Let $\overrightarrow{G}$ be a Pfaffian orientation of G.

Define the skew adjacency matrix $A_S(\overrightarrow{G}) = (a_{ij} : 0 \leq i, j \leq n-1)$ of G by

$$a_{ij} = \begin{cases} +1, & \text{if } (i,j) \in E(\overrightarrow{G}) \\ -1, & \text{if } (j,i) \in E(\overrightarrow{G}) \\ 0, & \text{otherwise} \end{cases}$$

Kasteleyn's Theorem

For any Pfaffian orientation $\overrightarrow{G}$ of G,

$$\#\text{Perfect Matchings in } G = \sqrt{\det(A_S(\overrightarrow{G}))}.$$

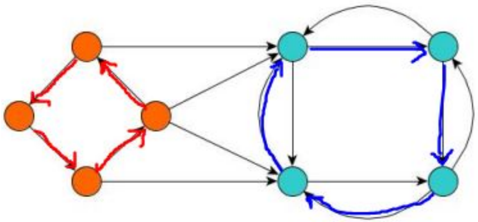
Proof. Let $\overleftrightarrow{G}$ be the directed graph obtained from G by replacing each undirected edge $\{i, j\}$ by the pair of directed edges (i, j), (j, i).

Step 1. $(\#\text{Perfect Matchings in } G)^2 = \#\text{Even cycle covers in } \overleftrightarrow{G}$.

Step 2. $\#\text{Even cycle covers in } \overleftrightarrow{G} = \det(A_S(\overrightarrow{G}))$.

Even Cycle cover

An even cycle cover of a directed graph G is a collection $\mathcal{C}$ of even length directed cycles s.t. every vertex of G is contained in exactly one cycle in $\mathcal{C}$.



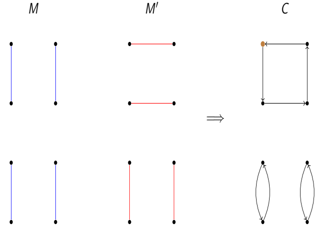
Proof cont.
Step 1. $(\#\text{Perfect Matchings in } G)^2 = \#\text{Even cycle covers in } \overleftrightarrow{G}$.

Lemma

There is a bijection between ordered pairs of perfect matchings in G and even cycle covers in $\overleftrightarrow{G}$.

Proof of lemma. Let (M, M') be an ordered pair of perfect matchings in G.

- For any edge $\{i, j\} \in M \cap M'$ take both $(i, j), (j, i)$ in C.
- Orient each cycle **c** in $M \cup M'$ according to the following convention: take the vertex with lowest number in **c** and orient the incident M-edge away from it.



□

Proof cont.

Step 2. #Even cycle covers in $\overleftrightarrow{G} = \det(A_S(\overrightarrow{G}))$.

- $\det(A_S(\overrightarrow{G})) = \sum_{\pi \in S_n} \operatorname{sgn} \pi \prod_{i=1}^{n} a_{i,\pi(i)}$ [2].

[2] $\operatorname{sgn} \pi$ is $+1$ if the cycle decomposition of π has an even number of even length cycles, and -1 otherwise.

Proof cont.

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- $\det(A_S(\overrightarrow{G})) = \sum_{\pi \in S_n} \operatorname{sgn} \pi \prod_{i=1}^{n} a_{i,\pi(i)}$ [2].

- Every permutation π has a unique decomposition into disjoint cycles, i.e. $\pi = \gamma_1 \cdots \gamma_k$.
  For example, $\pi = (2\,3)(1\,4\,5)$ corresponds to $\pi(2) = 3$, $\pi(3) = 2$, $\pi(1) = 4$, $\pi(4) = 5$ and $\pi(5) = 1$.

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- Let γ_j act on a certain subset $V_j \subseteq V$. The product $\prod_{i \in V_j} a_{i,\pi(i)}$ is non-zero iff edges $\{(i, \pi(i)) : i \in V_j\}$ form a cycle C_j in G.
  For example, $a_{1,4} \cdot a_{4,5} \cdot a_{5,1} \neq 0$ iff $\{(1,4), (4,5), (5,1)\}$ is a cycle.

[2] $\operatorname{sgn} \pi$ is $+1$ if the cycle decomposition of π has an even number of even length cycles, and -1 otherwise.

Proof cont.

- There is a one-to-one correspondence between permutations with non-zero contributions and cycle covers of $\overleftrightarrow{G}$: A permutation $\pi = \gamma_1 \cdots \gamma_k$ is mapped to the set of directed cycles (of even or odd length) corresponding to γ_1,...,γ_k.
  Each cycle C_j is given the direction determined by γ_j.

Proof cont.

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  Each cycle C_j is given the direction determined by γ_j.

