Counting Complexity

Computation and Reasoning Laboratory

Graduate course
Spring semester 2022

Πληροφορίες μαθήματος

Ειδικά Θέματα Αλγορίθμων: Μετρητική Πολυπλοκότητα (ΑΛΜΑ)
Προχωρημένα Θέματα Θεωρητικής Πληροφορικής (Θεωρία Αριθμών και Κρυπτογραφία) (ΣΗΜΜΥ)

- Διδάσκοντες: Σ. Ζάχος, Α. Παγουρτζής
- Βοηθός διδασκαλίας & επιμέλεια διαφανειών: Α. Χαλκή
- Διαλέξεις: Κάθε Πέμπτη: 16:00 – 20:00 (1.1.31, Παλαιά Κτίρια ΣΗΜΜΥ)
- Ώρες γραφείου: πριν και μετά από κάθε μάθημα
- Σελίδα: https://courses.corelab.ntua.gr/course/view.php?id=83
- Προαπαιτούμενα: Επιτυχής παρακολούθηση μαθήματος Υπολογιστικής Πολυπλοκότητας
- Εξέταση:
  - Ομιλία
  - Ασκήσεις

Course information

This course is designed for students of the graduate program "Algorithms, Logic, and Discrete Mathematics" and for PhD students of the School of Electrical and Computer Engineering.

- Teachers: S. Zachos, A. Pagourtzis
- Teaching assistant: A. Chalki
- Course material creators: S. Zachos, A. Pagourtzis, A. Chalki
- Course slides created by A. Chalki
- Website: https://courses.corelab.ntua.gr/course/view.php?id=83
- Lectures: Every Thursday 16:00 – 20:00 (room 1.1.31, old buildings)
- Prerequisites: Successful completion of a Computational Complexity course.
- To succeed in this course, students are required to submit two series of exercises and deliver a talk.

Motivation for this course

- 2021 Gödel Prize was jointly awarded to the three papers about the complexity of counting CSP (Bulatov 2013, Dyer & Richerby 2013, Cai & Chen 2017).
- Workshop 'Counting complexity and phase transitions' at Simons Institute for the Theory of Computing (2016).
- Two recent invited talks at conferences co-organized by CoReLab.
  - Richerby's talk
  - Cai's talk
- Research interests of CoReLab include Counting Complexity.

Plan for this course

- Introduction to counting complexity.
- Counting problems that are efficiently solved (counting perfect matchings in planar graphs and problems reducible to it).
- Dichotomy theorems for classes of counting problems (counting graph homomorphisms, $\#$CSPs).
- Approximating hard counting problems (sampling vs counting, intro to MCMC, FPRAS for counting problems).
- Descriptive complexity for counting (logical characterizations of $\#$P and some of its subclasses).

Why counting?

Why not? But also many interesting problems from different areas can be expressed as counting problems:

- Computing the partition function in statistical physics.
- Computing the volume of a convex body in computational geometry.
- Computing the permanent in linear algebra.
- Computing the social cost of a given mixed Nash equilibrium in selfish games in algorithmic game theory (in fact a probability problem can be reduced to this one).
- Optimizing an objective function under the presence of uncertainty requires counting approximate solutions for the corresponding decision problem.

Bibliography

Textbooks

1. Jin-Yi Cai, Xi Chen, *Complexity dichotomies for counting problems*, Cambridge University Press, 2017.
2. Mark Jerrum, *Counting and Markov chains*, CMU Math.
3. Sanjeev Arora, Boaz Barak, *Computational Complexity: A Modern Approach*, Cambridge University Press, 2009.

Overview

1. Introduction to Counting Complexity
   - The class #P
   - Three classes of counting problems
   - Holographic transformations
2. Matchgates and Holographic Algorithms
   - Kasteleyn's algorithm
   - Matchgates
   - Holographic algorithms
3. Polynomial Interpolation
4. Dichotomy Theorems for counting problems
5. Approximation of counting problems
   - Sampling and counting
   - Markov chains
   - Markov chain for sampling graph colorings
6. Appendix

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2 Matchgates and Holographic Algorithms
- Kasteleyn's algorithm
- Matchgates
- Holographic algorithms

3 Polynomial Interpolation

4 Dichotomy Theorems for counting problems

5 Approximation of counting problems
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6 Appendix

The class NP

- **Decision problem:** Is there an independent set of size at least k in the input graph G?
- **Membership in NP:** The set of graphs that have an independent set of size at least k.
- **NP**: $L \in \text{NP}$ iff there is an NPTM M s.t.

$$x \in L \Leftrightarrow M(x) \text{ has an accepting computation}$$

- **NP (alt def)**: $L \in \text{NP}$ iff $L = \{x \mid \exists y\, R(x, y)\}$ for a polynomially decidable and polynomially balanced relation R.

