Example (1)

1. DNF: A **DNF** formula ϕ can be encoded by the finite ordered structure $\mathcal{A} = \langle A = \{v_1, ..., v_n, d_1, ..., d_m\}, D, P, N\rangle$ over $\tau = \langle D^1, P^2, N^2\rangle$.

$$\phi \in \mathrm{DNF} \text{ iff } \mathcal{A} \models \exists T \exists d \, \forall v \Big(D(d) \wedge \big(P(d,v) \to T(v)\big) \wedge \big(N(d,v) \to \neg T(v)\big)\Big)$$

Exercise. Check this for $\phi = (x_1 \wedge x_2 \wedge \neg x_3 \wedge \neg x_4) \vee (\neg x_2 \wedge \neg x_4 \wedge x_3 \wedge x_5)$

Example (2)

2. $\mathrm{3CNF}$: A boolean formula ϕ in conjunctive normal form with three literals per clause can be encoded by the finite structure $\mathcal{A} = \{(v_1, ..., v_n), C_0, C_1, C_2, C_3\}$ over $\tau = \langle C_0^3, C_1^3, C_2^3, C_3^3\rangle$.

$$
\begin{aligned}
&\phi \in \mathrm{3CNF} \text{ iff} \\
\mathcal{A} \models &\exists T \forall x_1 \forall x_2 \forall x_3 \\
&\big[\big(C_0(x_1, x_2, x_3) \to (T(x_1) \wedge T(x_2) \wedge T(x_3))\big) \wedge \\
&\big(C_1(x_1, x_2, x_3) \to (\neg T(x_1) \wedge T(x_2) \wedge T(x_3))\big) \wedge \\
&\big(C_2(x_1, x_2, x_3) \to (\neg T(x_1) \wedge \neg T(x_2) \wedge T(x_3))\big) \wedge \\
&\big(C_3(x_1, x_2, x_3) \to (\neg T(x_1) \wedge \neg T(x_2) \wedge \neg T(x_3))\big)\big]
\end{aligned}
$$

Exercise. Check this for
$\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_2 \vee x_4 \vee x_3) \wedge (\neg x_3 \vee \neg x_4 \vee x_1) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_4)$

Example (3)

3. SAT: A boolean formula ϕ in conjunctive normal form can be encoded by the finite structure $\mathcal{A} = \langle\{v_1, ..., v_n, c_1, ..., c_m\}, C, P, N\rangle$ over $\tau = \langle C^1, P^2, N^2\rangle$.

$$\phi \in \mathrm{SAT} \text{ iff } \mathcal{A} \models \exists T \forall c \exists v \big[C(c) \rightarrow \big(P(c,v) \wedge T(v)\big) \vee \big(N(c,v) \wedge \neg T(v)\big)\big]$$

Overview

1. Descriptive complexity
   - The class NP
   - The class #P

The class $\#$FO

- Let τ be a vocabulary containing a relation symbol $\leq$. In other words we are considering finite ordered structures.

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A function $f : \text{STRUCT}(\tau) \to \mathbb{N}$ belongs to #FO iff there is a first-order formula ϕ with relation symbols from $\overrightarrow{T} \cup \tau$ and free first-order variables from $\overrightarrow{z}$ such that

$$f(\mathcal{A}) = |\{\langle \overrightarrow{T}, \overrightarrow{z} \rangle \mid \mathcal{A} \models \phi(\overrightarrow{T}, \overrightarrow{z})\}|.$$

$\#P = \#FO$ (Saluja, Sabrahmanyama & Thakur)

Theorem (Saluja, Sabrahmanyama & Thakur 1995)

The class $\#P$ coincides with the class $\#FO$.

Proof. $\#FO \subseteq \#P$: The NPTM nondeterministically chooses a tuple $\langle \overrightarrow{S}, \overrightarrow{a} \rangle$ and verifies in polynomial time that $\mathcal{A} \models \phi(\overrightarrow{T}/\overrightarrow{S}, \overrightarrow{z}/\overrightarrow{a})$.

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- There is a unique different value of $\overrightarrow{T}$ s.t. it satisfies $\mathcal{A} \models \exists \overrightarrow{T} \phi(\overrightarrow{T})$ for every different accepting computation of the corresponding NPTM $M_{\mathcal{A}}$ on input $\mathcal{A}$.

