Infinite Automata, Logics and Games

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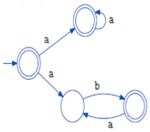
A nondeterministic finite automaton (*NFA*) is a quintuple, $(Q, \Sigma, \delta, q_0, F)$, consisting of

- a finite set of states Q,
- a finite set of input symbols Σ,
- a transition function $\delta : Q \times \Sigma \to Pow(Q)$,
- an initial state $q_0 \in Q$,
- a set of states F distinguished as accepting (or final) states $F \subseteq Q$.

NFA for $a^* + (ab)^*$:



REG is the class of languages recognised by a finite automaton.

An ω-automaton is a quintuple $(Q, \Sigma, \delta, q_0, Acc)$, where

- Q is a finite set of states,
- Σ is a finite alphabet,
- $\delta : Q \times \Sigma \to Pow(Q)$ is the state transition function,
- $q_0 \in Q$ is the initial state,
- Acc is the acceptance component (this corresponds to F in the case of finite automata).

In a deterministic ω-automaton, a transition function $\delta : Q \times \Sigma \to Q$ is used.

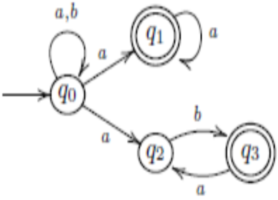
Let $A = (Q, \Sigma, \delta, q_0, Acc)$ be an ω-automaton. A run of A on an ω-word (stream) $\alpha = a_1a_2\ldots \in \Sigma^\omega$ is a countable infinite state sequence $\rho = \rho(0)\rho(1)\rho(2)\ldots \in Q^\omega$, such that the following conditions hold:

1. $\rho(0) = q_0$
2. $\rho(i) \in \delta(\rho(i-1), a_i)$ for $i \geqslant 1$ if A is nondeterministic,

For a run ρ of an ω-automaton, let $Inf(\rho) = \{q \in Q : \forall i \exists j > i \; \rho(j) = q\}$ (i.e. the set of states visited infinitely often).

An ω-automaton $A = (Q, \Sigma, \delta, q_0, Acc)$ is called

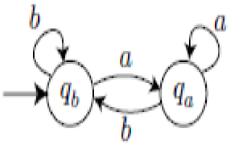
- **Büchi** automaton if $Acc = F \subseteq Q$ and the acceptance condition is the following: A stream $\alpha \in \Sigma^\omega$ is accepted by A iff there exists a run ρ of A on α satisfying the condition: $Inf(\rho) \cap F \neq \emptyset$.



Büchi automaton for $(a+b)^* a^\omega + (a+b)^* (ab)^\omega$ with $F = \{q_1, q_3\}$

An ω-automaton $A = (Q, \Sigma, \delta, q_0, Acc)$ is called

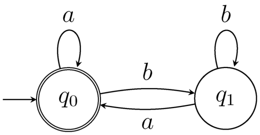
- **Muller** automaton if $Acc = \mathcal{F} \subseteq Pow(Q)$ and the acceptance condition is the following: A stream $\alpha \in \Sigma^\omega$ is accepted by A iff there exists a run ρ of A on α satisfying the condition: $Inf(\rho) \in \mathcal{F}$.



Muller automaton for $(a+b)^*a^\omega + (a+b)^*b^\omega$ with $\mathcal{F} = \{\{q_a\}, \{q_b\}\}$

An ω-automaton $A = (Q, \Sigma, \delta, q_0, Acc)$ is called

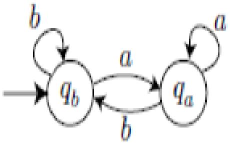
- **Rabin** automaton if $Acc = \{(E_1, F_1), ..., (E_k, F_k)\}$, with $E_i, F_i \subseteq Q$, $1 \leqslant i \leqslant k$, and the acceptance condition is the following: A stream $\alpha \in \Sigma^\omega$ is accepted by A iff there exists a run ρ of A on α satisfying the condition: $\exists(E, F) \in Acc(Inf(\rho) \cap E = \emptyset) \wedge (Inf(\rho) \cap F \neq \emptyset)$.



