Mini-Course: “Games, Dynamics and Learning”

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Follow the Regularized Leader

The algorithm of Follow the Regularized Leader is defined by the round-by-round recursive rule

$$
\begin{aligned}
X_{i,n} &= Q_i(Y_{i,n}) \\
Y_{i,n+1} &= Y_{i,n} + \gamma_n \hat{v}_{i,n}
\end{aligned}
\tag{FTRL}
$$

- $Q_i : Y_i \to X_i$ denotes the "choice map" of player i N.
- $\gamma_n > 0$ is a "learning rate" parameter such that $\sum_n \gamma_n = \infty$.
- $\hat{v}_{i,n}$ is a "payoff signal" that provides an estimate for the mixed payoffs of player i at stage n.

Regularization

The second component of FTRL is the choice map

$$Q_i(y_i) = \arg\max_{x_i \quad X_i} \{ \; y_i, x_i \quad - h_i(x_i)\}.$$

In the above, each player's ***regularizer*** $h_i : X_i \; \to R$ is defined as $h_i(x_i) = \sum_{\alpha_i \quad A_i} \theta_i(x_i)$ for some "kernel function" $\theta_i : [0,1] \to R$ with the following properties:

(*i*) θ_i is *continuous* on $[0,1]$;

(*ii*) C^2-smooth on $(0,1]$; and

(*iii*) $\inf_{[0,1]} \theta_i \; > 0$.

Examples

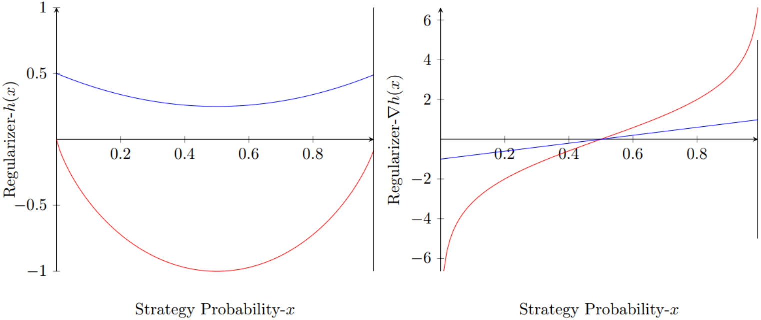
- Negative Shannon Entropy: $h(x) = \sum_i x_i \log(x_i)$
  - Exponential/Multiplicative Weight Updates

$$\Lambda_i(y) = \exp(y_i) / \sum_j \exp(y_j)$$

- Euclidean Regularizer: $h(x) = \sum_i {x_i}^2/2$
  - Euclidean Projection

$$\Pi(y) = \arg\min_{x \in \Delta} \|y - x\|^2$$

Dichotomy of regularizers



$$\begin{bmatrix} \text{steep} & h_1(x) & = & x\log(x) & + & (1-x)\log(1-x) \\ \text{non-steep} & h_2(x) & = & \frac{1}{2}x^2 & + & \frac{1}{2}(1-x)^2 \end{bmatrix}$$

The feedback model

We assume a "black-box" model for players' payoff vector of the form

$$\hat{v}_n = v(X_n) + Z_n \tag{1}$$

for some abstract error process $Z_n = (Z_{i,n})_{i \quad N}$.
We will further decompose Z_n as $Z_n = U_n + b_n$, where

- Random (zero-mean) error: $\mathrm{E}[U_n \,|\, F_n] = 0$.
- Systematic error: $b_n = \mathrm{E}[Z_n \,|\, F_n]$.

with F_n denoting the history of X_n up to stage n (inclusive).

Assumptions

We may then characterize the input signal $\hat{v}_n$ by means of the following statistics:

1. *Bias:* $\qquad \mathrm{E}[\; b_n \quad | \, F_n] \leq B_n$
2. *Variance:* $\qquad \mathrm{E}[\; U_n \;\; ^2 \, | \, F_n] \leq M_n^2$

In the above, B_n and M_n represent deterministic bounds on the bias and variance of the feedback signal $\hat{v}_n$.