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- So, the number of accepting paths of $M_{\mathcal{A}}$ is equal to $|\{\langle \overrightarrow{T} \rangle \mid \mathcal{A} \models \phi(\overrightarrow{T})\}|$.

□

Classes $\#\Sigma_i, \#\Pi_i$

- $\mathbf{\Sigma_0}$, $\mathbf{\Pi_0}$ formulas are unquantified **FO** formulas.
- $\mathbf{\Sigma_1}$ is a formula of the form $\exists \overrightarrow{x} \psi(\overrightarrow{x})$
- $\mathbf{\Pi_1}$ is a formula of the form $\forall \overrightarrow{x} \psi(\overrightarrow{x})$
- $\mathbf{\Sigma_2}$ is a formula of the form $\exists \overrightarrow{x} \forall \overrightarrow{y} \psi(\overrightarrow{x}, \overrightarrow{y})$
- $\mathbf{\Pi_2}$ is a formula of the form $\forall \overrightarrow{x} \exists \overrightarrow{y} \psi(\overrightarrow{x}, \overrightarrow{y})$

where ψ is unquantified.

A function $f : \text{STRUCT}(\tau) \to \mathbb{N}$ belongs to $\#\Sigma_i$ (resp. $\#\Pi_i$) iff there is a $\mathbf{\Sigma_i}$ (resp. $\mathbf{\Pi_i}$) formula ϕ s.t.

$$f(\mathcal{A}) = |\{\langle \overrightarrow{T}, \overrightarrow{z} \rangle \mid \mathcal{A} \models \phi(\overrightarrow{T}, \overrightarrow{z})\}|.$$

Example (1)

1. $\#\mathrm{DNF}$: A **DNF** formula can be encoded by the finite ordered structure $\mathcal{A} = \langle A = \{v_1, ..., v_n, d_1, ..., d_m\}, D, P, N\rangle$ over $\tau = \langle D^1, P^2, N^2\rangle$.

$$\#\mathrm{DNF}(\mathcal{A}) = |\{\langle T\rangle \mid \mathcal{A} \models \exists d\, \forall v \Big(D(d) \wedge \big(P(d, v) \to T(v)\big) \wedge \big(N(d, v) \to \neg T(v)\big)\Big)\}|$$

Hence $\#\mathrm{DNF} \in \#\Sigma_2$.

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$$\begin{aligned}
&\#3\mathrm{CNF}(\mathcal{A}) = |\{\langle T \rangle \mid \mathcal{A} \models (\forall x_1)(\forall x_2)(\forall x_3) \\
&\big[\big(C_0(x_1, x_2, x_3) \to (T(x_1) \wedge T(x_2) \wedge T(x_3))\big)\wedge \\
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Hence $\#3\mathrm{CNF} \in \#\Pi_1$.

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3. $\#\mathrm{SAT}$: A boolean formula in conjunctive normal form can be encoded by the finite structure $\mathcal{A} = \langle\{v_1, ..., v_n, c_1, ..., c_m\}, C, P, N\rangle$ over $\tau = \langle C^1, P^2, N^2\rangle$.

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$\#\Pi_2$ captures $\#P$

Proposition

$\#P = \#\Pi_2$.

Corollary

$\#\Pi_2 = \#FO$.

Hierarchy in #FO

Proposition 1

$$
\begin{array}{ccccc}
 & & \#\Pi_1 & & \\
 & \subseteq & & \subseteq & \\
\#\Sigma_0 = \#\Pi_0 & & & & \#\Sigma_2 \subseteq \#\Pi_2 = \#\mathrm{P}. \\
 & \subseteq & & \subseteq & \\
 & & \#\Sigma_1 & &
\end{array}
$$

Hierarchy in $\#$FO (2)

Proposition 2

$$\#\Sigma_0 = \#\Pi_0 \subset \#\Sigma_1 \subset \#\Pi_1 \subset \#\Sigma_2 \subset \#\Pi_2 = \#\text{FO}.$$

Proof. We prove here that $\#\Sigma_1 \subseteq \#\Pi_1$.

- Let $f \in \#\Sigma_1$ with $f(\mathcal{A}) = |\{\langle \overrightarrow{T}, \overrightarrow{z} \rangle \mid \mathcal{A} \models \exists \overrightarrow{x}\, \psi(\overrightarrow{x}, \overrightarrow{z}, \overrightarrow{T})\}|$.

