On the Relative Complexity of Approximate Counting problems [1]

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Introduction

- Exactly solvable counting problems are quite uncommon.
- There are problems that admit an FPRAS despite being complete in $\#P$. Like #MATCH and #DNF-SAT.

Main Problem

Can we characterize the hardness of FPRAS approximation of the problems in $\#P$.

Why FPRAS?

Could we allow more problems if we allowed ourselves approximations up to a constant factor?

- We know that for example, $\#\mathrm{IS}(k \cdot G) = (\#\mathrm{IS}(G))^K$
- So for large enough k, if we had a constant approximation algorithm for $(\#\mathrm{IS}(G))^K$, we would get an FPRAS for $\#\mathrm{IS}(k \cdot G)$
- Due to the nature of the problem, we need FPRAS reductions to get close enough.

How hard can approximate counting be?

Theorem (Valiant-Vasirani Bisection Technique)

There exists a FPRAS of $\#SAT$ with the use of an oracle for SAT.

- While exact counting complexity contains all the polynomial hierarchy from Toda's Theorem, FPRAS approximation requires only and NP-oracle.
- Assuming that $NP \neq RP$ then no $\#SAT$ equivalent problems admit an FPRAS.

"So it is not that hard!!!"

A Reminder

Fully Polynomial Randomized Approximation Scheme (FPRAS of f)

A probabilistic TM that with input $(x, \epsilon) \in \Sigma^* \times (0,1)$, runs in $\text{poly}(|x|, \frac{1}{\epsilon})$ and outputs a random variable Y that

$$\Pr[f(x)e^{-\epsilon} \leq Y \leq f(x)e^{\epsilon}] > \frac{3}{4}$$

Approximation-Preserving Reduction ($f \leq_{AP} g$)

Is a probabilistic oracle TM that with input (x, ϵ) that:

- Makes oracle calls of the form (w, δ) to g
- If the oracle is an FPRAS of g then the TM if an FPRAS of f
- runs in $\text{poly}(|x|, \frac{1}{\epsilon})$

Immediate Implications

- All problems that admit an FPRAS are clearly irreducible.
- All problems $A \in NP$ have a counting counterpart $\#A \in \#P$ and due to the parsimonious Cook-Levin reduction we have that

$$\#A \leq_{AP} \#SAT$$

- But can we hope for a dichotomy with respect to the AP-reduction?

#SAT AP-interedeucible problems

Theorem

Let A be a NP-complete problem. Then the corresponding counting problem, #A is complete for #P with respect to the AP-reduction.

Proof Sketch

Since A is NP-complete if we have a reduction from SAT to A. So if we used an FPRAS for $\#A$ with ϵ we can distinguish 0 from at least one. And so with we can solve SAT with an oracle from $\#A$ up to some probability of failure.
Using this as an oracle in the Valiant-Vasirani FPRAS we have a AP reduction from $\#SAT$ to $\#A$.

Notice that this is only a conjecture for exact counting.

Is a dichotomy possible?

No, like in NP we have an analog of Lardner's theorem:

Theorem (Indichotomy with respect to AP [2])

Let Π be a problem in $\#P$ that does not admit an FPRAS. Then there is a problem $\Pi' \in \#P$ such that:

- *there is no FPRAS for Π'*
- *and $\Pi \not\leq_{AP} \Pi'$*

Three Main Classes

We will concern ourselves with 3 classes:

- Problems that admit an FPRAS
- Problems that are interedducible to $\#SAT$
- Problems that are interedducible to $\#BIS$

But why study just three classes when there exists an infinite hierarchy?

- The theorem like Ladner's does not give us natural problems that we can use as candidates.
- There are a lot of natural problems interedducible to $\#BIS$.

Equivalent problems

Problems that are approximately interreducible to $\#BIS$:

$\#P_4$-Col

Input:A graph G.
Output: Number of colorings to the P_4 graph, path of length 3.

#Downsets

Input: A partially ordered set $(X, \leq)$.
Output: The number of downsets in $(X, \leq)$.

#1P1NSAT

Input: A CNF formula that has at most one positive/negative literal per clause.
Output: The number of satisfying assignments.

Equivalent problems

Theorem

$\#P_4$-Col is AP-interreducible to $\#BIS$.

Proof Sketch

Notice that a graph is P_4-Collorable iff it is bipartite. Also:

- For every P_4 coloring of G the end vertices correspond to an independent set of G.
- For every independent set of a connected bipartite graph we can construct two P_4 colourings.

Hence

$$\#P_4Col(G) = 2\#BIS(G)$$

Equivalent problems

Theorem

#Downsets is AP-interreducible to #1P1NSAT.

Proof Sketch

- From all relations of a downset we can make a formula that is a conjuction of clauses $x \implies y$ iff $x \geq y$ for every $x, y \in X$.
- From each 1P1N formula by fixing the loop variables and all except one positive and one negative variable per clause.
- We can construct a formula that the number of satisfying assignments is equal to the number of corresponding downsets. So by using oracle calls we can solve #1P1NSAT

Logical Characterization of the #BIS family

$$f_{IS}(A) = |\{I : A \vDash \forall x, y : x \sim y \rightarrow \neg I(x) \vee \neg I(y)\}|$$

As seen in class:

- We can express problems as first order logic sentences ϕ and instances of a problem as a model A for the sentence.
- While distinguishing the different solutions of a problem as free relations (I).
- Hence a counting problems can be seen expressed as the number of those relations.
- This yields a hierarchy for the #P problems subclasses with respect to the syntactic freedom of the respected sentences.

