Blockchain Study group: Group Notions of Knowledge & Logics for Belief

Corelab
Computation and Reasoning Laboratory

Ex. 2.37.1

$$\vdash_{\mathrm{S5C}} C_B\phi \leftrightarrow C_B C_B\phi$$

Ex. 2.37.1

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B\phi \leftrightarrow C_B C_B\phi}$$

$\varphi_1 : \ C_B\phi \rightarrow E_B C_B\ \phi$ mix, *Prop*

Ex. 2.37.1

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B\phi \leftrightarrow C_B C_B\phi}$$

φ_1 : $C_B\phi \rightarrow E_B C_B\ \phi$ mix, *Prop*

φ_2 : $C_B\ (C_B\phi \rightarrow E_B C_B\ \phi)$ *Nec* C_B

Ex. 2.37.1

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B\phi \leftrightarrow C_B C_B\phi}$$

$\varphi_1 : C_B\phi \rightarrow E_B C_B\, \phi$	mix, *Prop*
$\varphi_2 : C_B\, (C_B\phi \rightarrow E_B C_B\, \phi)$	*Nec* C_B
$\varphi_3 : C_B\, (C_B\phi \rightarrow E_B C_B\, \phi) \rightarrow C_B\, \phi \rightarrow C_B C_B\, \phi$	induction of CK

Ex. 2.37.1

$$\vdash_{S5C} \boldsymbol{C_B\phi \leftrightarrow C_B C_B\phi}$$

$\varphi_1 : C_B\phi \rightarrow E_B C_B \phi$	mix, *Prop*
$\varphi_2 : C_B (C_B\phi \rightarrow E_B C_B \phi)$	*Nec* C_B
$\varphi_3 : C_B (C_B\phi \rightarrow E_B C_B \phi) \rightarrow C_B \phi \rightarrow C_B C_B \phi$	induction of CK
$\varphi_4 : C_B \phi \rightarrow C_B C_B \phi$	*MP* 2, 3

Ex. 2.37.1

$$\vdash_{S5C} \boldsymbol{C_B\phi \leftrightarrow C_B C_B \phi}$$

$\varphi_1 : C_B\phi \rightarrow E_B C_B \phi$	mix, *Prop*
$\varphi_2 : C_B (C_B\phi \rightarrow E_B C_B \phi)$	*Nec* C_B
$\varphi_3 : C_B (C_B\phi \rightarrow E_B C_B \phi) \rightarrow C_B \phi \rightarrow C_B C_B \phi$	induction of CK
$\varphi_4 : C_B \phi \rightarrow C_B C_B \phi$	*MP* 2, 3
$\varphi_5 : C_B C_B \phi \rightarrow C_B \phi$	mix, *Prop*

Ex. 2.37.1

$$\vdash_{S5C} C_B\phi \leftrightarrow C_B C_B\phi$$

φ_1 :	$C_B\phi \rightarrow E_B C_B\,\phi$	mix, *Prop*
φ_2 :	$C_B\,(C_B\phi \rightarrow E_B C_B\,\phi)$	*Nec* C_B
φ_3 :	$C_B\,(C_B\phi \rightarrow E_B C_B\,\phi) \rightarrow C_B\,\phi \rightarrow C_B C_B\,\phi$	induction of CK
φ_4 :	$C_B\,\phi \rightarrow C_B C_B\,\phi$	*MP* 2, 3
φ_5 :	$C_B C_B\,\phi \rightarrow C_B\,\phi$	mix, *Prop*
φ_6 :	$C_B\,\phi \leftrightarrow C_B C_B\,\phi$	*Prop*

□

Ex. 2.37.3

$$\vdash_{\mathrm{S5C}} C_B\phi \leftrightarrow K_a C_B\phi$$

Ex. 2.37.3

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B\phi \leftrightarrow K_a C_B\phi}$$

φ_1 : $C_B\phi \to E_B C_B\ \phi$ mix, *Prop*

Ex. 2.37.3

$$\vdash_{\mathbf{S5C}} \boldsymbol{C_B\phi \leftrightarrow K_a C_B \phi}$$

φ_1 :	$C_B\phi \rightarrow E_B C_B\, \phi$	mix, *Prop*
φ_2 :	$E_B C_B\, \phi \rightarrow K_a C_B \phi$	*Prop*

Ex. 2.37.3

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B\phi \leftrightarrow K_a C_B\phi}$$

$\varphi_1 : \; C_B\phi \rightarrow E_B C_B\, \phi$	mix, *Prop*
$\varphi_2 : E_B C_B\, \phi \rightarrow K_a C_B\phi$	*Prop*
$\varphi_3 : \; C_B\phi \rightarrow K_a C_B\, \phi$	*HS* 1, 2

Ex. 2.37.3

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B\phi \leftrightarrow K_aC_B\phi}$$

φ_1 :	$C_B\phi \rightarrow E_BC_B\,\phi$	mix, *Prop*
φ_2 :	$E_BC_B\,\phi \rightarrow K_aC_B\phi$	*Prop*
φ_3 :	$C_B\phi \rightarrow K_aC_B\,\phi$	$HS\,1,2$
φ_4 :	$K_aC_B\phi \rightarrow C_B\phi$	axiom T

Ex. 2.37.3

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B\phi \leftrightarrow K_a C_B\phi}$$

φ_1 :	$C_B\phi \rightarrow E_B C_B\ \phi$	mix, *Prop*
φ_2 :	$E_B C_B\ \phi \rightarrow K_a C_B\phi$	*Prop*
φ_3 :	$C_B\phi \rightarrow K_a C_B\ \phi$	*HS* 1, 2
φ_4 :	$K_a C_B\phi \rightarrow C_B\phi$	axiom T
φ_5 :	$C_B\phi \leftrightarrow K_a C_B\phi$	*Prop* 3, 4