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- Instead of counting the tuples $\langle \overrightarrow{T}, \overrightarrow{z} \rangle$, we count the tuples $\langle \overrightarrow{T}, (\overrightarrow{z}, \overrightarrow{x^*}) \rangle$ where $\overrightarrow{x^*}$ is the lexicographically smallest $\overrightarrow{x}$ such that $\mathcal{A} \models \psi(\overrightarrow{x}, \overrightarrow{z}, \overrightarrow{T})$.

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- Let $\theta(\overrightarrow{x}, \overrightarrow{x^*})$ be the quantifier-free formula which expresses that $\overrightarrow{x^*}$ is lexicographically smaller than $\overrightarrow{x}$ under $\leq$.

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- Let $\theta(\overrightarrow{x}, \overrightarrow{x^*})$ be the quantifier-free formula which expresses that $\overrightarrow{x^*}$ is lexicographically smaller than $\overrightarrow{x}$ under $\leq$.
- Then,

$$f(\mathcal{A}) = |\{\langle \overrightarrow{T}, (\overrightarrow{z}, \overrightarrow{x^*})\rangle \mid \mathcal{A} \models \psi(\overrightarrow{x^*}, \overrightarrow{z}, \overrightarrow{T}) \wedge (\forall \overrightarrow{x})\big(\psi(\overrightarrow{x}, \overrightarrow{z}, \overrightarrow{T}) \to \theta(\overrightarrow{x}, \overrightarrow{x^*})\big)\}|$$

Proof cont.

The second part of the proof includes the following:

- $\#\text{3DNF} \in \#\Sigma_1 \setminus \#\Sigma_0$
- $\#\text{3CNF} \in \#\Pi_1 \setminus \#\Sigma_1$
- $\#\text{DNF} \in \#\Sigma_2 \setminus \#\Pi_1$
- $\#\textsc{HamiltonCycles} \in \#\Pi_2 \setminus \#\Sigma_2$

□

Proof cont.

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- $\#\textsc{HamiltonCycles} \in \#\Pi_2 \setminus \#\Sigma_2$

□

The above classes are not closed under parsimonious reductions.
For example, $\#\mathrm{3CNF} \in \#\Pi_1$, but $\#\textsc{HamiltonCycles} \notin \#\Pi_1$.

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- This hierarchy can help us determine classes of approximable counting problems.
- We denote by FPRAS the class of #P functions that admit an fpras.
- We expect that problems in FPRAS have easy decision version.
- For any function $f \in \#\mathrm{P}$, let $L_f = \{x \mid f(x) > 0\}$ be the corresponding decision problem.
- The class of #P functions with decision version in P is

$$\#\mathrm{PE} = \{f \mid f \in \#\mathrm{P} \text{ and } L_f \in \mathrm{P}\}$$

defined by Pagourtzis (2001).

We are interested in a subclass of #PE, namely TotP.

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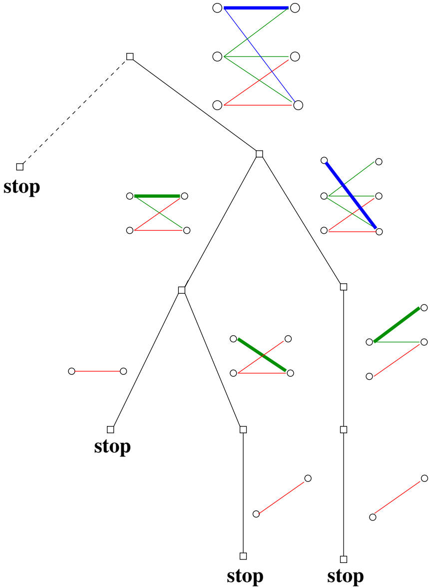
TotP is the Karp-closure of all self-reducible functions in #PE.

Definition (Kiayias, Pagourtzis, Sharma & Zachos 2001)

A function $f : \{0,1\}^* \to \mathbb{N}$ belongs to TotP if there is an NPTM M s.t.