- The permutations with some odd length cycle can be paired so they cancel each other: a permutation $\pi = \gamma_1 \cdots \gamma_j \cdots \gamma_k$ cancels out $\pi = \gamma_1 \cdots (\gamma_j)^{-1} \cdots \gamma_k$, where γ_j is of odd length.
  They have the same sign but opposite corresponding products.

Proof cont.

- We consider only permutations with even length cycles which correspond to even cycle covers of $\overleftrightarrow{G}$.

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- So, γ_j contributes a factor -1 to $\prod_{i=1}^{n} a_{i,\pi(i)}$, and a factor -1 to $\operatorname{sgn} \pi$, being an even cycle.

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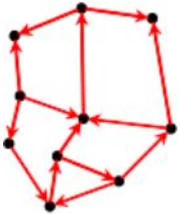
- So, γ_j contributes a factor -1 to $\prod_{i=1}^{n} a_{i,\pi(i)}$, and a factor -1 to $\operatorname{sgn} \pi$, being an even cycle.
- Therefore, overall π contributes 1 to the sum.

$\square$

Lemma

Let $\overrightarrow{G}$ be a connected planar directed graph, embedded in the plane. Suppose every face, except the outer infinite face, has an odd number of edges that are oriented clockwise. Then, $\overrightarrow{G}$ is Pfaffian.



Theorem

Every planar graph has a Pfaffian orientation.

Proof. W.l.o.g. assume G is connected.
By induction on the number m of edges.

- $m = n - 1$: G is a tree and every orientation is Pfaffian.
- $m \geq n$: Fix an edge on the exterior. By the induction hypothesis, $G \setminus e$ has a Pfaffian orientation. Adding e creates just one more face. Orient e such that this face has an odd number of edges oriented clockwise.

□

Overview

Matchgates and Holographic algorithms

- In the context of holographic algorithms based on matchgates, counting problems are reduced to counting perfect matchings in planar graphs.

- 2002: Valiant initially introduced matchgates as a way to show that a nontrivial, though restricted, fragment of quantum computation can be simulated in classical polynomial time (he presented a way to simulate certain quantum gates by matchgates).

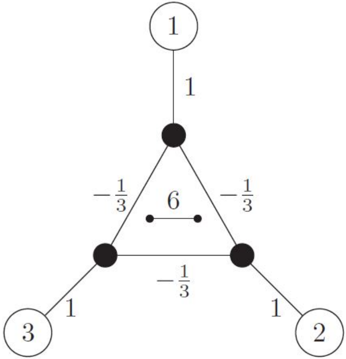
Matchgates

- A plane graph is a planar graph given with a particular planar embedding.
- W.l.o.g. we assume all edge weights are nonzero.

Definition

A matchgate is an undirected weighted plane graph G with a subset of distinguished nodes on its outer face, called the external nodes, ordered in a clockwise order.

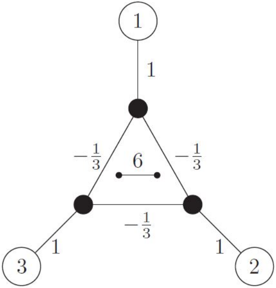
Example of a matchgate



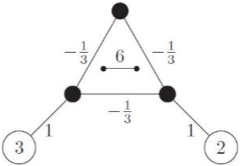
Some notation

- Let G be a matchgate with k external nodes. For each length-k bitstring α, G^α is obtained from G by the following operation: For all $1 \leq i \leq k$, if the i-th bit of α is 1, then we remove the i-th external node and all its incident edges.

G



G^{100}



Some notation

- For any graph G, we denote by $\mathcal{M}(G)$ the set of perfect matchings of G.
- For any weighted graph G, we denote by $\textsc{PerfMatch}(G)$ the weighted sum of perfect matchings of G, i.e.

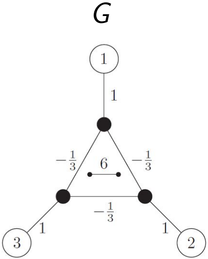
$$\textsc{PerfMatch}(G) = \sum_{M \in \mathcal{M}(G)} \prod_{e \in M} w(e).$$

The signature of a matchgate

Definition

Let G be a matchgate with k external nodes.
We define the signature of G as the vector $\Gamma_G = (\Gamma_G^\alpha)$, indexed by $\alpha \in \{0,1\}^k$ in lexicographic order, as follows:

$$\Gamma_G^\alpha = \textsc{PerfMatch}(G^\alpha) = \sum_{M \in \mathcal{M}(G^\alpha)} \prod_{e \in M} w(e).$$



For example,
$\Gamma_G^{000} = 6,$
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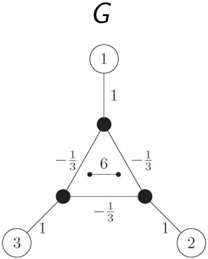

For example,
$\Gamma_G^{000} = 6$,
$\Gamma_G^{100} = 0$,
$\Gamma_G^{110} = 6 \cdot 1 \cdot (-\frac{1}{3}) = -2$,
$\Gamma_G = (6, 0, 0, -2, 0, -2, -2, 0)$.