Ex. 2.37.3

$$\vdash_{S5C} \boldsymbol{C_B\phi \leftrightarrow K_a C_B\phi}$$

φ_1 :	$C_B\phi \rightarrow E_B C_B\,\phi$	mix, *Prop*
φ_2 :	$E_B C_B\,\phi \rightarrow K_a C_B\phi$	*Prop*
φ_3 :	$C_B\phi \rightarrow K_a C_B\,\phi$	*HS* 1, 2
φ_4 :	$K_a C_B\phi \rightarrow C_B\phi$	axiom *T*
φ_5 :	$C_B\phi \leftrightarrow K_a C_B\phi$	*Prop* 3, 4

□

Ex. 2.37.4

$$\vdash_{\mathrm{S5C}} \neg C_B\phi \leftrightarrow K_a\neg C_B\phi$$

Ex. 2.37.4

$$\vdash_{\mathrm{S5C}} \neg C_B\phi \leftrightarrow K_a\neg C_B\phi$$

$\varphi_1 : K_a \neg C_B \phi \rightarrow \neg C_B \phi$ axiom T

Ex. 2.37.4

$$\vdash_{S5C} \neg C_B\phi \leftrightarrow K_a\neg C_B\phi$$

$\varphi_1 : K_a \neg C_B \phi \rightarrow \neg C_B \phi$ axiom T

$\varphi_2 : C_B \phi \leftrightarrow K_a C_B \phi$ 2.37.3

Ex. 2.37.4

$$\vdash_{\mathrm{S5C}} \neg C_B\phi \leftrightarrow K_a\neg C_B\phi$$

$\varphi_1 : K_a \neg C_B \phi \rightarrow \neg C_B \phi$	axiom T
$\varphi_2 : C_B \phi \leftrightarrow K_a C_B \phi$	2.37.3
$\varphi_3 : \neg C_B \phi \rightarrow \neg K_a C_B \phi$	*Prop* 2

Ex. 2.37.4

$$\vdash_{S5C} \neg C_B\phi \leftrightarrow K_a\neg C_B\phi$$

$\varphi_1 : K_a \neg C_B \phi \rightarrow \neg C_B \phi$	axiom T
$\varphi_2 : C_B \phi \leftrightarrow K_a C_B \phi$	2.37.3
$\varphi_3 : \neg C_B \phi \rightarrow \neg K_a C_B \phi$	*Prop* 2
$\varphi_4 : \neg K_a C_B \phi \rightarrow \neg C_B \phi$	*Prop* 2

Ex. 2.37.4

$$\vdash_{S5C} \neg C_B\phi \leftrightarrow K_a\neg C_B\phi$$

$\varphi_1 : K_a \neg C_B \phi \to \neg C_B \phi$	axiom T
$\varphi_2 : C_B \phi \leftrightarrow K_a C_B \phi$	2.37.3
$\varphi_3 : \neg C_B \phi \to \neg K_a C_B \phi$	$Prop$ 2
$\varphi_4 : \neg K_a C_B \phi \to \neg C_B \phi$	$Prop$ 2
$\varphi_5 : K_a (\neg K_a C_B \phi \to \neg C_B \phi)$	Nec for K_a, 4

Ex. 2.37.4

$$\vdash_{S5C} \neg C_B\phi \leftrightarrow K_a\neg C_B\phi$$

$\varphi_1 : K_a \neg C_B \phi \rightarrow \neg C_B \phi$	axiom T
$\varphi_2 : C_B \phi \leftrightarrow K_a C_B \phi$	2.37.3
$\varphi_3 : \neg C_B \phi \rightarrow \neg K_a C_B \phi$	*Prop* 2
$\varphi_4 : \neg K_a C_B \phi \rightarrow \neg C_B \phi$	*Prop* 2
$\varphi_5 : K_a (\neg K_a C_B \phi \rightarrow \neg C_B \phi)$	*Nec* for K_a, 4
$\varphi_6 : K_a (\neg K_a C_B \phi \rightarrow \neg C_B \phi) \rightarrow K_a \neg K_a C_B \phi \rightarrow K_a \neg C_B \phi$	ax. K for K_a

Ex. 2.37.4

$$\vdash_{S5C} \neg C_B\phi \leftrightarrow K_a\neg C_B\phi$$

$\varphi_1 : K_a \neg C_B \phi \rightarrow \neg C_B \phi$	axiom T
$\varphi_2 : C_B \phi \leftrightarrow K_a C_B\phi$	2.37.3
$\varphi_3 : \neg C_B \phi \rightarrow \neg K_a C_B\phi$	$Prop$ 2
$\varphi_4 : \neg K_a C_B\phi \rightarrow \neg C_B \phi$	$Prop$ 2
$\varphi_5 : K_a (\neg K_a C_B\phi \rightarrow \neg C_B \phi)$	Nec for K_a, 4
$\varphi_6 : K_a (\neg K_a C_B\phi \rightarrow \neg C_B \phi) \rightarrow K_a \neg K_a C_B\phi \rightarrow K_a \neg C_B \phi$	ax. K for K_a
$\varphi_7 : K_a \neg K_a C_B\phi \rightarrow K_a \neg C_B \phi$	MP 5, 6

Ex. 2.37.4

$$\vdash_{\mathrm{S5C}} \neg C_B\phi \leftrightarrow K_a\neg C_B\phi$$

φ_1 :	$K_a \neg C_B \phi \to \neg C_B \phi$	axiom T
φ_2 :	$C_B \phi \leftrightarrow K_a C_B\phi$	2.37.3
φ_3 :	$\neg C_B \phi \to \neg K_a C_B\phi$	$Prop\, 2$
φ_4 :	$\neg K_a C_B\phi \to \neg C_B \phi$	$Prop\, 2$
φ_5 :	$K_a (\neg K_a C_B\phi \to \neg C_B \phi)$	Nec for K_a, 4
φ_6 :	$K_a (\neg K_a C_B\phi \to \neg C_B \phi) \to K_a \neg K_a C_B\phi \to K_a \neg C_B \phi$	ax. K for K_a
φ_7 :	$K_a \neg K_a C_B\phi \to K_a \neg C_B \phi$	$MP\, 5, 6$
φ_8 :	$\neg K_a C_B\phi \to K_a \neg K_a C_B\phi$	axiom 5