Rabin automaton for $(a + b)^* a^\omega$ with $Acc = \{(\{q_1\}, \{q_0\})\}$

An ω-automaton $A = (Q, \Sigma, \delta, q_0, Acc)$ is called

- **Streett** automaton if $Acc = \{(E_1, F_1), ..., (E_k, F_k)\}$, with $E_i, F_i \subseteq Q$, $1 \leqslant i \leqslant k$, and the acceptance condition is the following: A stream $\alpha \in \Sigma^\omega$ is accepted by A iff there exists a run ρ of A on α satisfying the condition: $\neg(\exists(E,F) \in Acc(Inf(\rho) \cap E = \emptyset) \wedge (Inf(\rho) \cap F \neq \emptyset))$, i.e. $\forall(E,F) \in Acc(Inf(\rho) \cap E \neq \emptyset) \vee (Inf(\rho) \cap F = \emptyset)$ $\big($or $\forall(E,F) \in Acc(Inf(\rho) \cap F \neq \emptyset) \rightarrow (Inf(\rho) \cap E \neq \emptyset)\big)$.



Streett automaton with $Acc = \{(\{q_b\}, \{q_a\})\}$.
Each stream in the accepted language contains infinitely many a's only if it contains infinitely many b's (or equivalently they have finitely many a's or infinitely many b's), e.g. $(a+b)^* b^\omega + (a^* b)^\omega$

The Büchi recognizable ω-languages are the ω-languages of the form

$$L = U_1 V_1^\omega + U_2 V_2^\omega \ldots U_k V_k^\omega \text{ with } k \in \omega \text{ and } U_i, V_i \in REG \text{ for } i = 1, \ldots, k.$$

This family of ω-languages is also called the **ω-Kleene closure** of the class of regular languages and is commonly referred to as ω-REG.

The **emptiness problem** for Büchi automata is decidable.

Muller automata are equally expressive as nondeterministic Büchi automata.

Proof: On the board.

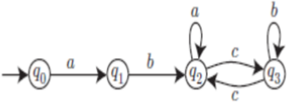
Rabin automata and Streett automata are equally expressive as Muller automata.

Proof:

- For a Rabin automaton $A = (Q, \Sigma, \delta, q_0, Acc)$, define the Muller automaton $A' = (Q, \Sigma, \delta, q_0, \mathcal{F})$, where $\mathcal{F} = \{G \in Pow(Q) | \exists (E, F) \in Acc.\, G \cap E = \emptyset \wedge G \cap F \neq \emptyset\}$.
  For a Streett automaton $A = (Q, \Sigma, \delta, q_0, Acc)$, define the Muller automaton $A' = (Q, \Sigma, \delta, q_0, \mathcal{F})$, where $\mathcal{F} = \{G \in Pow(Q) | \forall (E, F) \in Acc.\, G \cap E \neq \emptyset \vee G \cap F = \emptyset\}$.
- Conversely, given a Muller automaton, transform it into a nondeterministic Büchi automaton.
  Büchi acceptance can be viewed as a special case of Rabin acceptance, where $Acc = \{(\emptyset, F)\}$, as well as a special case of Streett acceptance, where $Acc = \{(F, Q)\}$.

An ω-automaton $A = (Q, \Sigma, \delta, q_0, c)$ with acceptance component $c : Q \to \{1, ..., k\}$ (where $k \in \omega$) is called **parity** automaton if it is used with the following acceptance condition:
A stream $\alpha \in \Sigma^\omega$ is accepted by A iff there exists a run ρ of A on α with

$$min\{c(q)|q \in Inf(\rho)\} \text{ is even}$$



Parity automaton A with colouring function c defined by $c(q_i) = i$.
$L(A) = ab(a^* cb^* c)^* a^\omega$

.

Parity automata can be converted into Rabin automata.

Proof: Let $A = (Q, \Sigma, \delta, q_0, c)$ be a parity automaton with $c : Q \to \{0, ..., k\}$. An equivalent Rabin automaton $A' = (Q, \Sigma, \delta, q_0, Acc)$ has the acceptance component $Acc = \{(E_0, F_0), ..., (E_r, F_r)\}$, $r = \lfloor \frac{k}{2} \rfloor$,
$E_i = \{q \in Q | c(q) < 2i\}$ and $F_i = \{q \in Q | c(q) \leqslant 2i\}$.

Muller automata can be converted into parity automata (a special case of Rabin automata).

Proof: On the board.