$$f(x) = \#(\text{all paths of } M \text{ on input } x) - 1.$$

Self-reducibility & easy decision ⇒ membership in TotP



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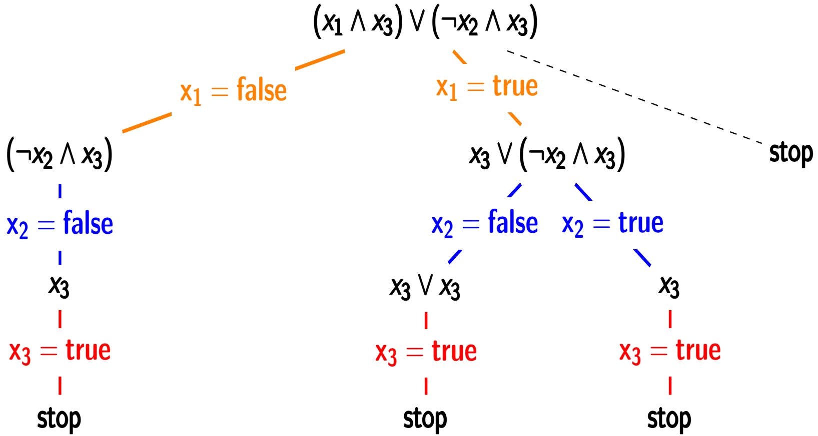

Robust subclasses of TotP

In the context of descriptive complexity we would like to define classes that are both

1. subclasses of TotP and
2. *robust*, i.e.
   - either they have natural complete problems under parsimonious reductions,
   - or they are closed under addition, multiplication and subtraction by one.

The class $\#\Sigma_0$

Proposition 1

Every problem in $\#\Sigma_0$ is computable in polynomial time.

The class $\#\Sigma_1$

Proposition 2

Every problem in $\#\Sigma_1$ has an fpras.

The class $\#\Sigma_1$

Proposition 2

Every problem in $\#\Sigma_1$ has an fpras.

1. Every $\#\Sigma_1$ function is reducible to a restricted version of $\#\mathrm{DNF}$ under a reducibility which preserves approximability.
2. $\#\mathrm{DNF}$ has an fpras.

Poly-time product reductions

The reductions used here are the following special case of parsimonious reductions.

Poly-time product reduction

$$f \leqslant_{pr} g : \ \exists h_1, h_2 \in \mathrm{FP}, \forall x \, f(x) = g(h_1(x)) \cdot h_2(x)$$

Proof.
Let $f(\mathcal{A}) = |\{\langle \overrightarrow{T}, \overrightarrow{z} \rangle \mid \mathcal{A} \models \exists \overrightarrow{y}\, \psi(\overrightarrow{y}, \overrightarrow{z}, \overrightarrow{T}\}|$, where

- ψ is in DNF,
- $\overrightarrow{y} = (y_1, \ldots, y_p)$, $\overrightarrow{z} = (z_1, \ldots, z_m)$,
- $\overrightarrow{T} = (T_1, \ldots, T_r)$ and T_i has arity a_i, $1 \leq i \leq r$.

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We make the following transformations:

1. We fix a $\overrightarrow{z_i} \in A^m$ and we write $\exists \overrightarrow{y}\, \psi(\overrightarrow{y}, \overrightarrow{z_i}, \overrightarrow{T}\}$ as a disjunct

$$\bigvee_{j=1}^{|A|^p} \psi(\overrightarrow{y_j}, \overrightarrow{z_i}, \overrightarrow{T}\}.$$

Proof.

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$$\bigvee_{j=1}^{|A|^p} \psi(\overrightarrow{y_j}, \overrightarrow{z_i}, \overrightarrow{T}\}.$$

2. We replace every subformula that is satisfied by $\mathcal{A}$ by true and every subformula that is not satisfied by $\mathcal{A}$ by false and we obtain $\psi'(\overrightarrow{z_i}, \overrightarrow{T})$.
   - Note that formula $\psi'(\overrightarrow{z_i}, \overrightarrow{T})$ is a propositional formula in DNF with variables of the form $T_i(\overrightarrow{w})$, $\overrightarrow{w} \in A^{a_i}$, $1 \leq i \leq r$.

Proof cont.

3. We introduce ℓ new variables $x_1, ..., x_\ell$, where $\ell = \log(|A|^m)$.
   - We can encode binary strings by conjunctions of these variables (and their negations), e.g. 0010 is encoded by $\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4$.

Proof cont.

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   - The binary representation s of any integer between 0 and $2^\ell - 1$ is encoded by a conjunction $x(s)$ of these variables (and their negations) in which x_i appears negated iff the i-th bit of s is 0.

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   - The binary representation s of any integer between 0 and $2^\ell - 1$ is encoded by a conjunction $x(s)$ of these variables (and their negations) in which x_i appears negated iff the i-th bit of s is 0.