Ex. 2.37.4

$$\vdash_{S5C} \neg C_B\phi \leftrightarrow K_a\neg C_B\phi$$

$\varphi_1 : K_a \neg C_B \phi \rightarrow \neg C_B \phi$	axiom T
$\varphi_2 : C_B \phi \leftrightarrow K_a C_B\phi$	2.37.3
$\varphi_3 : \neg C_B \phi \rightarrow \neg K_a C_B\phi$	$Prop$ 2
$\varphi_4 : \neg K_a C_B\phi \rightarrow \neg C_B \phi$	$Prop$ 2
$\varphi_5 : K_a (\neg K_a C_B\phi \rightarrow \neg C_B \phi)$	Nec for K_a, 4
$\varphi_6 : K_a (\neg K_a C_B\phi \rightarrow \neg C_B \phi) \rightarrow K_a \neg K_a C_B\phi \rightarrow K_a \neg C_B \phi$	ax. K for K_a
$\varphi_7 : K_a \neg K_a C_B\phi \rightarrow K_a \neg C_B \phi$	MP 5, 6
$\varphi_8 : \neg K_a C_B\phi \rightarrow K_a \neg K_a C_B\phi$	axiom 5
$\varphi_9 : \neg C_B\phi \rightarrow K_a \neg C_B\phi$	HS 3, 8, 7

Ex. 2.37.4

$$\vdash_{S5C} \neg C_B\phi \leftrightarrow K_a\neg C_B\phi$$

$\varphi_1 : K_a \neg C_B \phi \rightarrow \neg C_B \phi$	axiom T
$\varphi_2 : C_B \phi \leftrightarrow K_a C_B\phi$	2.37.3
$\varphi_3 : \neg C_B \phi \rightarrow \neg K_a C_B\phi$	$Prop$ 2
$\varphi_4 : \neg K_a C_B\phi \rightarrow \neg C_B \phi$	$Prop$ 2
$\varphi_5 : K_a (\neg K_a C_B\phi \rightarrow \neg C_B \phi)$	Nec for K_a, 4
$\varphi_6 : K_a (\neg K_a C_B\phi \rightarrow \neg C_B \phi) \rightarrow K_a \neg K_a C_B\phi \rightarrow K_a \neg C_B \phi$	ax. K for K_a
$\varphi_7 : K_a \neg K_a C_B\phi \rightarrow K_a \neg C_B \phi$	MP 5, 6
$\varphi_8 : \neg K_a C_B\phi \rightarrow K_a \neg K_a C_B\phi$	axiom 5
$\varphi_9 : \neg C_B\phi \rightarrow K_a \neg C_B\phi$	HS 3, 8, 7
$\varphi_{10} : \neg C_B\phi \leftrightarrow K_a\neg C_B\phi$	$Prop$ 1, 9

$\square$

Ex. 2.37.2

$$\vdash_{\mathrm{S5C}} \neg C_B \phi \leftrightarrow C_B \neg C_B \phi$$

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$$\vdash_{\text{S5C}} \neg C_B\phi \leftrightarrow C_B\neg C_B\phi$$

$\varphi_1 : C_B \neg C_B \phi \rightarrow \neg C_B \phi$ mix, *Prop*

Ex. 2.37.2

$$\vdash_{\mathrm{S5C}} \neg C_B\phi \leftrightarrow C_B\neg C_B\phi$$

$\varphi_1 : C_B \neg C_B \phi \rightarrow \neg C_B \phi$ — mix, *Prop*

$\varphi_2 : \neg C_B \phi \rightarrow K_{a'} \neg C_B\phi$ — 2.37.4, *Prop*, $a' \in B$

Ex. 2.37.2

$$\vdash_{\mathrm{S5C}} \neg C_B\phi \leftrightarrow C_B\neg C_B\phi$$

$\varphi_1 : C_B \neg C_B \phi \to \neg C_B \phi$	mix, *Prop*
$\varphi_2 : \neg C_B \phi \to K_{a'}\neg C_B\phi$	2.37.4, *Prop*, $a' \in B$
$\varphi_3 : \neg C_B \phi \to E_B \neg C_B\phi$	*Prop* 2

Ex. 2.37.2

$$\vdash_{\mathrm{S5C}} \neg C_B\phi \leftrightarrow C_B\neg C_B\phi$$

$\varphi_1 : C_B \neg C_B \phi \rightarrow \neg C_B \phi$	mix, *Prop*
$\varphi_2 : \neg C_B \phi \rightarrow K_{a'} \neg C_B\phi$	2.37.4, *Prop*, $a' \in B$
$\varphi_3 : \neg C_B \phi \rightarrow E_B \neg C_B\phi$	*Prop* 2
$\varphi_4 : \neg C_B \phi \rightarrow C_B \neg C_B\phi$	*Nec*, ind., *MP*

Ex. 2.37.2

$$\vdash_{S5C} \neg C_B\phi \leftrightarrow C_B\neg C_B\phi$$

$\varphi_1 : C_B \neg C_B \phi \rightarrow \neg C_B \phi$	mix, *Prop*
$\varphi_2 : \neg C_B \phi \rightarrow K_{a'} \neg C_B\phi$	2.37.4, *Prop*, $a' \in B$
$\varphi_3 : \neg C_B \phi \rightarrow E_B \neg C_B\phi$	*Prop* 2
$\varphi_4 : \neg C_B \phi \rightarrow C_B \neg C_B\phi$	*Nec*, ind., *MP*
$\varphi_5 : \neg C_B\phi \leftrightarrow C_B\neg C_B\phi$	*Prop* 1, 4