Assumptions

For concreteness, we will also make the following blanket assumptions:

1. *Bias control:* $\lim_{n\to\infty} B_n = 0$ and $\sum_n \gamma_n B_n < \infty$.
2. *Variance control:* $\sum_n \gamma_n^2 M_n^2 < \infty$.
3. *Generic observation errors at equilibrium:* For every mixed Nash equilibrium x of Γ and for all $n = 0, 1, \ldots,$ there exists a player i N and strategies a, b $\operatorname{supp}(x_i)$ such that

   $$P(|\hat{v}_{ia,n} - \hat{v}_{ib,n}| \geq \beta | F_n) > 0 \quad \text{for all sufficiently small } \beta > 0.$$

Examples

Model 1 - Oracle based feedback

I At each round n, every player i N picks an action $\alpha_{i,n}$ A_i based on $X_{i,n}$ X_i.

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- At each round n, every player $i \in N$ picks an action $\alpha_{i,n} \in A_i$ based on $X_{i,n} \in X_i$.
- An oracle reveals to each player the pure payoff vector $v_i(\alpha_n) \equiv (u_i(\alpha_i; \alpha_{-i,n}))_{\alpha_i \in A_i}$.

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- An oracle reveals to each player the pure payoff vector $v_i(\alpha_n) \equiv (u_i(\alpha_i; \alpha_{-i,n}))_{\alpha_i \quad A_i}$.
- Then the player's feedback signal is $\hat{v}_{i,n} = v_i(\alpha_n)$.

Model 1 - Oracle based feedback

Special case of our general model with

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- Assumption for bias is trivial because $\mathsf{E}[\hat{v}_n \,|\, F_n] = \mathsf{E}_{X_n}[v(\alpha_n)] = v(X_n)$, i.e., $b_n = 0$.

Model 1 - Oracle based feedback

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- Assumption for noise is satisfied as long as $\sum_n \gamma_n^2 < \infty$, since $U_n \leq 2 \max_X v(X)$.

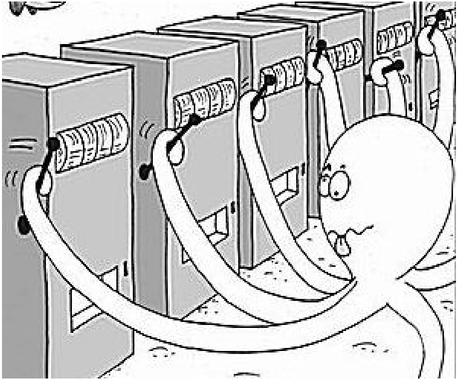
Model 1 - Oracle based feedback

Special case of our general model with

- (A1) is trivial because $\mathsf{E}[\hat{v}_n \,|\, F_n] = \mathsf{E}_{X_n}[v(\alpha_n)] = v(X_n)$, i.e., $b_n = 0$.
- (A2) is satisfied as long as $\sum_n \gamma_n^2 < \infty$, since $U_n \leq 2 \max_X v(X)$.
- (A3) is an immediate consequence of genericity. Otherwise, the game should have pure Nash equilibria.

Model 2 - Payoff based feedback (Bandit)

Google Ads



Bandit Case

- At each round n, every player $i \in N$ picks an action $\alpha_{i,n} \in A_i$ based on $X_{i,n} \in X_i$.

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Bandit Case

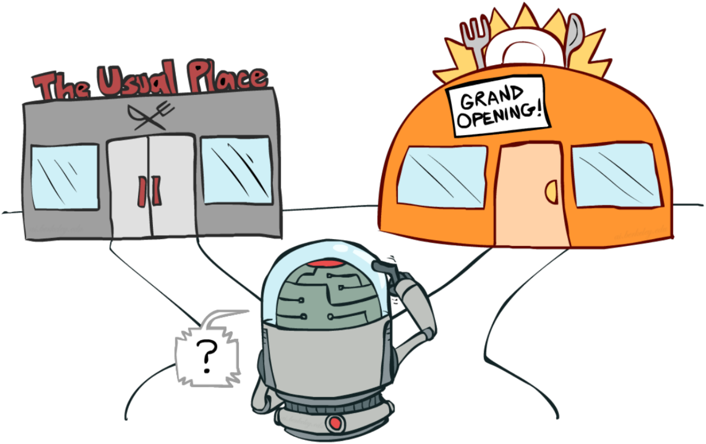
- At each round n, every player $i \quad N$ picks an action $\alpha_{i,n} \quad A_i$ based on $X_{i,n} \quad X_i$.
- Players observe their realized payoffs $u_i(\alpha_{i,n}, \alpha_{-i,n})$
- Players need to somehow estimate their payoffs!