The class $\#$P

- **Counting problem:** How many satisfying assignments does a 3CNF formula ϕ have?
- **Membership in #P:** The function that on input a 3CNF formula returns the number of its satisfying assignments.

Definition

A function $f : \{0,1\}^* \to \mathbb{N}$ is in $\#$P if there exists a polynomially decidable and polynomially balanced relation R such that for every $x \in \{0,1\}^*$,

$$f(x) = \left|\{y \in \{0,1\}^* : R(x,y)\}\right|.$$

- **#P** is a class of functions that take values in $\mathbb{N}$.

- An NPTM outputs 'yes' or 'no' and we want to compute the number of accepting computation paths.
- $f \in \#\mathrm{P}$ iff there is an NPTM M s.t. $M(x)$ has $f(x)$ accepting paths.
- For an NPTM M, we define the function $acc_M(x) : \{0,1\}^* \to \mathbb{N}$ as follows:

$$acc_M(x) = \# \text{ accepting paths of M on input } x$$

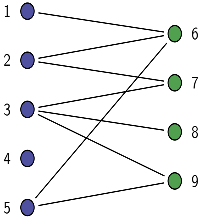
Then,

$$\#\mathrm{P} = \{acc_M \,|\, M \text{ is an NPTM } \}$$

Examples of counting problems in #P (1)

Every decision problem in NP has a counting version in #P.

Decision version: Is a graph 2-colourable?

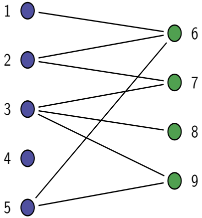


Counting version: How many 2-colorings does a graph have?

Examples of counting problems in #P (1)

Every decision problem in NP has a counting version in #P.

Decision version: Is a graph 2-colourable? in P



Counting version: How many 2-colorings does a graph have? in FP[1]

[1] FP is the class of functions computable in deterministic polynomial time.

Examples of counting problems in #P (2)

Decision version: Is a 3CNF formula satisfiable?

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_4 \vee x_2) \wedge (x_4 \vee x_1 \vee x_2)$$

Counting version: How many satisfying assignments does a 3CNF formula have?

Examples of counting problems in #P (2)

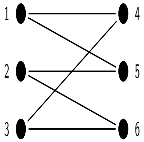
Decision version: Is a 3CNF formula satisfiable? NP-complete

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_4 \vee x_2) \wedge (x_4 \vee x_1 \vee x_2)$$

Counting version: How many satisfying assignments does a 3CNF formula have? NP-hard (we will see it is #P-complete)

Examples of counting problems in #P (3)

Decision version: Does a bipartite graph have a perfect matching?

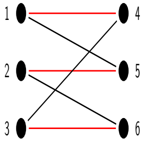


Counting version: How many perfect matchings does a bipartite graph have?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Examples of counting problems in #P (3)

Decision version: Does a bipartite graph have a perfect matching?



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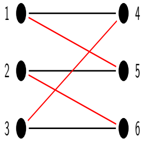
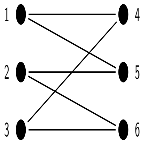


Counting version: How many perfect matchings does a bipartite graph have?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Examples of counting problems in #P (3)

Decision version: Does a bipartite graph have a perfect matching? in P



Counting version: How many perfect matchings does a bipartite graph have? #P-complete

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\textsc{Permanent}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

Examples of counting problems in #P (4)

Decision version: Is there an independent set (of any size) in a graph? trivial

Counting version: How many independent sets (of any size) are there in a graph? #P-complete

Reductions between counting functions

- Cook (poly-time **Turing**)

$$f \leqslant_T g: \; f \in FP^g$$

- Karp / **parsimonious** (poly-time many one)

$$f \leqslant_m g: \; \exists h \in FP, \; \forall x \; f(x) = g(h(x))$$

We write $f \equiv_T g$ (resp. $f \equiv_m g$) to denote that f, g are Turing equivalent (resp. parsimoniously equivalent).