□

2.37.5

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B\phi \to K_{a_1} \cdots K_{a_n}\phi} \quad \text{where } \{a_i\}_{[n]} \subseteq B$$

2.37.5

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B \phi \to K_{a_1} \cdots K_{a_n} \phi} \quad \text{where } \{a_i\}_{[n]} \subseteq B$$

Proposition (1)

$\vdash \varphi \to \psi \Rightarrow \vdash E_B\, \varphi \to E_B\, \psi$

2.37.5

$$\vdash_{\text{S5C}} \boldsymbol{C_B\phi \to K_{a_1}\cdots K_{a_n}\phi} \quad \text{where } \{a_i\}_{[n]} \subseteq B$$

Proposition (1)

$\vdash \varphi \to \psi \Rightarrow \vdash E_B\, \varphi \to E_B\, \psi$

Hint: $\vdash (\varphi \to \psi) \wedge (x \to y) \to (\varphi \wedge x \to \psi \wedge y)$ *& Ex. 2.18.2.*

2.37.5

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B \phi \to K_{a_1} \cdots K_{a_n} \phi} \quad \text{where } \{a_i\}_{[n]} \subseteq B$$

Proposition (1)

$\vdash \varphi \to \psi \Rightarrow \vdash E_B\, \varphi \to E_B\, \psi$

Hint: $\vdash (\varphi \to \psi) \wedge (x \to y) \to (\varphi \wedge x \to \psi \wedge y)$ *& Ex. 2.18.2.*

Proposition (2)

$\vdash E_B C_B\, \phi \to E_B^n \phi$, *where* $n \in \mathbb{N}$

2.37.5

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B \phi \to K_{a_1} \cdots K_{a_n} \phi} \quad \text{where } \{a_i\}_{[n]} \subseteq B$$

Proposition (1)

$\vdash \varphi \to \psi \Rightarrow \vdash E_B\, \varphi \to E_B\, \psi$

Hint: $\vdash (\varphi \to \psi) \wedge (x \to y) \to (\varphi \wedge x \to \psi \wedge y)$ *& Ex. 2.18.2.*

Proposition (2)

$\vdash E_B C_B\, \phi \to E_B^n \phi$, *where* $n \in \mathbb{N}$

Proposition (3)

$\vdash E_B^n \phi \to K_{a_1} \cdots K_{a_n} \phi$ *where* $\{a_i\}_{[n]} \subseteq B$

2.37.5 Cont'd

Proposition (2)

$C_B\phi \rightarrow \phi$

2.37.5 Cont'd

Proposition (2)

$$C_B\phi \rightarrow \phi \overset{(1)}{\Rightarrow} E_B C_B\phi \rightarrow E_B\phi$$

2.37.5 Cont'd

Proposition (2)

$$C_B\phi \rightarrow \phi \overset{(1)}{\Rightarrow} E_B C_B\phi \rightarrow E_B\phi$$

$$C_B\phi \rightarrow E_B C_B\phi$$

2.37.5 Cont'd

Proposition (2)

$$C_B\phi \to \phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B\phi$$

$$C_B\phi \to E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \to E_B\phi$$

2.37.5 Cont'd

Proposition (2)

$$C_B\phi \to \phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B\phi$$

$$C_B\phi \to E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \to E_B\phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B^2\phi$$

2.37.5 Cont'd

Proposition (2)

$C_B\phi \rightarrow \phi \overset{(1)}{\Rightarrow} E_B C_B \phi \rightarrow E_B \phi$

$C_B\phi \rightarrow E_B C_B \phi \overset{HS}{\Rightarrow} C_B\phi \rightarrow E_B \phi \overset{(1)}{\Rightarrow} E_B C_B \phi \rightarrow E_B^2 \phi$

$C_B\phi \rightarrow E_B C_B \phi$

2.37.5 Cont'd

Proposition (2)

$$C_B\phi \rightarrow \phi \overset{(1)}{\Rightarrow} E_B C_B\phi \rightarrow E_B\phi$$

$$C_B\phi \rightarrow E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \rightarrow E_B\phi \overset{(1)}{\Rightarrow} E_B C_B\phi \rightarrow E_B^2\phi$$

$$C_B\phi \rightarrow E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \rightarrow E_B^2\phi$$

2.37.5 Cont'd

Proposition (2)

$$C_B\phi \to \phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B\phi$$

$$C_B\phi \to E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \to E_B\phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B^2\phi$$

$$C_B\phi \to E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \to E_B^2\phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B^3\phi$$

2.37.5 Cont'd

Proposition (2)

$$C_B\phi \rightarrow \phi \overset{(1)}{\Rightarrow} E_B C_B\phi \rightarrow E_B\phi$$

$$C_B\phi \rightarrow E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \rightarrow E_B\phi \overset{(1)}{\Rightarrow} E_B C_B\phi \rightarrow E_B^2\phi$$

$$C_B\phi \rightarrow E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \rightarrow E_B^2\phi \overset{(1)}{\Rightarrow} E_B C_B\phi \rightarrow E_B^3\phi$$

$$\vdots$$

2.37.5 Cont'd

Proposition (2)

$$C_B\phi \to \phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B\phi$$

$$C_B\phi \to E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \to E_B\phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B^2\phi$$

$$C_B\phi \to E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \to E_B^2\phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B^3\phi$$

$$\vdots$$

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B\phi \to K_{a_1} \cdots K_{a_n}\phi}, \quad \text{where } \{a_i\}_{[n]} \subseteq B$$