Importance Weighted Estimator

$$\hat{v}_{ia,n} = \begin{cases} 0 & , \text{ if } a = a_{i,n} \\ \dfrac{u_i(a; a_{-i,n})}{x_{ia,n}} & , \text{ if } a = a_{i,n} \end{cases}$$

- I Unbiased: $E[\hat{v}_{i,n}] = v_i(X_n)$
- I Unbounded Variance: $E[\ (\hat{v}_{i,n})^2 \,|\, F_n] \quad \dfrac{1}{\min x_{ia,n}}$

Exploitation-Exploration



Let's leave our options open...

FTRL-exploration

- Idea: We do not limit from the beginning other options, we regularize the probabilities with an exploitation parameter that goes to zero in the infinity.

$$
\begin{aligned}
Y_{ia,n+1} &= Y_{ia,n} + \gamma_n \hat{v}_{ia,n} \\
X_{i,n} &= \arg\max_{X \quad \Delta(A_i)} \{ \; Y_{i,n}, X \quad - h_i(X) \} \\
\hat{X}_{i,n} &= (1 - \epsilon_n) X_{i,n} + \frac{\epsilon_n}{A_i}
\end{aligned}
$$

- Unbiased: $\mathrm{E}[\hat{v}_{i,n}] = v_i(\hat{X}_n)$
- Bounded Variance: $\mathrm{E}[\; U_{i,n} \;^2 \,|\, F_n] \qquad \dfrac{1}{\min \hat{X}_{ia,n}} = O(1/\epsilon_n)$
- Bias: $b_n \quad = \quad v(\hat{X}_n) - v(X_n) \quad = O(\varepsilon_n)$

The Bandit Case

- (A1) is satisfied as long as $\varepsilon_n \to 0$ and $\sum_n \gamma_n \varepsilon_n < \infty$.
- (A2) is satisfied $\sum_n \gamma_n^2 \varepsilon_n^{-1} < \infty$.
- (A3) is an immediate consequence of genericity. Otherwise, the game should have pure Nash equilibria.

Asymptotic Stability

A point x $\quad$ X is said to be

1. *Stochastically stable* under (FTRL): If for all $\delta > 0$ and all neighborhoods U of x $\quad$ there exists open set of initial conditions W_0 $\quad$ Y such that

$$\mathrm{P}(X_n \quad U \text{ for all } n = 0, 1, \ldots) \geq 1 - \delta$$

   whenever Y_0 $\quad$ W_0.

2. *Stochastically attracting* under (FTRL): If for all $\delta > 0$, there exists open set of initial conditions W_0 $\quad$ Y such that

$$\mathrm{P}(\lim_{n \to \infty} X_n = x \quad) \geq 1 - \delta$$

   whenever Y_0 $\quad$ W_0.

3. *Stochastically asymptotically stable* under (FTRL): if it is stochastically stable and attracting.

Main Results

Main Theorem. Suppose that Assumptions 1–3 hold.
Then:
x is a strict Nash equilibrium x is stochastically asymptotically stable under (FTRL)

Main Results

Theorem

Let x X be a strict Nash equilibrium of Γ. If (FTRL) is run with inexact payo feedback satisfying Assumptions 1 and 2, then x is stochastically asymptotically stable.

Theorem

Let x be a mixed Nash equilibrium of Γ. If (FTRL) is run with inexact payo feedback satisfying assumption 3, then x is not stochastically asymptotically stable.

Proof techniques - Instability

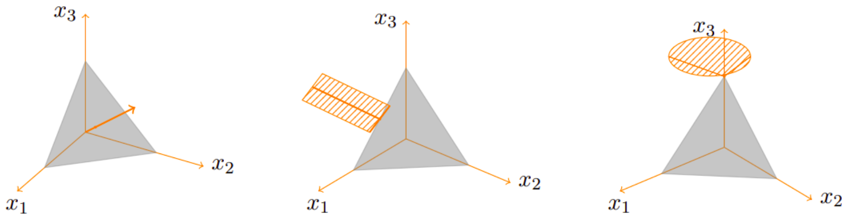


Figure: Polar cone

1. $x = Q(y) \Leftrightarrow y \quad \partial h(x)$
2. $\partial h(x) = \quad h(x) + \mathrm{PC}(x)$ for all $x \quad X$,
   where $PC(x) = \{y \quad Y : y_a \geq y_b$ for all $a, b \quad A\}$.