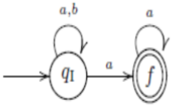
- Nondeterministic Büchi, Muller, Rabin, Streett, and parity automata are all equivalent in expressive power, i.e. they recognize the same ω-languages.
- The ω-languages recognized by these ω-automata form the class ω-KC(REG), i.e. the ω-Kleene closure of the class of regular languages.

- NFAs are equivalent to DFAs.
- NPDAs are not equivalent to DPDAs.
- Nondeterministic ω-automata are equivalent to deterministic ones?

Deterministic vs Nondeterministic Büchi Automata

There exist languages which are accepted by some nondeterministic Büchi-automaton but not by any deterministic Büchi automaton.

Proof. The following automaton is a nondeterministic Büchi automaton for $L = (a+b)^* a^\omega$.



Assume that there is a deterministic Büchi automaton A for the language L. Then there exist $n_0, n_1, n_2, \ldots$ such that A accepts the stream $w = a^{n_0} b a^{n_1} b a^{n_2} b \ldots \notin L$.

- Deterministic Muller, Rabin, Streett, and parity automata recognize the same ω-languages.
- The class of ω-languages recognized by any of these types of ω-automata is closed under complementation.

Proof:

- The transformations between nondeterministic automata work for deterministic ones except for those that use nondeterministic Büchi automata.

  NRabin $\longrightarrow$ NStreett: NRabin $\longrightarrow$ NMuller $\longrightarrow$ NBüchi $\longrightarrow$ NStreett

  DRabin $\longrightarrow$ DStreett: DRabin for L $\longrightarrow$ DMuller for L $\longrightarrow$ DMuller for $\overline{L}$ $\longrightarrow$ DRabin for $\overline{L}$ $\longrightarrow$ DStreett for L
- The languages recognizable by deterministic Muller automata are closed under union, intersection and complementation.

$$DMuller = DRabin = DStreett = NBuchi = NMuller = NRabin = NStreett$$

$$\uparrow$$

$$DBuchi$$

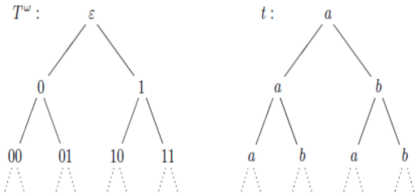
Determinization of Büchi Automata

Every nondeterministic Büchi automaton can be transformed into an equivalent deterministic Muller automaton (or a deterministic Rabin automaton).

- The powerset construction fails in case of Büchi automata.
- Muller ('63) presented a faulty construction.
- McNaughton ('66) showed that a Büchi automaton can be transformed effectively into an equivalent deterministic Muller automaton.

- Safra's construction ('88) leads to deterministic Rabin or Muller automata: given a nondeterministic Büchi automaton with n states, the equivalent deterministic automaton has $2^{\mathcal{O}(nlogn)}$ states.
- For Rabin automata, Safra's construction is optimal. The question whether it can be improved for Muller automata is open.
- Muller and Schupp ('95) presented a 'more intuitive' alternative, which is also optimal for Rabin automata.

- The **infinite binary tree** T^ω is the set $\{0,1\}^*$ of all strings on $\{0,1\}$.
- The elements $u \in T^\omega$ are the **nodes** of T^ω where ϵ is the root and $u0, u1$ are the immediate left and right successors of node u.
- A stream $\pi \in \{0,1\}^\omega$ is called a **path** of the binary tree T^ω.
- The set of all **Σ-labelled** trees, T^ω_Σ, contains trees where each node is labelled with a symbol of the alphabet Σ, i.e. trees with a mapping $t : T^\omega \to \Sigma$.



A **Muller tree automaton** is a quintuple $A = (Q, \Sigma, \delta, q_0, \mathcal{F})$, where

- Q is a finite set of states ,
- Σ is a finite alphabet,
- $\delta : Q \times \Sigma \rightarrow Pow(Q \times Q)$ denotes the transition relation,
- q_0 is an initial state,
- $\mathcal{F} \subseteq Pow(Q)$ is a set of designated state sets.

- A **run** of A on an input tree $t \in T_\Sigma$ is a tree $\rho \in T_Q$, satisfying $\rho(\epsilon) = q_0$ and for all $w \in \{0,1\}^*$: $\delta(\rho(w), t(w)) = (\rho(w0), \rho(w1))$.
- A run is called **successful** if for each path $\pi \in \{0,1\}^\omega$ the Muller acceptance condition is satisfied, that is, if $Inf(\rho|\pi) \in \mathcal{F}$.
- A accepts the tree t if there is a successful run of A on t.
- The tree language recognized by A is the set $T(A) = \{t \in T^\omega | A \text{ accepts } t\}$.