φ_1 : $C_B\phi \to E_B C_B\,\phi$ mix, *Prop*

2.37.5 Cont'd

Proposition (2)

$$C_B\phi \to \phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B\phi$$

$$C_B\phi \to E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \to E_B\phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B^2\phi$$

$$C_B\phi \to E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \to E_B^2\phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B^3\phi$$

$$\vdots$$

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B\phi \to K_{a_1}\cdots K_{a_n}\phi}, \quad \text{where } \{a_i\}_{[n]} \subseteq B$$

$\varphi_1 : C_B\phi \to E_B C_B \phi$ — mix, *Prop*

$\varphi_2 : E_B C_B\phi \to E_B^n\phi$ — Proposition (2)

2.37.5 Cont'd

Proposition (2)

$$C_B\phi \to \phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B\phi$$

$$C_B\phi \to E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \to E_B\phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B^2\phi$$

$$C_B\phi \to E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \to E_B^2\phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B^3\phi$$

$$\vdots$$

$$\vdash_{\text{S5C}} \boldsymbol{C_B\phi \to K_{a_1}\cdots K_{a_n}\phi}, \quad \text{where } \{a_i\}_{[n]} \subseteq B$$

φ_1 : $C_B\phi \to E_B C_B\,\phi$ — mix, *Prop*

φ_2 : $E_B C_B\phi \to E_B^n\phi$ — Proposition (2)

φ_3 : $E_B^n\phi \to K_{a_1}\cdots K_{a_n}\phi$ — Proposition (3)

2.37.5 Cont'd

Proposition (2)

$$C_B\phi \to \phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B\phi$$

$$C_B\phi \to E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \to E_B\phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B^2\phi$$

$$C_B\phi \to E_B C_B\phi \overset{HS}{\Rightarrow} C_B\phi \to E_B^2\phi \overset{(1)}{\Rightarrow} E_B C_B\phi \to E_B^3\phi$$

$$\vdots$$

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B\phi \to K_{a_1}\cdots K_{a_n}\phi}, \quad \text{where } \{a_i\}_{[n]} \subseteq B$$

φ_1 :	$C_B\phi \to E_B C_B\,\phi$	mix, *Prop*
φ_2 :	$E_B C_B\phi \to E_B^n\phi$	Proposition (2)
φ_3 :	$E_B^n\phi \to K_{a_1}\cdots K_{a_n}\phi$	Proposition (3)
φ_4 :	$C_B\phi \to K_{a_1}\cdots K_{a_n}\phi$	*HS* 1, 2, 3

Ex. 2.37.6.

$$\vdash_{\mathbf{S5C}} \boldsymbol{C_B\phi \rightarrow C_{B'}\phi} \quad \text{iff } B \subseteq B'$$

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(⇐)

Ex. 2.37.6.

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B\phi \to C_{B'}\phi} \quad \text{iff } B \subseteq B'$$

$(\Leftarrow)$

φ_1 : $C_B\phi \to E_B C_B\,\phi$ — mix, *Prop*

Ex. 2.37.6.

$$\vdash_{\mathbf{S5C}} \boldsymbol{C_B\phi \rightarrow C_{B'}\phi} \quad \text{iff } B \subseteq B'$$

($\Leftarrow$)

φ_1 : $C_B\phi \rightarrow E_B C_B\, \phi$ mix, *Prop*
φ_2 : $E_B C_B\phi \rightarrow E_{B'}\, C_B\phi$ *Prop*

Ex. 2.37.6.

$$\vdash_{\mathbf{S5C}} \boldsymbol{C_B\phi \rightarrow C_{B'}\phi} \quad \text{iff } B \subseteq B'$$

$(\Leftarrow)$

φ_1 :	$C_B\phi \rightarrow E_B C_B\, \phi$	mix, *Prop*
φ_2 :	$E_B C_B\phi \rightarrow E_{B'}\, C_B\phi$	*Prop*
φ_3 :	$C_B\phi \rightarrow E_{B'}\, C_B\phi$	*HS* 1, 2

Ex. 2.37.6.

$$\vdash_{S5C} \boldsymbol{C_B\phi \rightarrow C_{B'}\phi} \quad \text{iff } B \subseteq B'$$

$(\Leftarrow)$

φ_1 :	$C_B\phi \rightarrow E_B C_B\, \phi$	mix, *Prop*
φ_2 :	$E_B C_B\phi \rightarrow E_{B'}\, C_B\phi$	*Prop*
φ_3 :	$C_B\phi \rightarrow E_{B'}\, C_B\phi$	*HS* 1, 2
φ_4 :	$C_{B'}\,(C_B\phi \rightarrow E_{B'}\, C_B\phi)$	*Nec* for $C_{B'}$

Ex. 2.37.6.

$$\vdash_{S5C} \boldsymbol{C_B\phi \to C_{B'}\phi} \quad \text{iff } B \subseteq B'$$

$(\Leftarrow)$

φ_1 :	$C_B\phi \to E_B C_B\ \phi$	mix, *Prop*
φ_2 :	$E_B C_B\phi \to E_{B'}\ C_B\phi$	*Prop*
φ_3 :	$C_B\phi \to E_{B'}\ C_B\phi$	*HS* 1, 2
φ_4 :	$C_{B'}\ (C_B\phi \to E_{B'}\ C_B\phi)$	*Nec* for $C_{B'}$
φ_5 :	$C_B\phi \to C_{B'}\ C_B\phi$	ind., *MP*

Ex. 2.37.6.

$$\vdash_{\mathbf{S5C}} \boldsymbol{C_B\phi \rightarrow C_{B'}\phi} \quad \text{iff } B \subseteq B'$$

($\Leftarrow$)