Logical Characterization of the #BIS family

$$\#\Sigma_0 = \#\Pi_0 \subset \#\Sigma_1 \subset \#\Pi_1 \subset \#\Sigma_2 \subset \#\Pi_2 = \#\mathcal{FO} = \#\mathbf{P}$$

- Of course such a hierarchy can not express the different AP-interreducibility classes as we know that they are infinite.
- But it is provable that all problems in $\#\Sigma_1$ admit an FPRAS.
- And all the #BIS interreducible problems we have seen belong in Π_1.

Logical Characterization of the #BIS family

Definition: Counting Resticted Horn problems ($\#RH\Pi_1$)

A problem belongs in $\#RH\Pi_1$ if it can be expressed in the form

$$f(A) = |\{(T, z) : A \vDash \forall y : \phi(y, z, T)\}|$$

Where ϕ is a CNF formula that has at most one positive occurance of T and at most one negative occurance of T in every clause.

Logical Characterization of the #BIS family

Theorem

#1P1NSAT is complete for $\#RH\Pi_1$ with respect to the parsimonious reduction.

Theorem

The problems #BIS, $\#P_4$-Col, #Downsets (and else) are all AP-complete for $\#RH\Pi_1$.

Logical Characterization of the #BIS family

Theorem

The problems #BIS, #P_4-Col, #Downsets (and else) are all AP-complete for #$RH\Pi_1$.

Proof Sketch

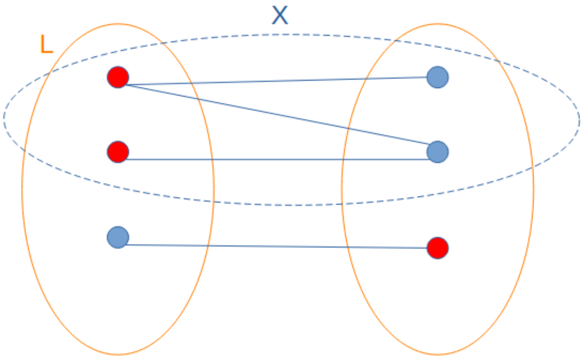
- Given the fact that #1P1NSAT is complete for #$\mathrm{RH}\Pi_1$ with respect to the parsimonious reductions as the problems are AP-interreducible with #1P1NSAT we already have hardness.
- So the only thing we need to prove is that they belong in this class.

$$f_{BIS}(\mathbf{A}) = |\{X : \mathbf{A} \vDash \forall x, y \in A : L(x) \wedge x \sim y \wedge X(x) \rightarrow X(y)\}|$$

Here X is the relation describing the independents sets and it is true for all left vertices in the independent set and false for all right vertices not in the independent set.

Logical Characterization of the #BIS family

$$f_{BIS}(\mathbf{A}) = |\{X : \mathbf{A} \vDash \forall x, y \in A : L(x) \land x \sim y \land X(x) \to X(y)\}|$$



q-Wrench Colourings

Theorem

It is true that:

- *For $q \leq 1$, #q-Wrench-Col, is AP-interreducible with #SAT.*
- *#2-Wrench-Col, is AP-interreducible with #BIS.*
- *For $q \geq 3$, #q-Wrench-Col, is AP-interreducible with #SAT.*

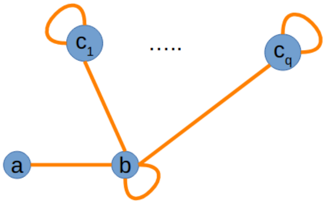


Figure 1: q-Wrench Graph

Non monotonicity of approximate counting complexity

Theorem

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- *#2-Wrench-Col, is AP-interreducible with #BIS.*
- *For $q \geq 3$, #q-Wrench-Col, is AP-interreducible with #SAT.*

This theorem implies that :

- #BIS is AP-interreducible with SAT
- or the approximate counting complexity of H-colourings is non-monotonic.

No life Bellow #BIS

Is there a graph H such that $\#H$-Col belongs between P and #BIS?

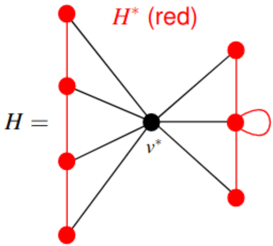
Theorem (Galanis, Goldberg, Jerrum 2015 [3])

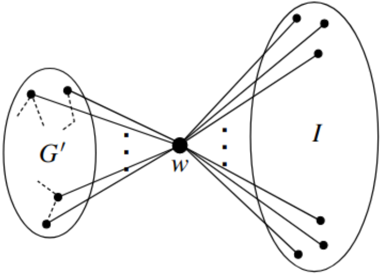
Let H be a graph whose connected components are not trivial. Then $\#BIS \leq_{AP} \#H$-Col.

Proof Idea

- Find a subgraph of H^* of H. Such that $\#H^*$-Col$\leq_{AP}\#H$-Col and has no trivial components.
- Then by induction on $|V(H)|$ we have

$$\#\text{BIS} \leq_{AP} \#H^*\text{-Col} \leq_{AP} \#H'\text{-Col}$$





The graph G

- What happens to the theorem if we allow trivial componets?
- Why of the problems are #SAT hard?

Thank you for your attention.
Are there any questions?

Bibliography I

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[3] A. Galanis, L. A. Goldberg, and M. Jerrum, "Approximately counting h-colorings is $\#\mathrm{BIS}$-hard," *SIAM Journal on Computing*, vol. 45, no. 3, pp. 680–711, 2016. DOI: 10.1137/15m1020551. [Online]. Available: https://doi.org/10.1137%2F15m1020551.