Example: $A = (\{q_0, q_a, q_b, q_d\}, \{a, b\}, \delta, q_0, \mathcal{F})$, where δ includes:

$$\delta(q_0, a) = (q_a, q_d), \delta(q_0, a) = (q_d, q_a), \delta(q_0, b) = (q_b, q_d), \delta(q_0, b) = (q_d, q_b),$$
$$\delta(q_d, a) = (q_d, q_d), \delta(q_d, b) = (q_d, q_d),$$
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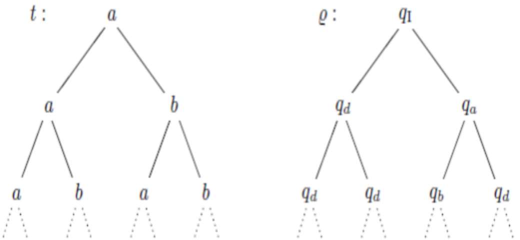

First transitions of ρ

Example: The Muller tree automaton $A = (\{q_0, q_a, q_b, q_d\}, \{a, b\}, \delta, q_0, \mathcal{F})$, where δ includes:

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and $\mathcal{F} = \{\{q_a, q_b\}, \{q_d\}\}$ recognizes the tree language
$T = \{t \in T_{\{a,b\}} |$ there is a path π through t such that $t|\pi \in (a+b)^*(ab)^\omega\}$.

Example: The Muller tree automaton $A = (\{q_0, q_1, q_2\}, \{a, b\}, \delta, q_0, \{\{q_0\}\})$, where δ includes the transitions:

$$\delta(q_0, a) = (q_0, q_0), \delta(q_0, b) = (q_1, q_1),$$
$$\delta(q_1, b) = (q_1, q_1), \delta(q_1, a) = (q_0, q_0).$$

recognizes the tree language
$T = \{t \in T_{\{a,b\}} |$ any path through t carries only finitely many $b's\}$.

The above language T can not be recognized by a Büchi tree automaton.

Büchi tree automata are strictly weaker than Muller tree automata.

Muller, Rabin, Streett, and parity tree automata all recognize the same tree languages.

- **Games on Sets**
  Let A, B be sets, i.e. $\sigma = \emptyset$. Let also $|A|, |B| \geqslant n$.
  Then $A \equiv_n B$.

Proof. Suppose after i rounds that the position is $((a_1, ..., a_i), (b_1, ..., b_i))$.
When the spoiler picks an element $a_{i+1} \in |A|$, then if

1. $a_{i+1} = a_j$ for $j \leqslant i$, then the duplicator responds with $b_{i+1} = b_j$.
2. otherwise, the duplicator responds with any $b_{j+1} \in |B| - \{b_1, ..., b_i\}$, which exists since $\|B\| \geqslant n$.

- **Games on Linear Orders**
  Let $k > 0$, and let L_1, L_2 be linear orders of length at least 2^k.
  Then $L_1 \equiv_k L_2$.

Proof. Let $L_1 = \{1, ..., n\}$ and $L_2 = \{1, ..., m\}$, with $n, m \geqslant 2^k + 1$, and $\sigma' = \{<, min, max\}$.
Let $\mathbf{a} = (a_{-1}, a_0, a_1, ..., a_i)$ and $\mathbf{b} = (b_{-1}, b_0, b_1, ..., b_i)$ after round i. Then, the duplicator can play in such a way that the following hold for $-1 \leqslant j, l \leqslant i$ after each round i:

1. 1 if $d(a_j, a_l) < 2^{k-i}$, then $d(b_j, b_l) = d(a_j, a_l)$.
2. if $d(a_j, a_l) \geqslant 2^{k-i}$, then $d(b_j, b_l) \geqslant 2^{k-i}$.
3. $a_j \leqslant a_l \Leftrightarrow b_j \leqslant b_l$.