$\varphi_1 : \ C_B\phi \rightarrow E_B C_B\ \phi$	mix, $Prop$
$\varphi_2 : E_B C_B\phi \rightarrow E_{B'}\ C_B\phi$	$Prop$
$\varphi_3 : \ C_B\phi \rightarrow E_{B'}\ C_B\phi$	$HS\ 1, 2$
$\varphi_4 : \ C_{B'}\ (C_B\phi \rightarrow E_{B'}\ C_B\phi)$	Nec for $C_{B'}$
$\varphi_5 : \ C_B\phi \rightarrow C_{B'}\ C_B\phi$	ind., MP
$\varphi_6 : \ C_B\phi \rightarrow \phi$	mix, $Prop$

Ex. 2.37.6.

$$\vdash_{\mathrm{S5C}} \boldsymbol{C_B\phi \rightarrow C_{B'}\phi} \quad \text{iff } B \subseteq B'$$

$(\Leftarrow)$

φ_1 :	$C_B\phi \rightarrow E_B C_B\,\phi$	mix, *Prop*
φ_2 :	$E_B C_B\phi \rightarrow E_{B'}\, C_B\phi$	*Prop*
φ_3 :	$C_B\phi \rightarrow E_{B'}\, C_B\phi$	*HS* 1, 2
φ_4 :	$C_{B'}\,(C_B\phi \rightarrow E_{B'}\, C_B\phi)$	*Nec* for $C_{B'}$
φ_5 :	$C_B\phi \rightarrow C_{B'}\, C_B\phi$	ind., *MP*
φ_6 :	$C_B\phi \rightarrow \phi$	mix, *Prop*
φ_7 :	$C_{B'}\, C_B\phi \rightarrow C_{B'}\phi$	*Nec*, distr., *MP*

Ex. 2.37.6.

$$\vdash_{\mathbf{S5C}} \boldsymbol{C_B\phi \rightarrow C_{B'}\phi} \quad \text{iff } B \subseteq B'$$

$(\Leftarrow)$

φ_1 :	$C_B\phi \rightarrow E_B C_B\,\phi$	mix, $Prop$
φ_2 :	$E_B C_B\phi \rightarrow E_{B'}\,C_B\phi$	$Prop$
φ_3 :	$C_B\phi \rightarrow E_{B'}\,C_B\phi$	$HS\ 1,2$
φ_4 :	$C_{B'}\,(C_B\phi \rightarrow E_{B'}\,C_B\phi)$	Nec for $C_{B'}$
φ_5 :	$C_B\phi \rightarrow C_{B'}\,C_B\phi$	ind., MP
φ_6 :	$C_B\phi \rightarrow \phi$	mix, $Prop$
φ_7 :	$C_{B'}\,C_B\phi \rightarrow C_{B'}\phi$	Nec, distr., MP
φ_8 :	$C_B\phi \rightarrow C_{B'}\phi$	$HS\ 5,7$

□

Ex. 2.37.6.

$$\vdash_{\mathbf{S5C}} \boldsymbol{C_B\phi \to C_{B'}\phi} \quad \text{iff } B \subseteq B'$$

$(\Leftarrow)$

φ_1 :	$C_B\phi \to E_B C_B\, \phi$	mix, *Prop*
φ_2 :	$E_B C_B\phi \to E_{B'}\, C_B\phi$	*Prop*
φ_3 :	$C_B\phi \to E_{B'}\, C_B\phi$	*HS* 1, 2
φ_4 :	$C_{B'}\, (C_B\phi \to E_{B'}\, C_B\phi)$	*Nec* for $C_{B'}$
φ_5 :	$C_B\phi \to C_{B'}\, C_B\phi$	ind., *MP*
φ_6 :	$C_B\phi \to \phi$	mix, *Prop*
φ_7 :	$C_{B'}\, C_B\phi \to C_{B'}\phi$	*Nec*, distr., *MP*
φ_8 :	$C_B\phi \to C_{B'}\phi$	*HS* 5, 7

□

$(\Rightarrow)$..?

Ex. 2.39

Axiom D : $\neg B_a\,(\phi \wedge \neg\phi)$

Axiom D' : $B_a\phi \rightarrow \neg B_a \neg\phi$

Ex. 2.39

Axiom D : $\neg B_a (\phi \wedge \neg \phi)$
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Axioms D, D' are equivalent with respect to K

Ex. 2.39

Axiom D : $\neg B_a (\phi \wedge \neg\phi)$
Axiom D' : $B_a\phi \rightarrow \neg B_a\neg\phi$

Axioms D, D' are equivalent with respect to K

$\vdash_{\mathrm{KD}} \boldsymbol{D'}$

φ_1 : $\neg B_a (\phi \wedge \neg\phi)$ axiom D

Ex. 2.39

Axiom D : $\neg B_a (\phi \wedge \neg\phi)$

Axiom D' : $B_a\phi \rightarrow \neg B_a \neg\phi$

Axioms D, D' are equivalent with respect to K

$$\vdash_{\mathrm{KD}} \boldsymbol{D'}$$

φ_1 : $\neg B_a (\phi \wedge \neg\phi)$ axiom D

φ_2 : $\neg B_a (\phi \wedge \neg\phi) \rightarrow \neg (B_a\, \phi \wedge B_a\, \neg\phi)$ 2.18.4, *Prop*

Ex. 2.39

Axiom D : $\neg B_a (\phi \wedge \neg\phi)$
Axiom D' : $B_a\phi \rightarrow \neg B_a\neg\phi$

Axioms D, D' are equivalent with respect to K

$$\vdash_{\mathrm{KD}} \boldsymbol{D'}$$

φ_1 : $\neg B_a (\phi \wedge \neg\phi)$ axiom D
φ_2 : $\neg B_a (\phi \wedge \neg\phi) \rightarrow \neg (B_a\, \phi \wedge B_a\, \neg\phi)$ 2.18.4, $Prop$
φ_3 : $B_a\phi \rightarrow \neg B_a\neg\phi$ $MP\ 1, 2,\ Prop$