4. Instead of taking the formula

$$\psi'(\overrightarrow{z_0}, \overrightarrow{T}) \vee \ldots \vee \psi'(\overrightarrow{z}_{|A|^m-1}, \overrightarrow{T})$$

we define the following formula

$$\theta_{\mathcal{A}} = [\psi'(\overrightarrow{z_0}, \overrightarrow{T}) \wedge x(0)] \vee \ldots \vee [\psi'(\overrightarrow{z}_{|A|^m-1}, \overrightarrow{T}) \wedge x(|A|^m - 1)].$$

Proof cont.

Observe that

- $\theta_{\mathcal{A}}$ is in DNF with variables $T_i(\overrightarrow{w})$, $\overrightarrow{w} \in A^{a_i}$, $1 \leq i \leq r$, and $x_1, ..., x_\ell$.

Proof cont.

Observe that

- $\theta_{\mathcal{A}}$ is in DNF with variables $T_i(\overrightarrow{w})$, $\overrightarrow{w} \in A^{a_i}$, $1 \leq i \leq r$, and $x_1, ..., x_\ell$.
- Variables $T_i(\overrightarrow{w})$ can be replaced by propositional variables t_{ik}, $1 \leq k \leq |A|^{a_i}$.

Proof cont.

Observe that

- $\theta_{\mathcal{A}}$ is in DNF with variables $T_i(\overrightarrow{w})$, $\overrightarrow{w} \in A^{a_i}$, $1 \leq i \leq r$, and $x_1, \ldots, x_\ell$.
- Variables $T_i(\overrightarrow{w})$ can be replaced by propositional variables t_{ik}, $1 \leq k \leq |A|^{a_i}$.
- Let $c(\mathcal{A})$ be the variables of the form $T_i(\overrightarrow{w})$ that do not appear in $\theta_{\mathcal{A}}$. It holds that:

$$f(\mathcal{A}) = 2^{c(\mathcal{A})} \cdot \text{ (the number of satisfying assignments of } \theta_{\mathcal{A}}\text{)}.$$

□

The classes $\#\Pi_1$ and $\#\Sigma_2$

We don't expect that either $\#\Pi_1$ or $\#\Sigma_2$ is a subclass of FPRAS, since $\#3\mathrm{CNF} \in \#\Pi_1$.

The class $\#\mathrm{R}\Sigma_2$

A function $f : \{0,1\} \to \mathbb{N}$ belongs to $\#\mathrm{R}\Sigma_2$ if there is a first-order formula ψ with relation symbols from $\overrightarrow{T} \cup \tau$ and free first-order variables from $\overrightarrow{z}$ such that

$$f(\mathcal{A}) = |\{\langle \overrightarrow{T}, \overrightarrow{z} \rangle \mid \mathcal{A} \models \exists \overrightarrow{x}\, \forall \overrightarrow{y}\, \phi(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{T}, \overrightarrow{z})\}|$$

where ψ is quantifier-free and when it is expressed in CNF, each conjunct has at most one occurrence of a relation symbol from $\overrightarrow{T}$.

The class $\#\mathrm{R}\Sigma_2$

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where ψ is quantifier-free and when it is expressed in CNF, each conjunct has at most one occurrence of a relation symbol from $\overrightarrow{T}$.

Proposition 3

Every function in $\#\mathrm{R}\Sigma_2$ has an fpras.

Proof. $\#\mathrm{DNF}$ is complete for $\#\mathrm{R}\Sigma_2$ under product reductions. The proof is similar to the previous one. □

- The decision version of every function in $\#\Sigma_0$, $\#\Sigma_1$ and $\#\mathsf{R}\Sigma_2$ is in P.
- $\#\textsc{Triangles} \in \#\Sigma_0$
- $\#\textsc{NonCliques}$, $\#\textsc{NonVertexCovers} \in \#\Sigma_1$,
- $\#\textsc{NonDominatingSets}$, $\#\textsc{NonEdgeDominatingSets} \in \#\mathsf{R}\Sigma_2$.

- Assuming NP $\neq$ RP, the following problem is undecidable: Given a first-order formula $\phi(\overrightarrow{z}, \overrightarrow{T})$ over $\tau \cup \overrightarrow{T}$, does the counting function defined by $\phi(\overrightarrow{z}, \overrightarrow{T})$ have an fpras?
- Assuming P $\neq$ P$^{\#\mathrm{P}}$, the following problem is undecidable: Given a first-order formula $\phi(\overrightarrow{z}, \overrightarrow{T})$ over $\tau \cup \overrightarrow{T}$, is the counting function defined by $\phi(\overrightarrow{z}, \overrightarrow{T})$ polynomial-time computable?

