\lambda})^D d\lambda = \frac{2}{(D+1)(D+2)}$$

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- $\mathbb{E}(S) = \frac{1}{D+1}$
- $\mathrm{Var}(S) \leq \frac{1}{(D+1)^2}$
- $Pr[|S - \mathbb{E}(S)| \geq \epsilon\sqrt{\mathrm{Var}(S)}] \leq \frac{1}{\epsilon^2}$ (too large!)

Consider identicaly distributed random variables $S_1, S_2, \dots S_r$, and let

$$\bar{S} := \frac{1}{r}(S_1 + S_2 + \dots + S_r).$$

- $\mathbb{E}(S) = \mathbb{E}(\frac{1}{r}(S_1 + S_2 + \dots + S_r)) = \frac{1}{r}(\mathbb{E}(S)_1 + \mathbb{E}(S)_2 + \dots + \mathbb{E}(S)_r) = \mathbb{E}(S)$.

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- $\mathrm{Var}(S) = \frac{1}{r} \cdot \mathrm{Var}(S)$, since
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  2. $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$.

The boosting of independent repetitions

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Thus, in our case take $r = O(\frac{1}{\epsilon^2})$ different instantiations of the algorithm, and output the average!

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We have $\mathbb{E}(S) = \frac{1}{D+1}, \mathrm{Var}(S) < \frac{1}{(D+1)^2 r}$, so applying Chebyshev's inequality yields

$$Pr[|S - \mathbb{E}(S)| \geq \frac{\epsilon}{D+1}] \leq \frac{1}{3}.$$

Thus, with probability $\frac{2}{3}$ our estimator will satisfy

$$(1 - 4\epsilon)D \leq \widetilde{D} \leq (1 + 4\epsilon)D.$$

So we need to run the algorith with $\epsilon' := \frac{\epsilon}{4}$.

How to boost the success probability?

Idea I: For target probability δ, we can set $r = \frac{16}{\epsilon^2 \delta}$ and obtain an estimate satisfying

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with probability $1-\delta$.
Why? $\mathrm{Var}(S)$ becomes $\frac{\epsilon^2\delta}{16\cdot(D+1)^2}$, so the same analysis yields

$$Pr[|S-\mathbb{E}(S)| \geq \frac{\epsilon}{4(D+1)}] \leq \delta.$$

But we can do waaay better!

For each $t \in [2\log(1/\delta)]$, keep a distinct elements data structures D_t with $O(\frac{1}{\epsilon^2})$ counters, and let $S^{(t)}$ be the estimate produced by each data structure.

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Then the quantity

$$S := \mathrm{median}_t S^{(t)}$$

satisfies the desired inequality with probability $1 - \delta$. Good, huh?

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Space complexity (besides storing the hash functions):
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Space complexity (besides storing the hash functions): $O(\log(1/\delta) \cdot \frac{1}{\epsilon^2} \cdot \log n)$ bits or $O(\log(1/\delta)/\epsilon^2)$ words. Compare with the trivial space of $O(D)$ words.

Avoiding taking $h : [n] \to [0, 1]$

$h(x_1)$	1010010
$h(x_2)$	1001100
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In the practical version of Flajolet-Martin (HyperLogLog) we estimate distinct elements based on maximum number of trailing zeros.

$$\Pr[h(x) \text{ has } \log D \text{ trailing zeros}] = \frac{1}{D}.$$

What happens for the practical Flajolet-Martin?

Total space: $O(\log\log D/\epsilon^2 + \log D)$ for an ϵ approximation with constant probability.

Quote from "Loglog Counting of Large Cardinalities

"Using an auxiliary memory smaller than the size of this abstract, the LogLog algorithm makes it possible to estimate in a single pass and within a few percents the number of different words in the whole of Shakespeare's works." - Flajolet, Durand.

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Using HyperLogLog to approximate 1 billion distinct items to 2% accuracy can be in approximately $1.6\text{KB} = 12800$ bits!

Distinct Elements in the Distributed Setting

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Estimate spam rate: Count number of distinct subject lines in emails sent by users that have registered in the last week, in comparison to number of emails sent overall.

Good news: Answering the above query can be done in 2 seconds in Google's distributed implementations!

F_2 ESTIMATION

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Output: A value V satisfying $(1 - \epsilon) \cdot \sum_{i=1}^n x_i^2 \leq V \leq (1 + \epsilon) \cdot \sum_{i=1}^n x_i^2$.

Alon-Mattias-Szegedy (AMS) sketch

There exists an algorithm which uses $O(\frac{\log n}{\epsilon^2})$ bits of space and returns an estimator as above with constant probability.

The algorithm and proof is just a couple of lines, yet the authors received the Goedel prize for that!

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Note that $V = \sum_{i=1}^{n} \sigma(i) x_i$ and

$$V^2 = \sum_{i,j} \sigma(i)\sigma(j) x_i x_j.$$

Let $X := V^2$.

So what about $\mathbb{E}(X)$?

It holds that

$$
\begin{aligned}
\mathbb{E}(V^2) = \mathbb{E}(\sum_{i,j} \sigma(i)\sigma(j)x_i x_j) = \\
\sum_{i,j} \mathbb{E}(\sigma(i) \cdot \sigma(j)) \cdot x_i x_j = \\
\sum_{i \neq j} \mathbb{E}(\sigma(i)) \cdot \mathbb{E}(\sigma(j)) \cdot x_i x_j + \sum_i x_i^2 = \\
\sum_{i \neq j} 0 \cdot 0 \cdot x_i x_j + \sum_i x_i^2 = \sum_i x_i^2.
\end{aligned}
$$

And what about $\mathrm{Var}(X^2)$?

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \mathbb{E}(V^4) - (\mathbb{E}(V^2))^2 =$$

$$\mathbb{E}\left(\sum_{i_1,i_2,i_3,i_4} \sigma(i_1)\sigma(i_2)\sigma(i_3)\sigma(i_4)x_{i_1}x_{i_2}x_{i_3}x_{i_4}\right) =$$

$$\left(\sum_{i_1,i_2,i_3,i_4} \mathbb{E}(\sigma(i_1)\sigma(i_2)\sigma(i_3)\sigma(i_4))x_{i_1}x_{i_2}x_{i_3}x_{i_4}\right) =$$

$$\sum_{i=1}^{n} x_i^4 + \sum_{i,j} 6x_i^2x_j^2 \leq 3 \cdot \sum_{i=1}^{n} x_i^2 = 3\mathbb{E}(X).$$

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$$\sum_{i=1}^{n} x_i^4 + \sum_{i,j} 6x_i^2x_j^2 \leq 3 \cdot \sum_{i=1}^{n} x_i^2 = 3\mathbb{E}(X).$$

If we apply Chebyshev's inequality, we run into the same issue as before (too large variance), so let's take $\frac{1}{\epsilon^2}$ different estimator and average them!

AMS SKETCH

- $r := \Theta(\frac{1}{\epsilon^2})$
- $V_j \leftarrow 0$ for all $j \in [r]$
- Pick hash functions $\sigma_j : [n] \rightarrow \{-1, 1\}$ (random signs) for all $j \in [n]$

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AMS sketch

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Update time: $O(\frac{1}{\epsilon^2})$ assuming operations in happen constant time.
Query time: $O(\frac{1}{\epsilon^2})$ assuming operations happen in constant time.

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- Applications of streaming algorithms.
- Distinct elements and the power of randomness.
- Estimating the F_2 moment.

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Thank you!