$\square$

Ex. 2.39 Cont'd

$$\vdash_{\mathbf{K+D'}} D$$

Ex. 2.39 Cont'd

$$\vdash_{\mathbf{K+D'}} D$$

$\varphi_1 : \; B_a\phi \rightarrow \neg B_a\neg\phi$ axiom D'

Ex. 2.39 Cont'd

$$\vdash_{\mathbf{K+D'}} \boldsymbol{D}$$

$\varphi_1 : \ B_a\phi \rightarrow \neg B_a\neg\phi$	axiom D'
$\varphi_2 : \neg(B_a\phi \wedge B_a\neg\phi)$	*Prop* 1

Ex. 2.39 Cont'd

$$\vdash_{\mathbf{K+D'}} \boldsymbol{D}$$

$\varphi_1 : B_a\phi \rightarrow \neg B_a \neg \phi$	axiom D'
$\varphi_2 : \neg (B_a\phi \wedge B_a \neg \phi)$	*Prop* 1
$\varphi_3 : \neg (B_a\phi \wedge B_a \neg \phi) \rightarrow \neg B_a (\phi \wedge \neg \phi)$	axiom K'', *Prop*

Ex. 2.39 Cont'd

$$\vdash_{\mathbf{K+D'}} \boldsymbol{D}$$

φ_1 :	$B_a\phi \rightarrow \neg B_a\neg\phi$	axiom D'
φ_2 :	$\neg(B_a\phi \wedge B_a\neg\phi)$	$Prop$ 1
φ_3 :	$\neg(B_a\phi \wedge B_a\neg\phi) \rightarrow \neg B_a(\phi \wedge \neg\phi)$	axiom K'', $Prop$
φ_4 :	$\neg B_a(\phi \wedge \neg\phi)$	MP 3, 4

□

Ex. 2.40

$$\vdash_{\mathrm{T}} \boldsymbol{D}$$

Ex. 2.40

$$\vdash_{\mathrm{T}} \boldsymbol{D}$$

According to 2.39, it is sufficient to show that

$$\vdash_{\mathrm{T}} \boldsymbol{D}'$$

Ex. 2.40

$$\vdash_{\mathrm{T}} \boldsymbol{D}$$

According to 2.39, it is sufficient to show that

$$\vdash_{\mathrm{T}} \boldsymbol{D'}$$

$\varphi_1 : \ \phi \to \neg B_a \neg\phi$ axiom T, $Prop$

Ex. 2.40

$$\vdash_{\mathrm{T}} \boldsymbol{D}$$

According to 2.39, it is sufficient to show that

$$\vdash_{\mathrm{T}} \boldsymbol{D'}$$

$\varphi_1 : \ \phi \to \neg\, B_a \neg\phi$ axiom T, $Prop$
$\varphi_2 : B_a\, \phi \to \phi$ axiom T

Ex. 2.40

$$\vdash_{\mathrm{T}} \boldsymbol{D}$$

According to 2.39, it is sufficient to show that

$$\vdash_{\mathrm{T}} \boldsymbol{D'}$$

$\varphi_1 : \ \phi \rightarrow \neg\, B_a\, \neg\phi$	axiom T, $Prop$
$\varphi_2 : B_a\, \phi \rightarrow \phi$	axiom T
$\varphi_3 : B_a\, \phi \rightarrow \neg\, B_a\, \neg\phi$	$HS\, 1, 2$

$\square$

Ex. 2.40 Cont'd

$$\nvdash_{\mathrm{KD}} \boldsymbol{T}$$

$\vdash_{\mathrm{KD}} T$

Ex. 2.40 Cont'd

$$\nvdash_{\mathrm{KD}} \boldsymbol{T}$$

$\vdash_{\mathrm{KD}} T \Rightarrow$
$\mathcal{KD} \models T$

Ex. 2.40 Cont'd

$$\nvdash_{\text{KD}} \boldsymbol{T}$$

$\vdash_{\text{KD}} T \Rightarrow$
$\mathcal{KD} \models T \Rightarrow$
$\forall M \in \mathcal{KD} \quad M \models T$

Ex. 2.40 Cont'd

$$\nvdash_{\mathrm{KD}} \boldsymbol{T}$$

$\vdash_{\mathrm{KD}} T \Rightarrow$

$\mathcal{KD} \models T \Rightarrow$

$\forall M \in \mathcal{KD} \quad M \models T \overset{2.12+}{\Rightarrow}$

$\mathcal{KD} \subseteq \mathcal{T}$

Ex. 2.40 Cont'd

$$\nvdash_{KD} \boldsymbol{T}$$

$\vdash_{KD} T \Rightarrow$
$\mathcal{KD} \models T \Rightarrow$
$\forall M \in \mathcal{KD} \quad M \models T \overset{2.12+}{\Rightarrow}$
$\mathcal{KD} \subseteq \mathcal{T}$ $\quad \rightarrow\!\leftarrow$

Ex. 2.41

$$\vdash_{\mathrm{KD45}} \neg B_a(\phi \wedge \neg B_a \phi)$$

Ex. 2.41

$$\vdash_{\mathrm{KD45}} \neg \boldsymbol{B_a} (\boldsymbol{\phi} \wedge \neg \boldsymbol{B_a \phi})$$

$$\varphi_1 : \neg (B_a\, \phi \wedge B_a \neg B_a\, \phi) \rightarrow \neg B_a\, (\phi \wedge \neg B_a\, \phi) \quad \text{axiom } K'', \mathit{Prop}$$

Ex. 2.41

$$\vdash_{\mathrm{KD45}} \neg \boldsymbol{B_a} (\boldsymbol{\phi} \wedge \neg \boldsymbol{B_a \phi})$$