Necessary condition

For any matchgate signature Γ_G, either for all α of odd Hamming weight, or for all α of even Hamming weight, $\Gamma_G^\alpha = 0$.

Symmetric matchgate signatures

Definition

A matchgate signature Γ_G is symmetric if, for all α and β of equal Hamming weight, $\Gamma_G^{\alpha} = \Gamma_G^{\beta}$.

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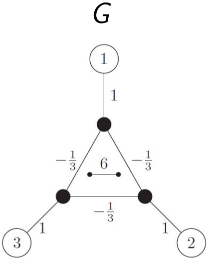

It holds that

$$\Gamma_G^{001} = \Gamma_G^{010} = \Gamma_G^{100} = 0$$

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Symmetric matchgate signatures

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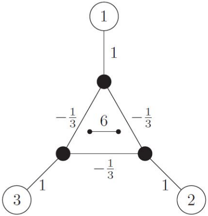

It holds that

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So, $\Gamma_G = [6, 0, -2, 0]$.

Characterization of symmetric matchgate signatures

Theorem

A symmetric signature is the signature of a matchgate iff it has the following form, for some $a, b \in \mathbb{C}$ and $k \in \mathbb{Z}$ [a]:

1. $[a^k b^0, 0, a^{k-1} b, 0, a^{k-2} b^2, 0, \dots, a^0 b^k]$ (arity $2k \geq 2$)
2. $[a^k b^0, 0, a^{k-1} b, 0, a^{k-2} b^2, 0, \dots, a^0 b^k, 0]$ (arity $2k + 1 \geq 1$)
3. $[0, a^k b^0, 0, a^{k-1} b, 0, a^{k-2} b^2, 0, \dots, a^0 b^k]$ (arity $2k + 1 \geq 1$)
4. $[0, a^k b^0, 0, a^{k-1} b, 0, a^{k-2} b^2, 0, \dots, a^0 b^k, 0]$ (arity $2k + 2 \geq 2$).

[a] We take the convention that $0^0 = 1$.

Overview

Holographic algorithms for some problems

- 2008: Valiant gave holographic algorithms for a series of problems in his paper "Holographic algorithms".
- The next problem is motivated by statistical physics.
  - An "ice problem" involves counting the number of orientations of an undirected graph such that certain local constraints are satisfied.
  - Pauling initially proposed such a model for planar square lattices, where the constraint was that an orientation assigned exactly two incoming and two outgoing edges at every node.

#PL-3-NAE-ICE

Input: A planar graph $G = (V, E)$ of maximum degree 3.

Output: The number of orientations such that no node has all incident edges directed toward it or all incident edges directed away from it.

- #PL-3-NAE-ICE is the problem of counting the number of no-sink-no-source orientations.
- We assume every node has degree 2 or 3, since a node of degree 1 will preclude such an orientation.

A holographic algorithm for $\#\text{PL-3-NAE-ICE}$
Step 1

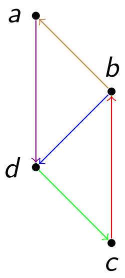
Let $G = (V, E)$ be an input to the problem.

To solve the problem by a holographic algorithm with matchgates, we make the following transformations:

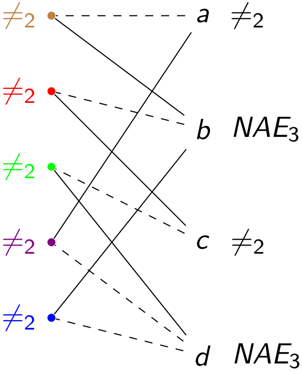
1. We design a signature grid $\Omega = (G', \pi)$ based on G:
   - We attach to each node of degree 3 a *Not-All-Equal* (or *NAE*) gate of arity 3, i.e. the signature $[0, 1, 1, 0]$, as a contravariant tensor (with only left dangling edges).
   - For any node of degree 2 we use a *NAE* gate of arity 2, i.e. a binary *Disequality* signature $(\neq_2) = [0, 1, 0]$, also as a contravariant tensor.
   - For each edge in E we use a binary *Disequality* signature $(\neq_2) = [0, 1, 0]$ as a covariant tensor (with only right dangling edges).

Example of Step 1 (edge-vertex incidence graph)

An orientation of G An assignment σ on the edges of $\Omega = (G', \pi)$



$\Longrightarrow$



- If G is planar, then G' is also planar.
- Here, a dashed edge e in Ω denotes that $\sigma(e) = 0$, whereas a solid edge e denotes that $\sigma(e) = 1$.