Quantitative Second-Order Logic

Given a relational vocabulary τ, the set of **Quantitative Second-Order** formulas (or just **QSO** formulas) over τ is given by the following grammar.

$$\alpha := \phi \mid s \mid (\alpha + \alpha) \mid (\alpha \cdot \alpha) \mid \Sigma x.\alpha \mid \Pi x.\alpha \mid \Sigma X.\alpha \mid \Pi X.\alpha$$

where ϕ is an **SO**-formula, $s \in \mathbb{N}$, x is a first-order variable and X is a second-order variable.

ΣQSO(FO) is the fragment of **QSO** where first- and second-order products ($\Pi x.$ and $\Pi X.$) are not allowed and ϕ is restricted to be in **FO**.

Semantics of **QSO** formulas

Let

- $\mathfrak{A}$ be a structure.
- v a first-order assignment for $\mathfrak{A}$
- V a second-order assignment for $\mathfrak{A}$

Then the evaluation of a **QSO** formula α over $(\mathfrak{A}, v, V)$ is defined as a function $[[\alpha]]$ that on input $(\mathfrak{A}, v, V)$ returns a number in $\mathbb{N}$.

$$[[\phi]](\mathcal{A}, v, V) = \begin{cases} 1, & \text{if } \mathcal{A} \models \phi \\ 0, & \text{otherwise} \end{cases}$$

$$[[s]](\mathcal{A}, v, V) = s$$

$$[[\alpha_1 + \alpha_2]](\mathcal{A}, v, V) = [[\alpha_1]](\mathcal{A}, v, V) + [[\alpha_2]](\mathcal{A}, v, V)$$

$$[[\alpha_1 \cdot \alpha_2]](\mathcal{A}, v, V) = [[\alpha_1]](\mathcal{A}, v, V) \cdot [[\alpha_2]](\mathcal{A}, v, V)$$

$$[[\Sigma x.\alpha]](\mathcal{A}, v, V) = \sum_{a \in A} [[\alpha]](\mathcal{A}, v[a/x], V)$$

$$[[\Pi x.\alpha]](\mathcal{A}, v, V) = \prod_{a \in A} [[\alpha]](\mathcal{A}, v[a/x], V)$$

$$[[\Sigma X.\alpha]](\mathcal{A}, v, V) = \sum_{B \subseteq A^{\text{arity}(X)}} [[\alpha]](\mathcal{A}, v, V[B/X])$$

$$[[\Pi X.\alpha]](\mathcal{A}, v, V) = \prod_{B \subseteq A^{\text{arity}(X)}} [[\alpha]](\mathcal{A}, v, V[B/X])$$

Arenas, Muñoz and Riveros (2017)

Let **F** be a fragment of **QSO** and C a counting complexity class. Then **F** captures C over ordered structures if the following conditions hold:

1. for every $\alpha \in$ **F**, there exists $f \in \mathrm{C}$ such that $[[\alpha]](\mathcal{A}) = f(\mathcal{A})$ for every ordered structure $\mathcal{A}$.
2. for every $f \in \mathrm{C}$, there exists $\alpha \in$ **F** such that $f(\mathcal{A}) = [[\alpha]](\mathcal{A})$ for every ordered structure $\mathcal{A}$.

Theorem

ΣQSO(FO) captures $\#\mathrm{P}$ over ordered structures.

Example (1)

1. Counting triangles in a graph:

$$\alpha_1 = \Sigma x.\Sigma y.\Sigma z.\big(E(x,y) \wedge E(y,z) \wedge E(z,x) \wedge x < y \wedge y < z\big).$$

Example (2)

2. Counting cliques in a graph:

$$\alpha = \Sigma X.\forall x \forall y \big(X(x) \wedge X(y) \wedge x \neq y) \rightarrow E(x, y).$$

Example (3)

3. Computing the permanent of a $n \times n$ matrix A with entries in $\{0, 1\}$,

$$\mathrm{perm}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} A(i, \sigma(i))$$

$$\alpha_3 = \Sigma S.\mathrm{permut}(S) \cdot \Pi x.\big(\exists y(S(x, y) \wedge M(x, y))\big)$$

where permut(S) is a first-order sentence that is true iff S is a permutation.

Exercise. Write formula permut(S).