$\varphi_1 : \neg (B_a\ \phi \wedge B_a \neg B_a\ \phi) \to \neg B_a\ (\phi \wedge \neg B_a\ \phi)$ axiom K'', *Prop*

$\varphi_2 : (B_a\ \phi \to \neg B_a \neg B_a\ \phi) \to \neg (B_a\ \phi \wedge B_a \neg B_a\ \phi)$ *Prop*

Ex. 2.41

$$\vdash_{\mathrm{KD45}} \neg B_a (\phi \wedge \neg B_a \phi)$$

$\varphi_1 : \neg(B_a\, \phi \wedge B_a \neg B_a\, \phi) \rightarrow \neg B_a\, (\phi \wedge \neg B_a\, \phi)$ axiom K'', *Prop*
$\varphi_2 : (B_a\, \phi \rightarrow \neg B_a \neg B_a\, \phi) \rightarrow \neg(B_a\, \phi \wedge B_a \neg B_a\, \phi)$ *Prop*
$\varphi_3 : (B_a\, \phi \rightarrow \neg B_a \neg B_a\, \phi) \rightarrow \neg B_a\, (\phi \wedge \neg B_a\, \phi)$ *HS* 1, 2

Ex. 2.41

$$\vdash_{\mathrm{KD45}} \neg \boldsymbol{B_a} (\boldsymbol{\phi} \wedge \neg \boldsymbol{B_a \phi})$$

$$\begin{array}{lr}
\varphi_1 : \neg(B_a\,\phi \wedge B_a \neg B_a\,\phi) \rightarrow \neg B_a\,(\phi \wedge \neg B_a\,\phi) & \text{axiom } K'', Prop \\
\varphi_2 : (B_a\,\phi \rightarrow \neg B_a \neg B_a\,\phi) \rightarrow \neg(B_a\,\phi \wedge B_a \neg B_a\,\phi) & Prop \\
\varphi_3 : (B_a\,\phi \rightarrow \neg B_a \neg B_a\,\phi) \rightarrow \neg B_a\,(\phi \wedge \neg B_a\,\phi) & HS\,1,2 \\
\varphi_4 : B_a\,\phi \rightarrow B_a B_a\,\phi & \text{axiom } 4
\end{array}$$

Ex. 2.41

$$\vdash_{\mathrm{KD45}} \neg B_a (\phi \wedge \neg B_a \phi)$$

φ_1 :	$\neg (B_a\, \phi \wedge B_a \neg B_a\, \phi) \rightarrow \neg B_a\, (\phi \wedge \neg B_a\, \phi)$	axiom K'', *Prop*
φ_2 :	$(B_a\, \phi \rightarrow \neg B_a \neg B_a\, \phi) \rightarrow \neg (B_a\, \phi \wedge B_a \neg B_a\, \phi)$	*Prop*
φ_3 :	$(B_a\, \phi \rightarrow \neg B_a \neg B_a\, \phi) \rightarrow \neg B_a\, (\phi \wedge \neg B_a\, \phi)$	$HS\, 1, 2$
φ_4 :	$B_a\, \phi \rightarrow B_a B_a\, \phi$	axiom 4
φ_5 :	$B_a B_a\, \phi \rightarrow \neg B_a \neg B_a\, \phi$	axiom D'

Ex. 2.41

$$\vdash_{\mathrm{KD45}} \neg B_a (\phi \wedge \neg B_a \phi)$$

φ_1 :	$\neg(B_a\ \phi \wedge B_a \neg B_a\ \phi) \rightarrow \neg B_a\ (\phi \wedge \neg B_a\ \phi)$	axiom K'', *Prop*
φ_2 :	$(B_a\ \phi \rightarrow \neg B_a \neg B_a\ \phi) \rightarrow \neg(B_a\ \phi \wedge B_a \neg B_a\ \phi)$	*Prop*
φ_3 :	$(B_a\ \phi \rightarrow \neg B_a \neg B_a\ \phi) \rightarrow \neg B_a\ (\phi \wedge \neg B_a\ \phi)$	$HS\ 1, 2$
φ_4 :	$B_a\ \phi \rightarrow B_a B_a\ \phi$	axiom 4
φ_5 :	$B_a B_a\ \phi \rightarrow \neg B_a \neg B_a\ \phi$	axiom D'
φ_6 :	$B_a\ \phi \rightarrow \neg B_a \neg B_a\ \phi$	$HS\ 5, 6$

Ex. 2.41

$$\vdash_{\mathrm{KD45}} \neg B_a(\phi \wedge \neg B_a \phi)$$

φ_1 : $\neg(B_a\,\phi \wedge B_a\,\neg\, B_a\,\phi) \rightarrow \neg\, B_a\,(\phi \wedge \neg\, B_a\,\phi)$ axiom K'', $Prop$

φ_2 : $(B_a\,\phi \rightarrow \neg\, B_a\,\neg\, B_a\,\phi) \rightarrow \neg(B_a\,\phi \wedge B_a\,\neg\, B_a\,\phi)$ $Prop$

φ_3 : $(B_a\,\phi \rightarrow \neg\, B_a\,\neg\, B_a\,\phi) \rightarrow \neg\, B_a\,(\phi \wedge \neg\, B_a\,\phi)$ $HS\, 1, 2$

φ_4 : $B_a\,\phi \rightarrow B_a B_a\,\phi$ axiom 4

φ_5 : $B_a B_a\,\phi \rightarrow \neg\, B_a\,\neg\, B_a\,\phi$ axiom D'

φ_6 : $B_a\,\phi \rightarrow \neg\, B_a\,\neg\, B_a\,\phi$ $HS\, 5, 6$

φ_7 : $\neg\, B_a\,(\phi \wedge \neg\, B_a\,\phi)$ $MP\, 6, 3$

□