Proof techniques - Instability

Lemma (Informal)

Let $X_{i,n}$ be the sequence of play in (FTRL) i.e., $X_{i,n} = Q(Y_{i,n})$ X_i of player i N; and for some round $n \geq 0$ let a, b supp$(X_{i,n})$ be two pure strategies of player i N. Then it holds:

$$(\theta_i(X_{ia,n+1}) - \theta_i(X_{ia,n})) - (\theta_i(X_{ib,n+1}) - \theta_i(X_{ib,n})) = \gamma_n(\hat{v}_{ia,n} - \hat{v}_{ib,n})$$

Proof techniques - Instability

Lemma (Informal)
Let $X_{i,n}$ be the sequence of play in (FTRL) i.e., $X_{i,n} = Q(Y_{i,n})$ X_i of player i N; and for some round $n \geq 0$ let a, b $\text{supp}(X_{i,n})$ be two pure strategies of player i N. Then it holds:

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- Assume ad absurdum that a mixed Nash equilibrium x is stochastically asymptotically stable. Since x is mixed, there exist a, b $\text{supp}(x)$.

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Let $X_{i,n}$ be the sequence of play in (FTRL) i.e., $X_{i,n} = Q(Y_{i,n})$ X_i of player i N; and for some round $n \geq 0$ let a, b $\text{supp}(X_{i,n})$ be two pure strategies of player i N. Then it holds:

$$(\theta_i(X_{ia,n+1}) - \theta_i(X_{ia,n})) - (\theta_i(X_{ib,n+1}) - \theta_i(X_{ib,n})) = \gamma_n(\hat{v}_{ia,n} - \hat{v}_{ib,n})$$

- Assume ad absurdum that a mixed Nash equilibrium x is stochastically asymptotically stable. Since x is mixed, there exist a, b $\text{supp}(x)$.
- The stochastic stability implies that for all $\varepsilon, \delta > 0$ if X_0 belongs to an initial neighborhood U_ε, then $X_n - x$ $< \varepsilon$ for all $n \geq 0$, with probability at least $1 - \delta$.

Proof techniques - Instability

- By the triangle inequality for two consecutive instances of the sequence of play $X_{i,n}, X_{i,n+1}$ for any player $i \quad N$ it holds:

  $|X_{ia,n+1} - X_{ia,n}| + |X_{ib,n+1} - X_{ib,n}| < O(\varepsilon)$ with probability $1 - \delta$

Proof techniques - Instability

- By the triangle inequality for two consecutive instances of the sequence of play $X_{i,n}, X_{i,n+1}$ for any player $i \quad N$ it holds:

  $$|X_{ia,n+1} - X_{ia,n}| + |X_{ib,n+1} - X_{ib,n}| < O(\varepsilon) \text{ with probability } 1-\delta$$

- Consider ε sufficiently small, such that the probabilities of the strategies that belong to the support of the equilibrium are bounded away from 0, for all the points of the neighborhood. Since θ_i is continuously differentiable in $(0, 1]$, the differences decribed in the lemma above are bounded from $O(\varepsilon)$.

Proof techniques - Instability

- If the sequence of play X_n is contained to an $\varepsilon-$neighborhood of x , then the difference of the feedback, for any player i N, to two strategies of the equilibrium is $O(\varepsilon/\gamma_n)$ with probability at least $1 - \delta$:

$$P(|\hat{v}_{ia,n} - \hat{v}_{ib,n}| = O(\varepsilon/\gamma_n) \mid F_n) \geq 1 - \delta$$

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$$P(|\hat{v}_{ia,n} - \hat{v}_{ib,n}| = O(\varepsilon/\gamma_n) \mid F_n) \geq 1 - \delta$$

- From assumption 3 for a fixed round n and some player i N, there exist $\beta, \pi > 0$ such that: $P(|\hat{v}_{ia,n} - \hat{v}_{ib,n}| \geq \beta \mid F_n) = \pi > 0$.

Proof techniques - Instability

- If the sequence of play X_n is contained to an $\varepsilon-$neighborhood of x , then the difference of the feedback, for any player i N, to two strategies of the equilibrium is $O(\varepsilon/\gamma_n)$ with probability at least $1 - \delta$:

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- From assumption 3 for a fixed round n and some player i N, there exist $\beta, \pi > 0$ such that: $P(|\hat{v}_{ia,n} - \hat{v}_{ib,n}| \geq \beta \mid F_n) = \pi > 0$.
- Thus by choosing $\varepsilon = O(\beta\gamma_n)$ and $\delta = \pi/2$, we obtain a contradiction and our proof is complete.