Proof continued. The base case of $i = 0$ is immediate.
For the induction step, suppose the spoiler is making his $(i+1)$st move in L_1, such that $a_j < a_{i+1} < a_l$. By condition 3 of the inductive hypothesis $b_j < b_{i+1} < b_l$.
There are two cases:

- $d(a_j, a_l) < 2^{k-i}$. By the inductive hypothesis $d(b_j, b_l) = d(a_j, a_l)$. The duplicator finds b_{i+1} so that $d(a_j, a_{i+1}) = d(b_j, b_{i+1})$ and $d(a_{i+1}, a_l) = d(b_{i+1}, b_l)$.
- $d(a_j, a_l) \geqslant 2^{k-i}$. By inductive hypothesis $d(b_j, b_l) \geqslant 2^{k-i}$.
  We have three possibilities:
  1. $d(a_j, a_{i+1}) < 2^{k-(i+1)}$. Then $d(a_{i+1}, a_l) \geqslant 2^{k-(i+1)}$, and the duplicator chooses b_{i+1} so that $d(b_j, b_{i+1}) = d(a_j, a_{i+1})$ and $d(b_{i+1}, b_l) \geqslant 2^{k-(i+1)}$.
  2. $d(a_{i+1}, a_l) < 2^{k-(i+1)}$. This case is similar to the previous one.
  3. $d(a_j, a_{i+1}) \geqslant 2^{k-(i+1)}$, $d(a_{i+1}, a_l) \geqslant 2^{k-(i+1)}$. Since $d(b_j, b_l) \geqslant 2^{k-i}$, by choosing b_{i+1} to be the middle of the interval $[b_j, b_l]$, duplicator ensures that $d(b_j, b_{i+1}) \geqslant 2^{k-(i+1)}$ and $d(b_{i+1}, b_l) \geqslant 2^{k-(i+1)}$.

Ehrenfeucht-Fraïssé Theorem

Let A and B be two σ-structures, where σ is a relational vocabulary. Then the following are equivalent:

1. A and B agree on $FO[k]$.
2. $A \equiv_k B$.

Corollary

A property P of finite σ-structures is not expressible in FO if for every $k \in \mathbb{N}$, there exist two finite σ-structures, A_k and B_k, such that:

- $A_k \equiv_k B_k$, and
- A_k has property P, and B_k does not.

EVEN is not FO-expressible over linear orders.

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EVEN is not FO-expressible over linear orders.

Corollary2

A property P is expressible in FO iff there exists a number k such that for every two structures A, B, if $A \in P$ and $A \equiv_k B$, then $B \in P$.

Proof.

- If P is expressible by an FO sentence Φ, let $k = qr(\Phi)$. If $A \in P$, then $A \models \Phi$, and hence for B with $A \equiv_k B$, we have $B \models \Phi$. Thus, $B \in P$.
- If $A \in P$ and we force A to agree on all $FO[k]$ sentences with B, then $B \in P$. A and B have the same rank-k type, and hence P is a union of types, and thus definable by a disjunction of some of the α_K's.

Ehrenfeucht-Fraïssé Theorem

Let A and B be two σ-structures, where σ is a relational vocabulary. Then the following are equivalent:

1. A and B agree on $FO[k]$.
2. $A \simeq_k B$.

Proof. $1 \Rightarrow 2$: Assume A and B agree on all quantifier-rank $k+1$ sentences.
For the *forth* condition: Pick $a \in |A|$, and let α_i be its rank-k 1-type. Then $A \models \exists x \alpha_i(x)$, where $\exists x \alpha_i(x)$ is a sentence of quantifier-rank $k+1$. Hence $B \models \exists x \alpha_i(x)$. Let b be the witness for the existential quantifier, that is, $tp_k(A, a) = tp_k(B, b)$. Equivalently for every ψ with $qr(\psi) = k$, $A \models \psi$ iff $B \models \psi$.
By inductive hypothesis, $(A, a) \simeq_k (B, b)$.
$2 \Rightarrow 1$: Assume $A \simeq_{k+1} B$. Every $FO[k+1]$ sentence is a boolean combination of $\exists x \phi(x)$, where $\phi \in FO[k]$.
Assume that $A \models \exists x \phi(x)$, so $A \models \phi(a)$ for some $a \in |A|$. By forth, find $b \in |B|$ such that $(A, a) \simeq_k (B, b)$. By inductive hypothesis, (A, a) and (B, b) agree on $FO[k]$.
Hence, $B \models \phi(b)$, and thus $B \models \exists x \phi(x)$.