Nash equilibria - reminder

A point x is a *Nash equilibrium* of Γ if

$$u_i(x\ ) \geq u_i(x_i; x_{-i}) \quad \text{for all } x_i \quad X_i \text{ and all } i \quad N\,. \qquad \text{(NE)}$$

We call support of x the set: $\mathrm{supp}(x_i\ ) = \{\alpha_i \quad A_i : x_{i\alpha_i} > 0\}$.
Equivalently, Nash equilibria can be characterized by means of the variational inequality

$$v_{i\alpha_i^{□}}(x\ ) \geq v_{i\alpha_i}(x\ ) \quad \text{for all } \alpha_i \quad \mathrm{supp}(x_i\ ) \text{ and all } \alpha_i \quad A_i,\ i \quad N\,.$$

Proof techniques - Stability

- Let $x \;\; = (\alpha_1, \ldots, \alpha_N) \quad A$ be a strict Nash equilibrium. Then for every $\varepsilon \quad (0,1)$, there exist constants $M_{i,\varepsilon}$ and the corresponding score-dominant open sets for each player $i \quad N$ such that: $\prod_{i \;\; N} Q_i(W_i(M_{i,\varepsilon})) \quad U_\varepsilon$, where $U_\varepsilon = \{x \quad X : x_{i\alpha_i^\square} > 1 - \varepsilon$ for all $i \quad N\}$ and

$$W_i(M_i) = \{Y_i : Y_{i\alpha_i^\square} - Y_{i\alpha_i} > M_i \text{ for all } \alpha_i = \alpha_i, \alpha_i \quad A_i\}$$

for each player $i \quad N$

Proof techniques - Stability

- Fix a confidence level $\delta > 0$, focus on one player $i \quad N$ and drop the index i for simplicity; consider a neighborhood U of x that can be described as the one above and for which $u_\alpha(X) - u_{\alpha\square}(X) \leq -c$ for some $c > 0$, for all $\alpha = \alpha \;, \alpha \quad A_i$ and all $X \quad U$.
- We will prove by induction that there exists an open set of initial conditions W_0, such that whenever $Y_0 \quad W_0$ then $Y_n \quad W$ for all $n = 0, 1, \ldots$.

- Notice that whenever $X \quad U$, the payoffs belong to the set $W = Q^{-1}(U)$. Furthermore, the payoff differences $Y_\alpha - Y_{\alpha'}$ between every pure strategy $\alpha \quad A_i$, $\alpha = \alpha$ and the strategy of the equilibrium α can be expressed as

$$
\begin{aligned}
Y_{\alpha,n+1} - Y_{\alpha',n+1} = & Y_{\alpha,0} - Y_{\alpha',0} + \sum_{k=0}^{n} \gamma_k (u_\alpha(X_k) - u_{\alpha'}(X_k)) \\
& + \sum_{k=0}^{n} \gamma_k Noise_k + \sum_{k=0}^{n} \gamma_k Bias_k
\end{aligned}
$$

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$$Y_{\alpha,n+1} - Y_{\alpha^{□},n+1} = Y_{\alpha,0} - Y_{\alpha^{□},0} + \sum_{k=0}^{n} \gamma_k (u_\alpha(X_k) - u_{\alpha^{□}}(X_k)) + \sum_{k=0}^{n} \gamma_k Noise_k + \sum_{k=0}^{n} \gamma_k Bias_k$$

- Using martingale limit theory we control the terms $\sum_{k=0}^{n} \gamma_k Noise_k$, $\sum_{k=0}^{n} \gamma_k Bias_k$ as to be less than $\varepsilon_1 = \sqrt{2 \sum_{k=0}^{\infty} \gamma_k^2 M_k^2 / \delta}$, $\varepsilon_2 = 2 \sum_{k=0}^{\infty} \gamma_k B_k / \delta$ equivalently with probability at least $1 - \delta$.

- Let $R_n = \sum_{k=0}^{n} \gamma_k (U_{\alpha,k} - U_{\alpha^*,k})$, which is a martingale.
- Consider the event $D_{n,\varepsilon_1} = \{\sup_{0 \le k \le n} R_k \ge \varepsilon_1\}$, then

$$P(D_{n,\varepsilon_1}) \le \frac{E[R_n^2]}{{\varepsilon_1}^2} \le \frac{2 \sum_{k=0}^{n} \gamma_k^2 M_k^2}{\varepsilon_1^2}$$

- Notice that

$$E[R_n^2] = \sum_{k=0}^{n} \gamma_k^2 E[|U_{\alpha,k} - U_{\alpha^*,k}|^2] \le 2 \sum_{k=0}^{n} \gamma_k^2 E[\| U_k \|^2]$$
$$= 2 \sum_{k=0}^{n} \gamma_k^2 E[E[\| U_k \|^2 \,|\, F_k]] \le 2 \sum_{k=0}^{n} \gamma_k^2 M_k^2$$

and $E[U_{a,k} U_{b,l}] = E[E[U_{a,k} U_{b,l} \,|\, F_{k \vee l}]] = 0$ for all $k \neq l$ and a, b be either of the pure strategy α and the strategy of the equilibrium α^*, due to the noise being zero-mean.

- Let $\Gamma_1 = 2\sum_{k=0}^{\infty} \gamma_k^2 M_k^2$ and choose $\varepsilon_1 = \sqrt{2\Gamma_1/\delta}$.
- The event $D_{\varepsilon_1} = \quad {}_{n=0}^{\infty} D_{\varepsilon_1,n}$ will happen with probability at most $\delta/2$.

- Notice that

$$|\sum_{k=0}^{n} \gamma_k (b_{\alpha,k} - b_{\alpha\square,k})| \leq \sum_{k=0}^{n} \gamma_k |b_{\alpha,k} - b_{\alpha\square,k}| \leq 2 \sum_{k=0}^{n} \gamma_k \; b_k$$

- Let $S_n = 2 \sum_{k=0}^{n} \gamma_k \; b_k$, which is a submartingale.
- If $E_{n,\varepsilon_2} = \{\sup_{0 \leq k \leq n} S_k \geq \varepsilon_2\}$ then it holds

$$\mathrm{P}(E_{n,\varepsilon_1}) \leq \frac{\mathrm{E}[S_n]}{\varepsilon_2} = \frac{2 \sum_{k=0}^{n} \gamma_k \, \mathrm{E}[\mathrm{E}[\; b_k \quad | F_k]]}{\varepsilon_2} \leq \frac{2 \sum_{k=0}^{n} \gamma_k B_k}{\varepsilon_2}$$

- Let $\Gamma_2 = 2\sum_{k=0}^{\infty} \gamma_k B_k$ and choose $\varepsilon_2 = 2\Gamma_2/\delta$.
- Then the event $E_{\varepsilon_2} = \quad {}_{n=0}^{\infty} E_{n,\varepsilon_2}$ will occur with probability at most $\delta/2$.

- Choose $M_0 > M + \varepsilon_1 + \varepsilon_2$ and let $W_0 = \{Y : Y_\alpha < -M_0$ for all $\alpha = \alpha$ $\}$. If Y_0 W_0 then with probability at least $1 - \delta$ we prove that Y_n M for all $n = 1, 2, \ldots$ and thus the equilibrium is stochastically stable.

- Choose $M_0 > M + \varepsilon_1 + \varepsilon_2$ and let $W_0 = \{Y : Y_\alpha < -M_0 \text{ for all } \alpha = \alpha\ \}$. If $Y_0 \quad W_0$ then with probability at least $1 - \delta$ we prove that $Y_n \quad M$ for all $n = 1, 2, \ldots$ and thus the equilibrium is stochastically stable.
- Since with probability at least $1 - \delta$ the sequence remains in the neighborhood U we have

$$Y_{\alpha,n+1} - Y_{\alpha^{\square},n+1} \leq -c \sum_{k=0}^{n} \gamma_k + \varepsilon_1 + \varepsilon_2 \qquad (2)$$

which implies that the score differences go to $-\infty$, thus all the strategies except for the strategy of the equilibrium become dominated.As a result the point is stochastically asymptotically stable.

Permitted parameters

The above conditions for the method's learning rate and exploration parameters can be achieved by using schedules of the form

- $\gamma_n \quad 1/n^p$
- $\varepsilon_n \quad 1/n^q$

with $p + q > 1$ and $2p - q > 1$. A popular choice is $p = 2/3 + \delta$ and $q = 1/3 + \delta$ for some arbitrarily small $\delta > 0$ – or $\delta = 0$ and including an extra logarithmic factor.